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DOES GENERAL RELATIVITY ALLOW AN OBSERVER TO VIEW AN ETERNITY IN A FINITE TIME?

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I investigate whether there are general relativistic spacetimes that allow an observer μ to collect in a finite time all the data from the worldline of another observer λ, where the proper length of λ's worldline is infinite. The existence of these spacetimes has a bearing on certain problems in computation theory. A theorem shows that most standard spacetimes cannot accommodate this scenario. There are however spacetimes which can: anti-de Sitter spacetime is one example.

Key words: general relativity, eternity, anti-de Sitter, Pitowsky.

1. INTRODUCTION

Any computer primed to perform an infinite number of computational steps must take an eternity to complete the task, because completion in a finite time would imply an unbounded signal velocity — conflicting with relativity theory.[1] This would seem to suggest that the full potential of these computers is available only to immortal computer users. But, as Itamar Pitowsky has pointed out (private communication), there is no reason why the computer user must remain beside the computer. If he follows a different worldline his clock will tick at a rate different to that of the computer's clock, and perhaps an extreme case could be organized in which the rates are such that the finite proper time as measured by the computer user "corresponds" to an infinite proper time as measured by the computer. In this case, and granting also that the computer can always signal to the computer user, the computer user will take only a finite time to view the eternity of the computer's life and with it the results of its computations.

Are there relativistic spacetimes in which this is allowed to occur? This is Pitowsky's question, which I will now attempt to answer.

2. THE STORY OF DAVE, HAL, AND GOLDBACH

Let us spell out the details of the problem with the help of a little story, in order to clarify what is required of those spacetimes that permit the scenario described above. To begin with, there lives an immortal computer[2], called HAL, and a mortal computer user, called Dave. Suppose that Dave is itching to know whether the Goldbach conjecture is true or false. He turns to HAL for help. HAL is happy to sacrifice her eternal life to this great question and begins to test systematically all the even numbers to see if there exists one which is not the sum of two primes. Meanwhile Dave has done some calculations of his own and has discovered that there exists in the universe a special worldline of length[3] one hour, which contains a point to the future of HAL's worldline. It so happens that this worldline is close by, and Dave decides to follow it. "Do keep in touch," he calls to HAL as he leaves. Throughout his journey Dave receives a stream of messages from HAL telling him which even numbers have been tested and whether or not a counter-example to the conjecture has been found. As the hour draws on, the messages are received at an ever increasing rate until finally, in the hour's closing moments, the rate blows up forcing the messages to compact together. A moment later and the messages cease. Then, suddenly, Dave knows the truth of the Goldbach conjecture.

Let us call the above scenario with HAL and Dave, (S), and a spacetime which permits (S) to occur, *Pitowsky.* Before giving a precise definition of a Pitowsky spacetime, we need to gain a little more insight into these spacetimes by attempting to accommodate (S) in the simplest of relativistic arenas: Minkowski spacetime.

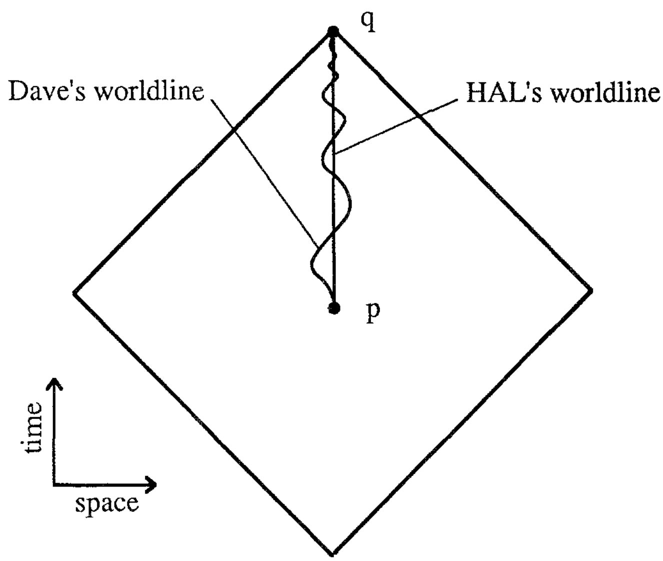

Fig. 1. Conformal diagram of Minkowski spacetime.

3. IS MINKOWSKI SPACETIME PITOWSKY?

Without any loss of generality, let us work in HAL's rest frame. We need to construct a worldline for Dave such that, after one hour of his proper time, he is to the future of the whole of HAL's worldline. However it is immediately clear that no such point exists, except perhaps the "point" at which HAL ceases to exist, i.e., at $x = 0, t = \infty$. I write "point" because it is added to the usual representation of Minkowski spacetime and consequently cannot be regarded as a being a member of it. This point at infinity is labelled q in the conformal diagram depicted in Fig. 1. If Dave oscillates to-and-fro about the line $x = 0$, along a worldline which approaches a null zig-zag line exponentially quickly, then it is easy to show that his total proper distance from $p(x = 0, t = 0)$ to $q(x = 0, t = \infty)$ is finite.

But even granting that Dave can reach q in finite time is not enough to capture the spirit of (S). For supposing Goldbach's conjecture to be true *de facto*, Dave will never come to know the truth within Minkowski spacetime (as against the point q). To guard against this possibility, I will demand of a Pitowsky spacetime that the point q is not a point at infinity but a normal spacetime point. Imposing this condition means Dave will live through the event q, thus allowing him to reflect upon all the data he received from HAL prior to then.

This proviso is implicit in the following definition of a Pitowsky spacetime and, as one might expect, Minkowski spacetime fails to satisfy its conditions.

4. THE DEFINITION OF A PITOWSKY SPACETIME

The discussion above suggests the following preliminary definition.

> (Prelim). Let (M, g_{ab}) be a spacetime. Then (M, g_{ab}) is Pitowsky if there exists two future-directed timelike curves $\lambda, \mu \subset M$ which share the same past endpoint, and a point $q \in \mu$ such that:
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> $$(1) \int_\lambda d\tau = \infty \quad (2) \int_\mu d\tau < \infty \quad (3)\ \lambda \subset J^-(q).$$

The curves λ and μ represent, respectively, the worldlines of HAL and Dave; q represents the point at which Dave has finally gathered all the data from HAL's worldline. That λ and μ share the same past endpoint reflects that HAL and Dave are initially together. Conditions (1) and (2) reflect the longevity of the two observers: HAL lives forever, while Dave lives for only a finite time.[4] Condition (3) ensures that HAL can always signal to Dave and, moreover, that all these signals reach Dave before Dave passes through q.

In fact, the definition of a Pitowsky spacetime has an equivalent but simpler formulation; call it (Definition).

> (Definition). Let (M, g_{ab}) be a spacetime. Then (M, g_{ab}) is Pitowsky if there exists a future-directed timelike curve $\lambda \subset M$ with past endpoint, and a point $q \in M$ such that:
>
> $$(i) \int_\lambda d\tau = \infty \quad (ii)\ \lambda \subset J^-(q).$$

This follows because all those spacetimes which satisfy the conditions of (Definition) admit a curve with the properties of the curve μ in (Prelim); i.e., a future-directed timelike curve of finite length which has the same past endpoint, p, say, as λ, and which passes through q. To see this, first notice that since λ is timelike (ii) implies that p can

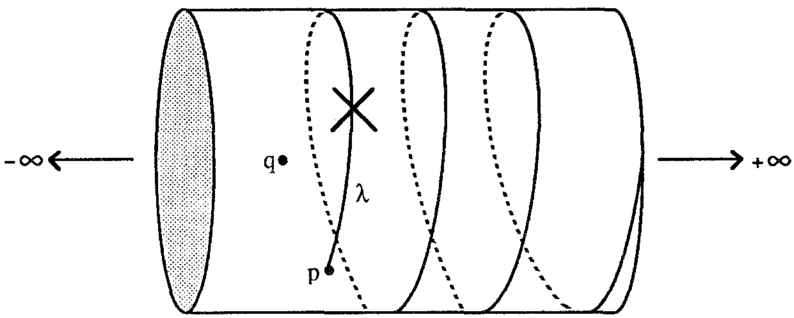

Fig. 2. A causally vicious Pitowsky spacetime.

be joined to q by a future-directed timelike curve, ν, say. If the length of ν is finite, then ν is the required curve. If the length is infinite, then one can construct the required curve by using the well-known result that any timelike curve with fixed endpoints can be deformed into a timelike curve of arbitrarily small length (by being deformed into a "zig-zag-stay-close-to-the-light-cone" curve; cf. Fig. 1).

(Definition) tells us that, roughly speaking, a spacetime is Pitowsky if it admits a point to the future of a curve of infinite length.

5. THREE EXAMPLES AND A THEOREM

A simple example of a Pitowsky spacetime is afforded by the cylindrical spacetime formed by rolling up a temporal segment of two dimensional Minkowski spacetime. It is easy to see from Fig. 2 that the conditions of (Definition) are satisfied.

This is perhaps not a very exciting example though, since the spacetime's closed timelike curves make it rather unphysical. We will shortly see an example of a causally well-behaved Pitowsky spacetime. However a great many causally well-behaved spacetimes are not Pitowsky, as shown by the following

Theorem. No globally hyperbolic spacetime is Pitowsky.

(Remark. A spacetime (M, g_{ab}) is *strongly causal* if about each point $p \in M$ every neighbourhood of p contains a neighbourhood of p which no causal curve intersects more than once (Hawking and Ellis, p.192). A spacetime (M, g_{ab}) is *globally hyperbolic* if (M, g_{ab}) is strongly causal and, for any two points $x, y \in M$, $J^-(x) \cap J^+(y)$ is compact or empty (ibid., p.206).)

Proof. Let (M, g_{ab}) be a globally hyperbolic spacetime. Suppose there exists a future-directed timelike curve λ and a point q such that $\lambda \subset J^-(q)$. (It will be shown that the length of λ *must* be finite — thus preventing (M, g_{ab}) being Pitowsky.) Since strong causality holds on M, the set $J^-(q) \cap J^+(p)$ has an open cover $\{U_\alpha\}$, where each U_α is a convex normal neighbourhood with compact closure such that λ enters each U_α at most once. From this cover construct another cover $\{V_\beta\}$ of $J^-(q) \cap J^+(p)$, where each $V_\beta \subset U_\alpha$ for some α and such that, for each V_β with $V_\beta \cap \lambda \neq \emptyset$, the length of λ in V_β does not exceed ϵ. Now, since (M, g_{ab}) is globally hyperbolic, $J^-(q) \cap J^+(p)$ is compact, which implies that $\{V_\beta\}$ has a finite subcover. This finite cover contains λ since $\lambda \subset J^-(q) \cap J^+(p)$. Therefore λ can be covered with a finite number of V_βs, each of which contains a segment of λ of finite length. Thus λ has finite length and so (M, g_{ab}) is not Pitowsky. □

Most of the standard spacetimes — Minkowski spacetime, Friedmann models, Schwarzschild solution — are globally hyperbolic and so, by the result above, are non-Pitowsky.

Looking at solutions farther afield, however, I claim: *anti-de Sitter spacetime is Pitowsky.*[5] This spacetime — which is, incidentally, stably causal and *ipso facto* causally well-behaved — can be covered (ibid., p.131) by a single "spherical polar" coordinate system (t, r, θ, ϕ), in which case the line element assumes the form

$$ds^2 = \cosh^2 r\, dt^2 - dr^2 - \sinh^2 r(d\theta^2 + \sin^2\theta\, d\phi^2).$$

As will now be shown, we may restrict attention to the plane $\theta = \phi = 0$. In this case the line element simplifies to

$$ds^2 = \cosh^2 r\, dt^2 - dr^2.$$

In keeping with the notation of (Definition), let p and q be the points $(0,0)$, $(0, \frac{3}{2}\pi)$, respectively, and let λ be the future-inextendible timelike curve with past endpoint p whose tangent vector satisfies

$$\frac{dt}{dr} = \frac{2^{\frac{1}{2}}}{\cosh r}.$$

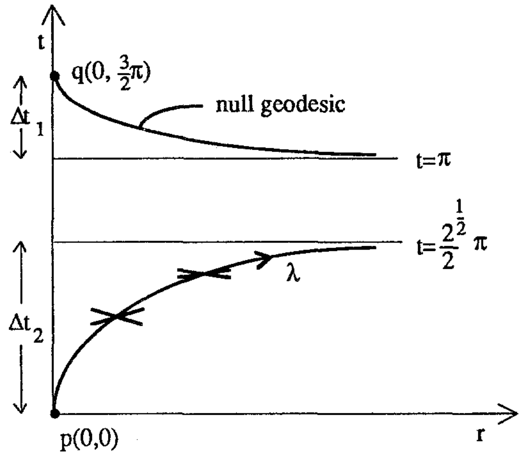

Fig. 3. Anti-de Sitter spacetime.

Claim: p, q, λ satisfy the conditions of (Definition). To satisfy (i) we must show that the length, $L(\lambda)$, of λ is infinite. We have

$$L(\lambda) \;=\; \int_0^\infty \frac{ds}{dr}\,dr = \int_0^\infty (\cosh^2 r\,(\frac{dt}{dr})^2 - 1)^{\frac{1}{2}}\,dr = \int_0^\infty dr = \infty.$$

Next, to satisfy (ii), we need to show that $\lambda \subset J^-(q)$. Setting $ds = 0$, we find that the light cone at q is given by $dt/dr = \pm 1/\cosh r$ (see Fig. 3). The total decrease in t along q's past light cone is

$$\triangle t_1 = \int_0^\infty \frac{dt}{dr}\,dr = \int_0^\infty \frac{1}{\cosh r}\,dr = [2\arctan(e^r)]_0^\infty = \frac{\pi}{2}.$$

Thus every null geodesic with future endpoint q is contained in the set $\{(t,r)|\pi < t \leq \frac{3}{2}\pi, 0 \leq r < \infty\}$. For any point $p = p(t,r)$ with $t < \pi$, the timelike curve through p given by $r = constant$ intersects the past light cone of q. Hence $p \in J^-(q)$. But this applies to every point $p \in \lambda$ since the t coordinate along λ never exceeds

$$\triangle t_2 = \int_0^\infty \frac{dt}{dr}\,dr = \int_0^\infty \frac{2^{\frac{1}{2}}}{\cosh r} = \frac{2^{\frac{1}{2}}}{2}\pi\,(< \pi).$$

(ii) is therefore also satisfied and so anti-de Sitter spacetime is Pitowsky.

The maximally extended Reissner-Nordström solution is also Pitowsky. This can be seen by examining Fig. 4, which shows part of

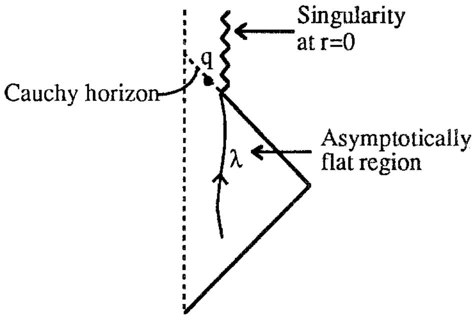

Fig. 4. Part of the conformal diagram of the maximally extended Reissner-Nordström solution.

the spacetime's conformal diagram. The future-inextendible timelike curve λ is contained in an asymptotically flat region and is of infinite length. The spacetime is Pitowsky because λ lies to the past of q, a point on the Cauchy horizon.[6]

6. TWO CONCLUDING REMARKS

It is curious to note that although HAL solves the immortal question, she never actually possesses the answer; that is left for Dave, the mortal bystander. Of course this is because HAL can only ever see part of her life (her past), while Dave can see the whole of it.

It would be interesting to add to our list of three Pitowsky spacetimes some spacetimes with distinctly physical features. However, I take it as agreed that the enactment of (S) would be hopelessly impractical in our universe even if the spacetime structure were sympathetic to it. It is therefore difficult to see how extending this list would add much, if anything at all, to the discussion. No matter: This negative point is eclipsed by the positive demonstration that the curious events of (S) can find a home in a causally well-behaved general relativistic model. This kind of demonstration — like Gödel's refutation of the necessity of absolute time — serves to show us that time has possibilities far beyond those imagined by common sense.

ACKNOWLEDGEMENTS

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REFERENCE

S.W. Hawking and G.F.R. Ellis, *The Large Scale Structure of Space-Time* (Cambridge University Press, Cambridge, 1973).

NOTES

1. It is conceivable that a computer whose dimensions shrunk sufficiently rapidly to zero could perform an infinite number of computations in a finite time — although presumably the ensuing singularity would prevent any data being transmitted beyond the event horizon. I will ignore this possibility.
2. If the concept of "an immortal computer" is thought too fantastic, replace it with "an infinite number of mortal computers lying temporally end-to-end."
3. Here and hereafter "length" means "proper length."
4. The exact length here is unimportant to the subsequent investigation; all that matters is that it is finite.
5. I am treating anti-de Sitter spacetime in its universal covering form.
6. This observation is actually nothing new. See ibid., p.161.