
On the radar method in general-relativistic spacetimes

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Summary. If a clock, mathematically modeled by a parametrized timelike curve in a general-relativistic spacetime, is given, the radar method assigns a time and a distance to every event which is sufficiently close to the clock. Several geometric aspects of this method are reviewed and their physical interpretation is discussed.

1 Introduction

When Einstein was asked about the meaning of time he used to say: “Time is the reading of a clock”. Taking this answer seriously, one is forced to accept that time is directly defined only at the position of a clock; if one wants to assign a time to events at some distance from the clock one needs an additional prescription. As such prescription, Einstein suggested the radar method with light rays.

Although originally designed for special relativity, the radar method works equally well in general relativity. What one needs is a clock in an arbitrary general-relativistic spacetime. Here and in the following, our terminology is as follows. A *general-relativistic spacetime* is a 4-dimensional manifold M with a smooth metric tensor field g of Lorentzian signature and a time orientation; the latter means that a globally consistent distinction between future and past has been made. A *clock* is a smooth embedding $\gamma : t \mapsto \gamma(t)$ from a real interval into M such that the tangent vector $\dot{\gamma}(t)$ is everywhere timelike with respect to g and future-pointing. This terminology is justified because we can interpret the value of the parameter t as the reading of a clock. Note that our definition of a clock does not demand that “its ticking be uniform” in any sense. Only smoothness and monotonicity is required.

The radar method assigns a time and a distance to an event q in the following way. One has to send a light ray from an event on the curve γ , say $\gamma(t_1)$, to q and receive the reflected light ray at another event on γ , say $\gamma(t_2)$, see Figure 1. The *radar time* T and the *radar distance* R of the event q with respect to γ are then defined by

$$T = \frac{1}{2}(t_2 + t_1) , \quad (1)$$

$$R = \frac{1}{2}(t_2 - t_1) . \quad (2)$$

Here and in the following, “light ray” tacitly means “freely propagating light ray”, i.e., it is understood that there is no optical medium and that mirrors or other appliances that deviate a light ray are not used. Adopting the standard formalism of general relativity, “light ray” is then just another word for “lightlike geodesic of the spacetime metric g ”.

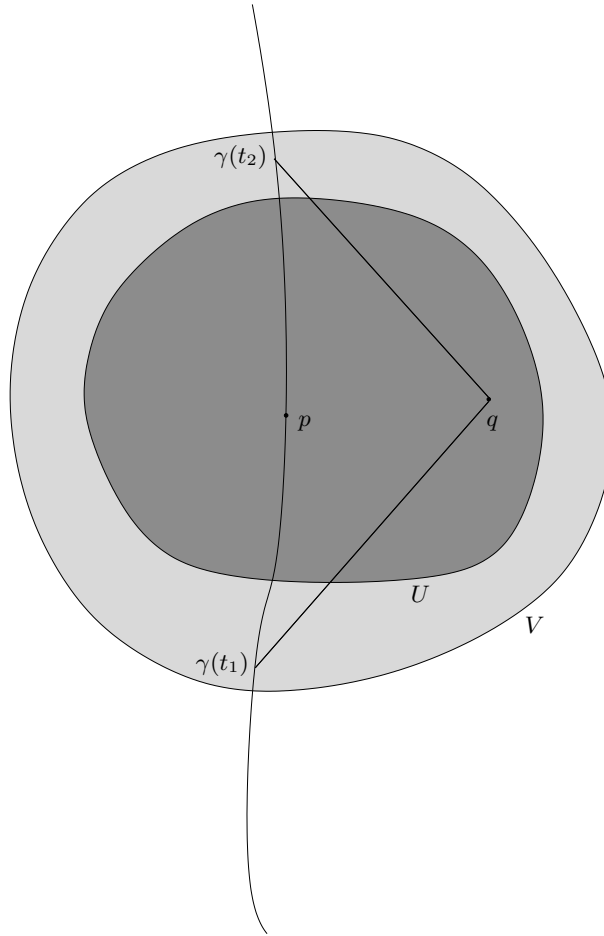


Fig. 1. The radar method.

In the following we discuss the radar method from a geometrical point of view, reviewing some known results and formulating a few new ones. The

radar method has obvious relevance for the communication with satellites in the solar system, because all such communication is made with the help of electromagnetic radiation that can be modeled, in almost all cases, in terms of light rays. By sending a light ray to a satellite and receiving the reflected signal the radar time T and the radar distance R of events at the satellite are directly measurable quantities. Note that we do not need an experimentalist at the event q where the light ray is reflected; a passive reflecting body, such as the LAGEOS satellites, would do.

2 Radar neighborhoods

The radar time T and the radar distance R of an event q with respect to a clock γ are well-defined if there is precisely one future-pointing and precisely one past-pointing light ray from q to γ . Neither existence nor uniqueness of such light rays is guaranteed.

It is possible that an event q cannot be connected to γ by any future-pointing (or any past-pointing) light ray. There are two physically different situations in which this occurs: First, q may be in a “shadow” of some obstacle that lies in the direction to γ ; second, q may be behind an “event horizon” of γ , see Figure 2.

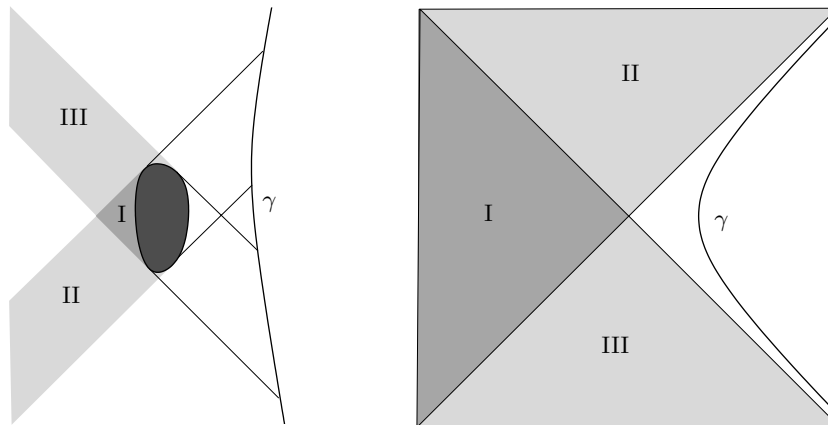


Fig. 2. Shadows (left) and horizons (right) are obstacles for the radar method. The example on the left shows a clock γ in Minkowski spacetime with a subset removed. The example on the right shows a clock γ with uniform acceleration in Minkowski spacetime. In both cases, events in the region II cannot be connected to γ by a future-pointing light ray, events in the region III cannot be connected to γ by a past-pointing light ray, and events in the region I cannot be connected to γ by any light ray.

It is also possible that an event q can be connected to γ by two or more future-pointing (or past-pointing) light rays. Whenever the future light cone (or the past light cone) of q has a caustic or a transverse self-intersection, it meets some timelike curves at least twice, see [26] or [27] for a detailed discussion. If the past light cone of q intersects γ at least twice, an observer at q sees two or more images of γ , i.e., we are in a gravitational lensing situation. Figure 5 shows an example of a past light cone that has two intersections with appropriately chosen timelike curves, as is geometrically evident from the picture.

These observations clearly show that, in an arbitrary general-relativistic spacetime, the radar method does not work globally. However, it always works locally. This is demonstrated by the following simple proposition.

Proposition 1. *Let γ be a clock in an arbitrary general-relativistic spacetime and $p = \gamma(t_0)$ some point on γ . Then there are open subsets U and V of the spacetime with $p \in U \subset V$ such that every point q in $U \setminus \text{image}(\gamma)$ can be connected to the worldline of γ by precisely one future-pointing and precisely one past-pointing light ray that stays within V , see Figure 1. In this case, U is called a radar neighborhood of p with respect to γ .*

To prove this, we just have to recall that every point in a general-relativistic spacetime admits a convex normal neighborhood, i.e., a neighborhood V such that any two points in V can be connected by precisely one geodesic that stays within V . Having chosen such a V , it is easy to verify that every sufficiently small neighborhood U of p satisfies the desired property.

As an aside, we mention that the existence of radar neighborhoods, in the sense of Proposition 1, was chosen as one of the axioms in the axiomatic approach to spacetime theory by Ehlers, Pirani and Schild [5].

If U is a radar neighborhood, the radar time T and the radar distance R are well-defined functions on $U \setminus \text{image}(\gamma)$. By continuous extension onto the image of γ one gets smooth hypersurfaces $T = \text{constant}$ that intersect γ orthogonally; hence, they are spacelike near γ . Note, however, that they need not be spacelike on the whole radar neighborhood. The hypersurfaces $R = \text{constant}$ have a cylindrical topology, see Figure 3. Incidentally, if one replaces (1) by $T = pt_1 + (1-p)t_2$ with any number p between 0 and 1, each hypersurface $T = \text{constant}$ gets a conic singularity at the intersection point with γ . This clearly shows that the choice of the factor $1/2$ is the most natural and the most convenient one. (If one allows for *direction-dependent* factors, one can get smooth hypersurfaces with factors other than $1/2$. This idea, which however seems a little bit contrived, was worked out by Havas [13] where the reader can find more on the “conventionalism debate” around the factor $1/2$.)

By covering γ with radar neighborhoods U (and the pertaining convex normal neighborhoods V), it is easy to verify that T and R coincide on the intersection of any two radar neighborhoods. Hence, T and R are well-defined on some tubular neighborhood of γ . We will now investigate how large this

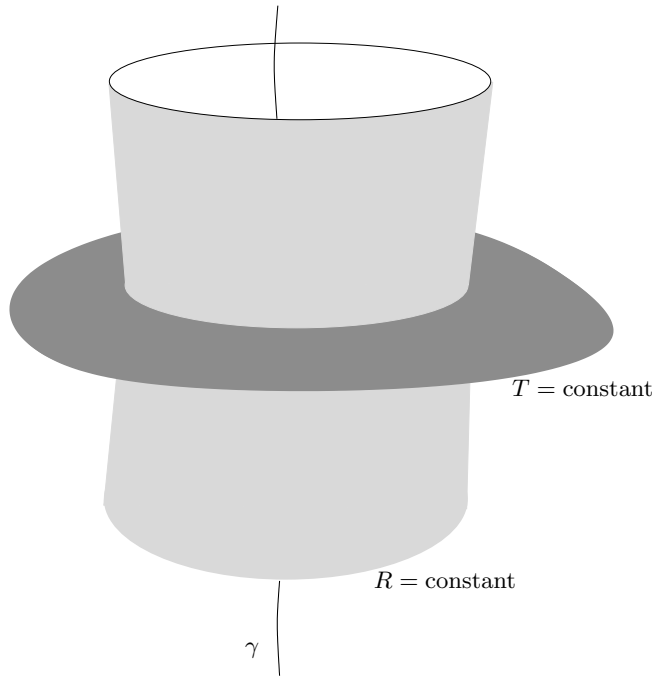


Fig. 3. Hypersurfaces $T = \text{constant}$ and hypercylinders $R = \text{constant}$ defined by the radar method.

neighborhood can be for the case of a clock moving in the Solar System, the latter being modeled by the Schwarzschild spacetime around the Sun.

To that end we consider the Schwarzschild spacetime around a non-transparent spherical body of radius r_* and mass m . (The radius is measured in terms of the radial Schwarzschild coordinate and for the mass we use geometrical units, i.e., the Schwarzschild radius is $2m$.) Using the standard deflection formula for light rays in the Schwarzschild spacetime, the following result can be easily verified. If a bundle of light rays comes in initially parallel from infinity, the rays that graze the surface of the central body will meet the axis of symmetry of the bundle at radius

$$r_f = \frac{r_*}{4\frac{m}{r_*} + O\left(\left(\frac{m}{r_*}\right)^2\right)} \approx \frac{r_*^2}{4m} \tag{3}$$

see Figure 4. This radius r_f is sometimes called the *focal length* of a non-transparent body of radius r_* and mass m . If we insert the values of our Sun we find

$$r_f \approx 550 \text{ a.u.} \tag{4}$$

where 1 a.u. = 1 astronomical unit is the average distance from the Earth to the Sun. From any event at $r < r_f$, the future-pointing and past-pointing

light rays spread out without intersecting each other. They cover the whole space $r > r_*$ with the exception of those points that lie in the “shadow” cast by the central body, see Figure 4. By contrast, light rays from an event at $r > r_f$ do intersect; the past light cone of such an event is shown in Figure 5.

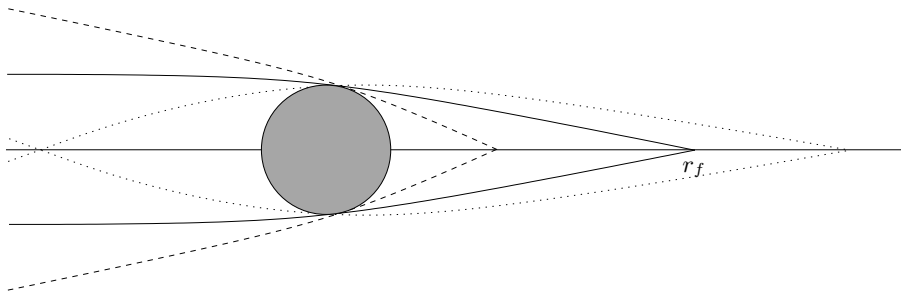


Fig. 4. The focal length r_f of a non-transparent spherical body.

As a consequence, for a clock γ moving arbitrarily in the region $r > r_*$, an event q at a radius r with $r_* < r < r_f$ can be connected to the worldline of γ by at most one future-pointing and at most one past-pointing light ray. We shall make the additional assumption that γ is inextendible and approaches neither the surface of the central body nor infinity in the future or in the past. This assures that there are no event horizons for γ . As a consequence, any event q at radius r with $r_* < r < r_f$ can be connected to γ by precisely one future-pointing and precisely one past-pointing light ray unless γ moves through the shadow cast by the central body for light rays issuing from q .

An event q at radius $r > r_f$, on the other hand, may be connected to the worldline of a clock by several future-pointing (or past-pointing) light rays. This is geometrically evident from Figure 5.

So for any clock in the Solar System the radar method assigns a unique time T and a unique distance R to any event at radius $r < r_f$, with the exception of those events for which the clock lies in the shadow of the central body. Note that for all existing spacecraft the distance from the Sun is considerably smaller than $r_f = 550$ a.u. (In October 2005, the spacecraft farthest away from the Sun was Pioneer 10 with a distance of 89 a.u.)

The idea of sending a spacecraft to $r > 550$ a.u. was brought forward by Eshleman [8] in 1979. What makes this idea attractive is the possibility of observing distant light sources strongly magnified by the focusing effect of the gravitational field of the Sun, see again Figure 4. For a detailed discussion of the perspectives of such a mission see Turyshev and Andersson [37].

It should be emphasized that our consideration applies only to a non-transparent body. If the central body is transparent, light rays passing through the central region of the body are focussed at a radius that is much smaller than the r_f given above. If the interior is modeled by a perfect fluid with

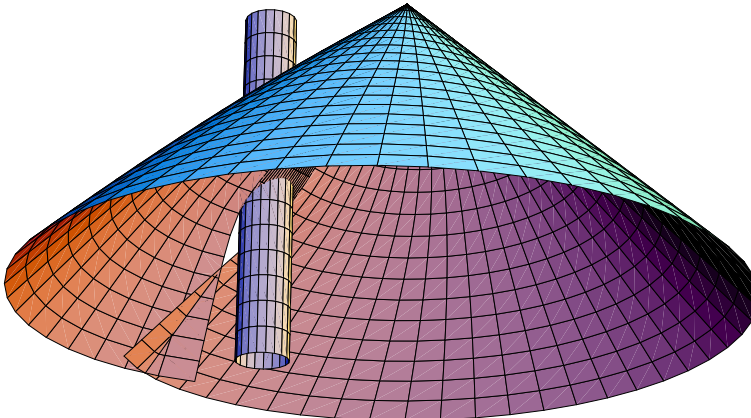


Fig. 5. Past light cone of an event at radius $r > r_f$ in the field of a non-transparent gravitating body. The “chimney” is the world tube of the gravitating body. The “shadow” is clearly seen as a gap in the light cone. In this 2 + 1-dimensional picture the light cone forms a transverse self-intersection. If the gravitating body is spherical symmetric, taking the missing spatial dimension into account shows that actually a sphere’s worth of light rays is focused at each point of the intersection set.

constant density, one finds for the Sun a focal length of 30 a.u., in comparison to the 550 a.u. for the non-transparent case, see Nemiroff and Ftacfas [22]. A transparent Sun is a reasonable model for neutrino radiation (which travels approximately, though not precisely, on lightlike geodesics) and for gravitational radiation (which travels along lightlike geodesics if modeled as a linear perturbation of the Schwarzschild background). So, the focusing at 30 a.u. might have some futuristic perspective in view of neutrino astronomy and gravitational wave astronomy.

3 Characterization of standard clocks with the radar method

If we reparametrize the curve γ , the hypersurfaces $T = \text{constant}$ and $R = \text{constant}$ change. Therefore, the radar method can be used to characterize distinguished parametrizations of worldlines, i.e., distinguished clocks. In a general-relativistic spacetime, the standard clock parametrization is defined by the condition

$$\frac{d}{dt}g(\dot{\gamma}(t), \dot{\gamma}(t)) = 0 \tag{5}$$

where g is the spacetime metric. This defines a parametrization along any timelike curve that is unique up to affine transformations, $t \mapsto at + b$ with real constants a and b . As we restrict to future-pointing parametrizations, a must be positive. Then the choice of a determines the unit and the choice of

b determines the zero on the dial. By choosing a appropriately, we can fix the unit of a standard clock such that $g(\dot{\gamma}(t), \dot{\gamma}(t)) = -1$. Then the parameter of the clock is called *proper time*. Note that under an affine reparametrization $t \mapsto at + b$ the radar time and the radar distance transform according to $T \mapsto aT + b$ and $R \mapsto aR$, i.e., the hypersurfaces $T = \text{constant}$ and $R = \text{constant}$ are relabeled but remain unchanged.

With the help of the radar method one can formulate an operational prescription that allows to test whether a clock is a standard clock. This prescription is now briefly reviewed, for details and proofs see [24]. Here we assume that the test is made in a general-relativistic spacetime; in [24] the more general case of a Weylian spacetime is considered.

To test whether a clock γ behaves like a standard clock in a particular event $\gamma(t_0)$, we emit at this event two freely falling particles in spatially opposite directions. These two freely falling particles are mathematically modeled by timelike geodesics μ and $\bar{\mu}$, and the condition that they are emitted in spatially opposite directions means that the future-oriented tangent vector to γ is a convex linear combination of the future-oriented tangent vectors to μ and $\bar{\mu}$. If we restrict to a radar neighborhood of $\gamma(t_0)$, the radar method assigns a time T and a distance R to each event on μ , and a time \bar{T} and a distance \bar{R} to each event on $\bar{\mu}$, see Figure 6. These quantities can be actually measured provided that the two freely falling particles are reflecting objects. From these measured quantities we can calculate the differential quotients dR/dT and d^2R/dT^2 along μ and the differential quotients $d\bar{R}/d\bar{T}$ and $d^2\bar{R}/d\bar{T}^2$ along $\bar{\mu}$, i.e., the *radar velocity* and the *radar acceleration* of the two freely falling particles. It is shown in [24] that the standard clock condition (5) holds at $t = t_0$ (which corresponds to $T = \bar{T} = t_0$) if and only if

$$\frac{\frac{d^2R}{dT^2}}{1 - \left(\frac{dR}{dT}\right)^2} \Big|_{T=t_0} = - \frac{\frac{d^2\bar{R}}{d\bar{T}^2}}{1 - \left(\frac{d\bar{R}}{d\bar{T}}\right)^2} \Big|_{\bar{T}=t_0}. \quad (6)$$

This prescription can be used, in particular, to directly test whether atomic clocks are standard clocks. All experiments so far are in agreement with this hypothesis, but a direct test has not been made.

There are alternative characterizations of standard clocks by Marzke and Wheeler [19] and Kundt and Hoffman [18] which also work with light rays and freely falling particles. The advantages of the method reviewed here in comparison to these two older methods are outlined in [25].

4 Radar coordinates, optical coordinates, and Fermi coordinates

Given any clock γ in any general-relativistic spacetime, the radar method assigns, as outlined above, to each event q in some tubular neighborhood of γ a radar time T and a radar distance R . In order to get a coordinate system

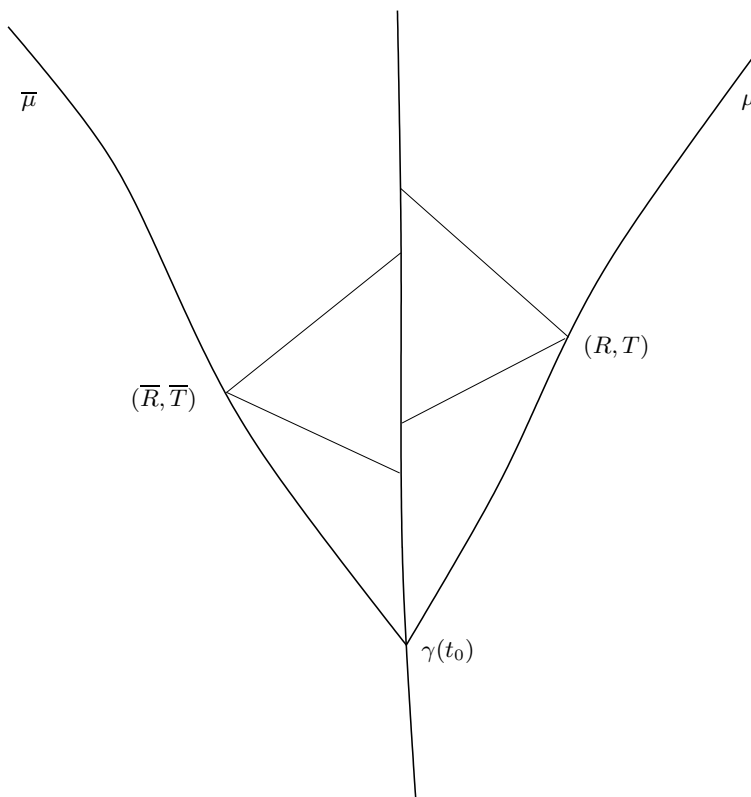


Fig. 6. Testing a clock as a standard clock with the radar method.

(*radar coordinates*) on this tubular neighborhood, we may add two angular coordinates ϑ and φ in the following way. Choose at each point $\gamma(t)$ an orthonormal tetrad $(E_0(t), E_1(t), E_2(t), E_3(t))$, smoothly dependent on t , such that $E_0(t)$ is future-pointing and tangent to γ . To each event q consider the past-oriented light ray, in the notation of Figure 1, from $\gamma(t_2)$ to q . The tangent vector to this light ray at $\gamma(t_2)$ must be proportional to a vector of the form $-E_0(t_2) + \cos \varphi \sin \vartheta E_1(t_2) + \sin \varphi \sin \vartheta E_2(t_2) + \cos \vartheta E_3(t_2)$ which defines ϑ and φ . Thus, ϑ and φ indicate at which point on the sky of γ the event q is seen. Just as with ordinary spherical coordinates, there are coordinate singularities at $R = 0$ and at $\sin \vartheta = 0$, and φ has to be identified with $\varphi + 2\pi$. Apart from these obvious pathologies, the radar coordinates $(T, R, \vartheta, \varphi)$ form a well-defined coordinate system on some tubular neighborhood of γ . There are two possibilities of modifying the radar coordinates without changing the information contained in them. First, one may replace T and R by t_1 and t_2 , according to (1) and (2), and use the modified radar coordinates $(t_1, t_2, \vartheta, \varphi)$. Second, one may switch to Cartesian-like coordinates (T, x, y, z) by introducing $x = R \cos \varphi \sin \vartheta$, $y = R \sin \varphi \sin \vartheta$, $z = R \cos \vartheta$ in order to remove the

coordinate singularities at $R = 0$ and $\sin \vartheta = 0$. Radar coordinates have been used as a tool, e.g., in the axiomatic approach to spacetime theory of Schröter and Schelb [33, 34, 32].

We will now compare radar coordinates with two other kinds of coordinate systems that can be introduced near the worldline of any clock γ : “optical coordinates” and “Fermi coordinates”. We will see that there are some similarities but also major differences between these three types of coordinate systems. For an alternative discussion of optical coordinates and Fermi coordinates see Synge [35].

Optical coordinates were introduced by Temple [36]. The alternative name *observational coordinates* is also common, see Ellis et al. [6, 7]. They assign to the event q the four-tuple $(t_2, s, \vartheta, \varphi)$, where t_2 , ϑ and φ have the same meaning as above and s is the “affine length” (or “projected length”) along the past-oriented light ray from $\gamma(t_2)$ to q . Using the exponential map \exp determined by the spacetime metric, s can be defined by the equation

$$q = \exp_{\gamma(t_2)} \left(s(-E_0(t_2) + \cos \varphi \sin \vartheta E_1(t_2) + \sin \varphi \sin \vartheta E_2(t_2) + \cos \vartheta E_3(t_2)) \right). \quad (7)$$

Just as radar coordinates, optical coordinates are well-defined, apart from the obvious coordinate singularities at $s = 0$ and $\sin \vartheta = 0$ on some tubular neighborhood of γ . The boundary of this neighborhood is reached when the past light cone of an event on γ develops a caustic or a transverse self-intersection. (Beyond such points, the optical coordinates are multi-valued. This does not mean that they are useless there; however, they do not define a coordinate system in the usual sense.) As radar coordinates require a similar condition not only on past light cones but also on future light cones, the domain of radar coordinates is always contained in the domain of optical coordinates. Also, there is an important advantage of optical coordinates in view of calculations: Optical coordinates only require to calculate the past-pointing lightlike geodesics issuing from points on γ ; radar coordinates require to calculate past-pointing and future-pointing lightlike geodesics from points on γ , and to determine their intersections. Nonetheless, for applications in the solar system radar coordinates are advantageous because they have an operational meaning. In principle, optical coordinates also have an operational meaning: $(t_2, \vartheta, \varphi)$ are the same as in radar coordinates, and for the affine (or projected) length s a prescription of measurement was worked out by Ruse [29] after this length measure had been introduced mathematically by Kermack, McCrea and Whittaker [16]. However, this prescription requires the distribution of assistants with rigid rods along each light ray issuing from γ into the past which is, of course, totally unrealistic in an astronomical situation. In this sense, optical coordinates have an operational meaning only in principle but not in practice, whereas radar coordinates have an operational meaning both in principle and in practice, at least in the Solar System. In cosmology, however, this is no longer true. Then the radar coordinates, just as the optical coordinates, have an operational meaning only in principle but not in prac-

tice: Sending a light ray to a distant galaxy and waiting for the reflected ray is a ridiculous idea. As a matter of fact, optical coordinates are much more useful in cosmology than radar coordinates. Although s is not directly measurable, it is related in some classes of spacetimes to other distance measures, such as the redshift or the angular diameter distance, which can be used to replace s . For applications of optical coordinates in cosmology see [7]. As the simplest example, one may consider optical coordinates and radar coordinates in Robertson-Walker spacetimes, cf. Jennison and McVittie [15] and Fletcher [10].

We now turn to *Fermi coordinates* which were introduced by Enrico Fermi [9]. Let us recall how they are defined. As above, we have to choose along γ an orthonormal tetrad $(E_0(t), E_1(t), E_2(t), E_3(t))$ with E_0 tangent to γ . Following Fermi, we require that the covariant derivative of each spatial axis E_μ ($\mu = 1, 2, 3$) is parallel to the tangent of γ . This *Fermi transport* law can be operationally realized by means of gyroscope axes [20] or Synge's *bouncing photon* method [35, 28]. (Actually, the construction below can be carried through equally well if the spatial axes are not Fermi parallel. What is needed is only smooth dependence on the foot-point, just as with radar coordinates and optical coordinates.) Then every event q in a sufficiently small tubular neighborhood of γ can be written in the form

$$q = \exp_{\gamma(\tau)} \left(\rho (\cos \phi \sin \theta E_1(\tau) + \sin \phi \sin \theta E_2(\tau) + \cos \theta E_3(\tau)) \right). \quad (8)$$

The Fermi coordinates of the point q are the four numbers $(\tau, \rho, \theta, \phi)$. Thus, each surface $\tau = \text{constant}$ is generated by the geodesics issuing orthogonally from the point $\gamma(\tau)$. The distance ρ is defined analogously to the affine length in the optical coordinates, but now along spacelike rather than lightlike geodesics. Also, the angular coordinates θ and ϕ are analogous to the angular coordinates ϑ and φ in the radar and optical coordinates, but now they indicate the direction of a spacelike vector, rather than the direction of the spatial part of a lightlike vector. Just as the other two coordinate systems, Fermi coordinates are well-defined only on some tubular neighborhood of γ . There are two reasons that limit this neighborhood. First, a hypersurface $\tau = \text{constant}$ might develop caustics or self-intersections. Second, two hypersurfaces $\tau = \text{constant}$ might intersect. In contrast to radar coordinates, Fermi coordinates are insensitive to reparametrizations of γ (apart from the fact that the surfaces $\tau = \text{constant}$ are relabeled). The difficulty involved in their calculation is the same as for optical coordinates which is considerably less than for radar coordinates, as already mentioned above. The essential drawback of Fermi coordinates is in the fact that they have absolutely no operational meaning: None of the four coordinates τ, ρ, θ and ϕ can be measured because there is no prescription for physically realizing a spacelike geodesic orthogonal to a worldline.

In spite of this fact, Fermi coordinates have found many applications because sometimes physically relevant effects can be conveniently calculated

in terms of Fermi coordinates. For a plea in favor of Fermi coordinates, in comparison to radar coordinates, see Bini, Lusanna and Mashhoon [1]. In Minkowski spacetime, e.g., it is fairly difficult to calculate the radar time hypersurfaces $T = \text{constant}$ for an accelerating clock. By contrast, the Fermi time hypersurfaces $\tau = \text{constant}$ are just the hyperplanes perpendicular to the worldline which are quite easy to determine. (Of course, for an accelerating clock these hyperplanes necessarily intersect, so they cannot form a smooth foliation on all of Minkowski spacetime.) It is an interesting question to ask for which clocks the radar time hypersurfaces $T = \text{constant}$ coincide with the Fermi time hypersurfaces $\tau = \text{constant}$. For standard clocks (recall Section 3) in Minkowski spacetime, Dombrowski, Kuhlmann and Proff [4] have found the following answer.

Proposition 2. *Let γ be a standard clock in Minkowski spacetime. Then the following two statements are equivalent.*

- (a) *The radar time hypersurfaces $T = \text{constant}$ are hyperplanes, i.e., they coincide with the Fermi time hypersurfaces $\tau = \text{constant}$.*
- (b) *The 4-acceleration of γ is constant (i.e., a Fermi-transported vector along γ).*

A worldline with constant 4-acceleration in Minkowski spacetime is either a straight line (“inertial observer”, for which the 4-acceleration is zero) or a hyperbola (“Rindler observer”, for which the 4-acceleration is a non-zero Fermi-transported vector, see Figure 2). It is easy to check that, indeed, in both cases the radar time hypersurfaces with respect to proper time parametrization are hyperplanes. The non-trivial statement of Proposition 2 is in the fact that these are the *only* cases for which the radar time hypersurfaces are hyperplanes.

We end this section with a remark on the fact that the term “radar coordinates” has been used in the literature also in another way. Instead of supplementing the radar time T and the radar distance R with two angular coordinates, one could choose a second clock $\tilde{\gamma}$ which defines a radar time \tilde{T} and a radar distance \tilde{R} . If the two clocks are sufficiently close, $(T, R, \tilde{T}, \tilde{R})$ can be used as coordinates on some open subset which is not a tubular neighborhood of either clock. Of course, one can replace (T, R) by (t_1, t_2) according to (1) and (2), and analogously (\tilde{T}, \tilde{R}) by $(\tilde{t}_1, \tilde{t}_2)$. In the coordinates $(t_1, t_2, \tilde{t}_1, \tilde{t}_2)$, which are used e.g. by Ehlers, Pirani and Schild [5], the coordinate hypersurfaces are light cones. Thus, the construction makes use of the fact that four light cones generically intersect in a point. (Two light cones generically intersect in a 2-dimensional manifold, where “generically” means that we have to exclude points where one of the light cones fails to be a submanifold and points where the two light cones are tangent. Similarly, three light cones generically intersect in a 1-dimensional manifold.) In this sense the radar coordinates of Ehlers, Pirani and Schild are similar to the GPS type coordinates of Blagojević, Garecki, Hehl and Obukhov [2]. The only difference is that the latter

characterize each point as intersection of four future light cones that issue from four given worldlines (“GPS satellites”), whereas the former characterize each point as intersection of two future and two past light cones that issue from two given worldlines.

5 Synchronization of clocks

Let γ be a clock in an arbitrary general-relativistic spacetime, and consider a second clock $\tilde{\gamma}$. If $\tilde{\gamma}$ is sufficiently close to γ , the radar method, carried through with respect to the clock γ , assigns a unique time $T(\tilde{t})$ and a unique distance $R(\tilde{t})$ to each event $\tilde{\gamma}(\tilde{t})$. We say that $\tilde{\gamma}$ is *synchronous* to γ if $T(\tilde{t}) = \tilde{t}$ for all \tilde{t} in the considered time interval. (Instead of synchronous one may say *Einstein synchronous* or *radar synchronous* to be more specific.) Clearly, for every worldline sufficiently close to γ there is a unique parametrization that is synchronous to γ . Selecting this particular parametrization is called *synchronization* with γ . Note that the relation of being synchronous is not symmetric: $\tilde{\gamma}$ may be synchronous to γ without γ being synchronous to $\tilde{\gamma}$. As an example, we may choose two affinely parametrized straight timelike lines γ and $\tilde{\gamma}$ in Minkowski spacetime that are not parallel. If we arrange the parameters such that $\tilde{\gamma}$ is synchronous to γ , the converse is not true. Also, the relation of being synchronous is not transitive: If $\tilde{\gamma}$ is synchronous to γ and $\hat{\gamma}$ is synchronous to $\tilde{\gamma}$, it is not guaranteed that $\hat{\gamma}$ is synchronous to γ . This non-transitivity is best illustrated with the *Sagnac effect*: Consider a family of clocks along the rim of a rotating circular platform in Minkowski spacetime. Starting with any one of these clocks, synchronize each clock with its neighbor on the right. Then there is a deficit time interval after completing the full circle.

The following proposition characterizes the special situation that two clocks are mutually synchronous.

Proposition 3. *Let $\gamma : \mathbb{R} \rightarrow M$ and $\tilde{\gamma} : \mathbb{R} \rightarrow M$ be two clocks, in an arbitrary spacetime, for which the parameter extends from $-\infty$ to ∞ . Assume that the worldlines of the two clocks have no intersection but are sufficiently close to each other such that the radar method can be carried through in both directions. If $\tilde{\gamma}$ is synchronous to γ and γ is synchronous to $\tilde{\gamma}$, then the radar distance R of $\tilde{\gamma}$ with respect γ is a constant R_0 , and the radar distance \tilde{R} of γ with respect to $\tilde{\gamma}$ is the same constant R_0 .*

Proof. The radar method carried through with γ assigns to each event $\tilde{\gamma}(\tilde{t})$ a time $T(\tilde{t})$ and a distance $R(\tilde{t})$. Analogously, the radar method carried through with respect to $\tilde{\gamma}$ assigns to each event $\gamma(t)$ a time $\tilde{T}(t)$ and a distance $\tilde{R}(t)$. This implies the following identities, see Figure 7.

$$\begin{aligned} t &= T(\tilde{T}(t) - \tilde{R}(t)) + R(\tilde{T}(t) - \tilde{R}(t)) , \\ \tilde{t} &= \tilde{T}(T(\tilde{t}) - R(\tilde{t})) + \tilde{R}(T(\tilde{t}) - R(\tilde{t})) . \end{aligned} \tag{9}$$

If the clocks are mutually synchronous, T and \tilde{T} are the identity maps, so (9) simplifies to

$$\begin{aligned}\tilde{R}(t) &= R(t - \tilde{R}(t)) , \\ R(\tilde{t}) &= \tilde{R}(\tilde{t} - R(\tilde{t})) .\end{aligned}\tag{10}$$

These equations hold for all t and all \tilde{t} in \mathbb{R} . By considering the special case $t = \tilde{t} - R(\tilde{t})$ we find

$$R(\tilde{t}) = R(\tilde{t} - 2R(\tilde{t}))\tag{11}$$

for all \tilde{t} in \mathbb{R} . To ease notation, we drop the tilde in the following. By induction, (11) yields

$$R(t) = R(t - 2nR(t)) \quad \text{for all } n \in \mathbb{N} .\tag{12}$$

It is now our goal to prove that (12) implies that R is a constant. By contradiction, assume there is a point where R has negative derivative, $R'(t_*) < 0$. Then we must have

$$\frac{t_* - (t_* + \varepsilon) + 2R(t_* + \varepsilon)}{2(R(t_*) - R(t_* + \varepsilon))} \longrightarrow \infty \quad \text{for } \varepsilon \rightarrow +0 ,\tag{13}$$

because, by our assumption that the worldlines of the two clocks do not intersect, $R(t_*) > 0$. Thus, there is an infinite sequence t_n that converges towards t_* from above, such that

$$n = \frac{t_* - t_n + 2R(t_n)}{2(R(t_*) - R(t_n))} \quad \text{for all sufficiently large } n \in \mathbb{N} .\tag{14}$$

As (14) can be rewritten as

$$t_n - 2(n+1)R(t_n) = t_* - 2nR(t_*) ,\tag{15}$$

our earlier result (12) yields $R(t_n) = R(t_*)$ for all members t_n of our sequence, which obviously contradicts the assumption $R'(t_*) < 0$. We have thus proven that $R'(t) \geq 0$ for all t . But then we must have $R'(t) = 0$ for all t , because, again by (12), to every t there is a smaller parameter value at which the function R takes the same value. Hence, R must be a constant, $R(t) = R_0$ for all t . It is obvious from (10) that then \tilde{R} must take the same constant value.

We illustrate this result with an example in Minkowski spacetime, using standard coordinates (x^0, x^1, x^2, x^3) such that the metric takes the form

$$g = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 .\tag{16}$$

We consider the two clocks

$$\gamma(t) = (t, 0, 0, 0) ,\tag{17}$$

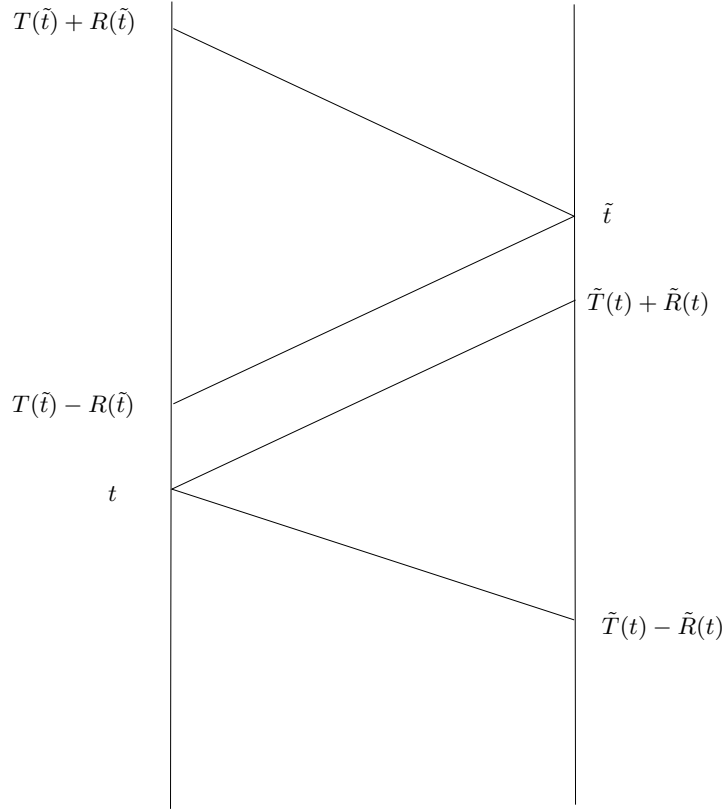


Fig. 7. Illustration of the proof of Proposition 3.

$$\tilde{\gamma}(\tilde{t}) = (\sqrt{1 - \omega^2 R_0^2} \tilde{t}, R_0 \cos \omega \tilde{t}, R_0 \sin \omega \tilde{t}, 0), \quad (18)$$

where ω and R_0 are constants such that $\omega^2 R_0^2 < 1$. In both cases the parameter is proper time, i.e., both clocks are standard clocks with the usual choice of the time unit. The first clock is at rest at the origin of the coordinate system, the other clock moves with constant angular velocity ω on a circle with radius R_0 around the origin. An elementary exercise shows that the radar method carried through with respect to γ assigns to each event $\tilde{\gamma}(\tilde{t})$ the time $T(\tilde{t}) = \sqrt{1 - \omega^2 R_0^2} \tilde{t}$ and the distance $R(\tilde{t}) = R_0$. On the other hand, the radar method carried through with respect to $\tilde{\gamma}$ assigns to each event $\gamma(t)$ the time $\tilde{T}(t) = t/\sqrt{1 - \omega^2 R_0^2}$ and the distance $\tilde{R}(t) = R_0/\sqrt{1 - \omega^2 R_0^2}$. Thus, neither clock is synchronous to the other, and they assign to each other constant but different distances. Now let us modify this example by changing the time unit for $\tilde{\gamma}$ according to the affine transformation $\tilde{t} \mapsto \hat{t} = \sqrt{1 - \omega^2 R_0^2} \tilde{t}$. This transformation replaces $\tilde{\gamma}$ with a new clock $\hat{\gamma}$,

$$\hat{\gamma}(\hat{t}) = \left(\hat{t}, R_0 \cos \frac{\omega \hat{t}}{\sqrt{1 - \omega^2 R_0^2}}, R_0 \sin \frac{\omega \hat{t}}{\sqrt{1 - \omega^2 R_0^2}}, 0 \right). \quad (19)$$

Note that $\hat{\gamma}$ is still a standard clock, but not with the usual time unit. We now find that the radar method carried through with respect to γ assigns to each event $\hat{\gamma}(\hat{t})$ the time $T(\hat{t}) = \hat{t}$ and the distance $R(\hat{t}) = R_0$. On the other hand, the radar method carried through with respect to $\hat{\gamma}$ assigns to each event $\gamma(t)$ the time $\hat{T}(t) = t$ and the distance $\hat{R}(t) = R_0$. This modified example illustrates that Proposition 3 may apply to situations where there is no symmetry between the two clocks.

6 Observer fields

By an *observer field* on a general-relativistic spacetime we mean a smooth vector field V which is everywhere timelike and future-pointing. An observer field V is called a *standard observer field* if $g(V, V) = -1$. According to our earlier terminology, integral curves of observer fields are clocks, and integral curves of standard observer fields are standard clocks with the usual choice of time unit. For the sake of brevity, we will refer to the integral curves of an observer field V as to “clocks in V ”. Note that V fixes the parametrization for each of its integral curves uniquely up to an additive constant, i.e., for each clock in V there is still the freedom of “choosing the zero point on the clock’s dial”.

In this section we consider the following four properties of an observer field V , and for each of them we give necessary and sufficient conditions on V under which it is satisfied.

- Property A:** For each clock γ in V , any other clock in V that is sufficiently close to γ such that the radar method can be carried through is synchronous with γ , provided that the additive constant has been chosen appropriately.
- Property B:** For each clock γ in V , any other clock in V that is sufficiently close to γ such that the radar method can be carried through has temporally constant radar distance from γ .
- Property C:** For any three clocks $\gamma_1, \gamma_2, \gamma_3$ in V which are sufficiently close to each other the following is true. If one light ray from γ_1 to γ_3 intersects the worldline of γ_2 , then all light rays from γ_1 to γ_3 intersect the worldline of γ_2 .
- Property D:** For any two clocks γ_1 and γ_2 in V that are sufficiently close to each other, the light rays from γ_1 to γ_2 and the light rays from γ_2 to γ_1 span the same 2-surface.

All four properties are obviously closely related to the radar method, and we will discuss them one by one. In the following we have to assume that the reader is familiar with the standard text-book decomposition of the covariant

derivative of an observer field into acceleration, rotation, shear and expansion, and with the related physical interpretation.

We begin with Property A. We emphasize that in the formulation of this property we restricted to clocks that are sufficiently close to each other such that the radar method can be carried through, but *not* to clocks that are infinitesimally close. The synchronizability condition for infinitesimally close clocks is a standard text-book matter, see e. g. Sachs and Wu [30], Sect. 2.3 and 5.3. One finds that this condition is satisfied, for an appropriately rescaled observer field $e^f V$, if and only if V is irrotational, i.e. locally hypersurface-orthogonal. The rescaling means that the clocks of the observers have to be changed appropriately. The synchronization condition for clocks that are not infinitesimally close to each other is less known. It is given in the following Proposition.

Proposition 4. (i) *A standard observer field V satisfies Property A if and only if V is an irrotational Killing vector field.*
(ii) *An arbitrary (not necessarily standard) observer field V satisfies Property A if and only if V is an irrotational conformal Killing vector field.*

Proof. The hard part of the proof is in a paper by Kuang and Liang [17] who proved the following. If V is a standard observer field, any point admits a neighborhood that can be sliced into hypersurfaces that are synchronization hypersurfaces for all clocks in V if and only if V is proportional to an irrotational Killing vector field. In this case, the flow of the Killing vector field maps synchronization hypersurfaces onto synchronization hypersurfaces. Clearly, Property A requires in addition that the hypersurfaces can be labeled such that along each integral curve of V the labeling coincides with proper time. Thus, the flow of V itself must map synchronization hypersurfaces onto synchronization hypersurfaces. This completes the proof of Proposition 4 (i). Now let V be an arbitrary observer field on the spacetime (M, g) . Then it is a standard observer field on the conformally rescaled spacetime $(M, -g(V, V)^{-1}g)$. Clearly, as a conformal factor does not affect the paths of lightlike geodesics, V satisfies Property A on the original spacetime if and only if it satisfies Property A on the conformally rescaled spacetime. By Proposition 4 (i), the latter is true if and only if V is a normalized irrotational Killing vector field of the metric $-g(V, V)^{-1}g$ and, thus, if and only if V is an irrotational conformal Killing vector field of the original metric g . This completes the proof of Proposition 4 (ii).

A spacetime that admits an irrotational Killing vector field normalized to -1 is called *ultrastatic*, and a spacetime that admits an irrotational conformal Killing vector field is called *conformally static*. Hence, we can summarize, that ultrastaticity is necessary and sufficient for the existence of a standard observer field that satisfies Property A, and conformal staticity is necessary and sufficient for the existence of a (not necessarily standard) observer field that satisfies Property A. A simple and instructive example is an expanding

Robertson-Walker spacetime. Such a spacetime admits a timelike conformal Killing vector field W orthogonal to hypersurfaces such that $g(W, W)$ is non-constant along the integral curves of W . The flow lines of W are often referred to as the ‘‘Hubble flow’’. By Proposition 4 (ii), the observer field W satisfies Property A, i.e., if we use on the Hubble flow lines a parametrization adapted to W (often called ‘‘conformal time’’), then the clocks are synchronous. However, the standard observer field V that results by normalizing W does not satisfy Property A, i.e., if we use on the Hubble flow lines the parametrization by proper time, the clocks are not synchronous (unless they are infinitesimally close to each other). This example demonstrates that it is sometimes mathematically convenient to use non-standard observer fields.

We now turn to Property B which may be viewed as a rigidity condition. Again, there is a well-known text-book result on the situation where only clocks that are infinitesimally close are considered: For a standard observer field, any two clocks that are infinitesimally close to each other have temporally constant radar distance if and only if V has vanishing shear and vanishing expansion. This is known as the *Born rigidity* condition, referring to a classical paper by Born [3] who introduced this rigidity notion in *special relativity*. The differential equations for Born rigid observer fields in *general relativity* were first written by Salzmann and Taub [31]. They have non-trivial integrability conditions, i.e., Born rigid observer fields do not exist on arbitrary spacetimes. The following important result is known as the *generalized Herglotz-Noether theorem*: If V is a Born-rigid, not hypersurface orthogonal standard observer field on a spacetime with constant curvature, then V is proportional to a Killing vector field. This was proven by Herglotz [14] and Noether [23] for the case of vanishing curvature (Minkowski spacetime) and generalized by Williams [38] to the case of positive or negative curvature (deSitter or anti-deSitter spacetime). As in the case of the synchronization condition, the rigidity condition for clocks that are not infinitesimally close to each other is less well known. It is given in the following proposition.

- Proposition 5.** (i) *A standard observer field V satisfies Property B if and only if V is proportional to a Killing vector field.*
(ii) *An arbitrary (not necessarily standard) observer field V satisfies Property B if and only if $V = e^f W$, where W is a conformal Killing vector field and f is a scalar function that is constant along each integral curve of V .*

Proof. For the proof of Proposition 5 (i) we refer to Müller zum Hagen [21]. To prove Proposition 5 (ii), let V be an arbitrary observer field. As in the proof of Proposition 4 (ii), we make use of the fact that V satisfies Property B with respect to the metric g if and only if V satisfies Property B with respect to the conformally rescaled metric $-g(V, V)^{-1}g$. The latter is true, by Proposition 5 (i), if and only if $V = e^f W$ where W is a Killing vector field of the metric $-g(V, V)^{-1}g$. The latter condition is true if W is a conformal Killing vector field of the original metric g and $-g(V, V)^{-1}g(W, W) = e^{-2f}$ is constant along each integral curve of V . This completes the proof.

A spacetime that admits a timelike Killing vector field is called *stationary* and a spacetime that admits a timelike conformal vector field is called *conformally stationary*. Hence, we can summarize that stationarity is necessary and sufficient for the existence of a standard observer field with Property B, and that conformal stationarity is necessary and sufficient for the existence of a (not necessarily standard) observer field with Property B. As an example, we may again consider the Hubble flow in an expanding Robertson-Walker spacetime. As with Property A, Property B is satisfied if we use conformal time but not if we use proper time.

We now turn to Property C. This property can be rephrased in the following way. If, from the position of one clock in V , two other clocks in V are seen at the same spot in the sky (i.e., one behind the other), then this will be true for all times. In a more geometric wording, Property C requires that the light rays issuing from any one integral curve of V into the past together with the integral curves of V are surface forming. In Hasse and Perlick [12], observer fields with this property were called *parallax free*, and the following proposition was proven.

Proposition 6. *An observer field V satisfies Property C if and only if V is proportional to a conformal Killing vector field.*

The “if” part follows from the well-known fact that the flow of a conformal Killing vector field maps light rays onto light rays. The proof of the “only if” part is more involved, see [12]. Clearly, Property C refers only to the motion of the clocks, but not to their “ticking” (i.e., not to the parametrization). Hence, it is irrelevant whether we consider standard observer fields or non-standard observer fields.

Finally we turn to Property D which is a way of saying that light rays from γ_1 to γ_2 take the same spatial paths as light rays from γ_2 to γ_1 . If this property is satisfied, there is a timelike 2-surface between γ_1 and γ_2 that is ruled by two families of lightlike geodesics. Note that *any* timelike 2-surface is ruled by two families of lightlike curves; in general, however, these will not be geodesics. Foertsch, Hasse and Perlick [11] have shown that a timelike 2-surface is ruled by two families of lightlike geodesics if and only if its second fundamental form is a multiple of its first fundamental form. In the mathematical literature, such surfaces are called *totally umbilic*. Some construction methods and examples of timelike totally umbilic 2-surfaces are discussed in Foertsch, Hasse and Perlick [11]. Note that in an arbitrary spacetime totally umbilic 2-surfaces need not exist. This shows that Property D, which requires such a 2-surface between any two sufficiently close integral curves of some observer field, is quite restrictive. A criterion is given in the following proposition.

Proposition 7. *An observer field V satisfies Property D if and only if V is proportional to an irrotational conformal Killing vector field.*

Proof. The proof of the “if” part follows from Foertsch, Hasse and Perlick [11], Proposition 3. To prove the “only if” part, fix any event p and let γ

be the clock in V that passes through p . On a neighborhood of p , with the worldline of γ omitted, consider two vector fields X and Y such that the integral curves of X are future-pointing light rays and the integral curves of Y are past-pointing light rays issuing from the worldline of γ . This condition fixes X and Y uniquely up to nowhere vanishing scalar factors. Property D requires X and Y to be surface forming and V to be tangent to these surfaces. The first condition is true, by the well-known Frobenius theorem, if and only if the Lie bracket of X and Y is a linear combination of X and Y , and the second condition is true if and only if V is a linear combination of X and Y . As a consequence, the Lie bracket of Y and V must be a linear combination of Y and V , i.e., Y and V must be surface forming. This proves that V must satisfy Property C. Hence, by Proposition 6, V must be proportional to a conformal Killing vector field. What remains to be shown is that this conformal Killing vector field is irrotational, i.e., hypersurface orthogonal. To that end we come back to the observation that V is a linear combination of X and Y . This means that, for any integral curve of V in the considered neighborhood, light rays from γ are seen in the same spatial direction in which light rays to γ are emitted. This is true, in particular, for integral curves of V that are infinitesimally close to γ . Synge [35] and, in a simplified way, Pirani [28] have shown that this “bouncing photon construction” implies that the connecting vector between the two worldlines is Fermi transported. This is true for *all* pairs of infinitesimally neighboring worldlines of V if and only if V is irrotational. This completes the proof of Proposition 7.

The important result to be kept in mind is that, in a spacetime that is not conformally stationary, it is impossible to find an observer field that satisfies any of the four Properties A, B, C, D. This demonstrates that several features of the radar method which intuitively might be taken for granted, are actually not satisfied in many cases of interest.

References

1. D. Bini, L. Lusanna, and B. Mashhoon. Limitations of radar coordinates. *Int. J. Modern Phys.*, D 14:1413–1429, 2005.
2. M. Blagojević, J. Garecki, F. W. Hehl, and Y. N. Obukhov. Real null coframes in general relativity and GPS type coordinates. *Phys. Rev.*, D 65:044018, 2002.
3. M. Born. Die Theorie des starren Elektrons in der Kinematik des Relativitätsprinzips. *Annalen der Physik*, 30:1–56, 1909.
4. P. Dombrowski, J. Kuhlmann, and U. Proff. On the spatial geometry of a non-inertial observer in special relativity. In T. J. Willmore and N. J. Hitchin, editors, *Global Riemannian Geometry*, pages 177–193, Chichester, UK, 1984. Horwood.
5. J. Ehlers, F. Pirani, and A. Schild. The geometry of free fall and light propagation. In L. O’Raifeartaigh, editor, *General Relativity*, pages 63–84, Oxford, UK, 1972. Clarendon.

6. G. F. R. Ellis. Limits to verification in cosmology. In J. Ehlers, editor, *Proceedings of the 9th Texas Symposium on Relativistic Astrophysics*, volume 336 of *Annals of the New York Academy of Sciences*, pages 130–160, New York, U.S.A., 1980. New York Academy of Sciences.
7. G. F. R. Ellis, S. D. Nel, R. Maartens, W. R. Stoeger, and A. P. Whitman. Ideal observational cosmology. *Phys. Rep.*, 124:315–417, 1985.
8. R. v. Eshleman. Gravitational lens of the sun – Its potential for observations and communications over interstellar distances. *Science*, 205:1133–1135, 1979.
9. E. Fermi. Sopra i fenomeni che avvengono in vicinanza di una linea ovaria. *Rendiconti Accad. Linc. Roma*, 31(1):21–23, 1922.
10. R. C. Fletcher. Light exchange in an expanding universe in fixed coordinates. *Amer. J. Phys.*, 62:648–656, 1994.
11. T. Foertsch, W. Hasse, and V. Perlick. Inertial forces and photon surfaces in arbitrary spacetimes. *Class. Quant. Grav.*, 20:4635–4651, 2003.
12. W. Hasse and V. Perlick. Geometrical and kinematical characterization of parallax-free world models. *J. Math. Phys.*, 29:2064–2068, 1988.
13. P. Havas. Simultaneity, conventionalism, general covariance and the special theory of relativity. *Gen. Relativ. Gravit.*, 19:435–453, 1987.
14. G. Herglotz. Über den vom Standpunkt des Relativitätsprinzips als “starr” zu bezeichnenden Körper. *Annalen der Physik*, 31:393–415, 1910.
15. R. C. Jennison and G. C. McVittie. Radar distance in Robertson-Walker spacetimes. *Royal Society Edinburgh Communications (Physical Sciences)*, 1:1–17, 1975.
16. W. O. Kermack, W. H. McCrea, and E. T. Whittaker. On properties of null geodesics and their application to the theory of radiation. *Proc. Roy. Soc. Edinburgh*, 53:31–47, 1932.
17. Z. Kuang and C. Liang. All space-times admitting strongly synchronizable reference frames are static. *J. Math. Phys.*, 34:1016–1021, 1993.
18. W. Kundt and B. Hoffmann. Determination of gravitational standard time. In Anonymous, editor, *Recent developments in general relativity*, pages 303–306, Oxford, UK, 1962. Pergamon.
19. R. F. Marzke and J. A. Wheeler. Gravitation as geometry. I: The geometry of space-time and the geometrodynamical standard meter. In H. Y. Chiu and W. F. Hoffmann, editors, *Gravitation and relativity*, pages 40–64, New York, USA, 1964. Benjamin.
20. C. Misner, K. Thorne, and J. A. Wheeler. *Gravitation*. Freeman, San Francisco, 1973.
21. H. Müller zum Hagen. A new physical characterization of stationary and static space-times. *Proc. Cambridge Philos. Soc.*, 71:381–389, 1972.
22. R. J. Nemiroff and C. Ftacilas. Our sun as a gravitational lens. *Bull. Amer. Astron. Soc.*, 29:827, 1997.
23. F. Noether. Zur Kinematik des starren Körpers in der Relativtheorie. *Annalen der Physik*, 31:919–944, 1910.
24. V. Perlick. Characterization of standard clocks by means of light rays and freely falling particles. *Gen. Relativ. Gravit.*, 19:1059–1073, 1987.
25. V. Perlick. Characterization of standard clocks in general relativity. In U. Majer and H.-J. Schmidt, editors, *Semantical aspects of spacetime theories*, pages 169–180, Mannheim, Germany, 1994. BI Wissenschaftsverlag.
26. V. Perlick. Criteria for multiple imaging in Lorentzian manifolds. *Class. Quant. Grav.*, 13:529–537, 1996.

27. V. Perlick. Gravitational lensing from a spacetime perspective. *Living Rev. Relativity*, 7(9), 2004. <http://www.livingreviews.org/lrr-2004-9>.
28. F. Pirani. A note on bouncing photons. *Bull. Acad. Polon. Sci., Ser. Sci. Mat. Astr. Phys.*, 13:239–242, 1965.
29. H. S. Ruse. On the measurement of spatial distance in a curved space-time. *Proc. Roy. Soc. Edinburgh*, 53:79–88, 1933.
30. R. K. Sachs and H.-H. Wu. *General relativity for mathematicians*. Springer, New York, 1977.
31. G. Salzmann and A. H. Taub. Born-type rigid motion in relativity. *Phys. Rev.*, 95:1659–1669, 1954.
32. U. Schelb. An axiomatic basis of space-time theory. III: Construction of a differentiable manifold. *Rep. Math. Phys.*, 31:297–309, 1992.
33. J. Schröter. An axiomatic basis of space-time theory. I: Construction of a causal space with coordinates. *Rep. Math. Phys.*, 26:303–333, 1988.
34. J. Schröter and U. Schelb. An axiomatic basis of space-time theory. II: Construction of a c^0 -manifold. *Rep. Math. Phys.*, 31:5–27, 1992.
35. J. L. Synge. *Relativity. The general theory*. North-Holland, Amsterdam, 1960.
36. G. Temple. New systems of normal co-ordinates for relativistic optics. *Proc. Roy. Soc. London*, A 168:122–148, 1938.
37. S. G. Turyshev and B. G. Andersson. The 550-au mission: a critical discussion. *Mon. Not. Roy. Astron. Soc.*, 341:577–582, 2003.
38. G. Williams. Motions in relativistic spaces. *J. Math. Anal. Appl.*, 22:646–657, 1968.