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A Discussion of Space-Time Metric Engineering

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The Alcubierre Warp Drive Metric, wherein a spacecraft can appear to vastly exceed the speed of light without locally ever doing so, derived in [1], is reconsidered. It is shown that the underlying driving physical mechanism (at least in a mathematical sense) is not the expansion/contraction of the space surrounding the spacecraft via the York Time T [2]. Rather, the driving mechanism is a boost that serves as a multiplier of the ship's initial velocity. This effect can in principle be likened to watching a movie in fast-forward. The expansion/contraction of space is merely a side effect of the warp drive's underlying mechanism - that can be viewed as sort of a Doppler effect, or stress/strain on space.

KEY WORDS: Boost; York time; metric engineering.

1. INTRODUCTION

Alcubierre [1], in 1994, derived a metric that satisfies Einstein's field equation of General Relativity Theory that would allow arbitrarily short travel times between two distant points in space. Alcubierre identified this metric as the realization of a "warp drive" as depicted in science fiction. Alcubierre remarked that the driving mechanism in his metric is the simultaneous expansion of space behind a spacecraft and a corresponding contraction of space in front of the spacecraft. In this fashion, a spacecraft can be seen by an external observer to have an arbitrarily large speed ($\gg$c) while locally the spacecraft stays within its own future light cone and never exceeds the speed of light. According to Alcubierre, this idea was an extension of the hypothesis that the early universe underwent a rapid inflationary phase immediately after the onset of the big bang. During this rapid inflationary

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phase, objects appear to recede from one another at speeds much larger than c, but locally all objects would have sub-c speeds and follow local light cones, thereby maintaining the laws of physics in their Local Inertial Frames (LIF's).

To illuminate the physics of the Alcubierre metric we will perform a gauge transformation on the metric to put it into canonical form. Using this canonical form, it will then be shown that the mathematical driving mechanism behind the Alcubierre Warp Drive Metric is not the York Time. Rather, it is a boost in an isometric spherical shell around the spacecraft generating the field that produces the metric. It is this isometric spherical shell-like boost that operates on an initial sub-c velocity of the spacecraft as a multiplier making the apparent final velocity as seen by both an external observer and a passenger on board the spacecraft to be arbitrarily large ($\gg$c).

2. THE ALCUBIERRE METRIC

Alcubierre first proves that the local proper time on board a spacecraft located at the origin of his warp field metric frame is equal to the coordinate time outside the warp field metric frame. Second, he also proves that local proper acceleration, α, on board the spacecraft located at the origin of the warp field metric frame is equal to zero. As he notes, this can only be achieved if the stress energy tensor, $T^{\mu\nu}$, that produces the warp field has negative energy density (exotic matter). Such $T^{\mu\nu}$ violate both the weak and dominant energy conditions. Pfenning and Ford [3] explored this negative energy density requirement further using quantum inequalities to show that the total integrated energy densities required to generate modest warp fields are physically unattainable (granting certain assumptions regarding the structure of the field that they adopt). While such considerations may render the Alcubierre warp drive metric practically unobtainable, it is still of interest to examine the physical basis on which it rests.

The Alcubierre warp drive metric is:

$$ds^2 = -dt^2 + (dx - v_s f(r_s)dt)^2 + dy^2 + dz^2 \tag{1}$$

where the interval ds^2 is the proper time, alternatively denoted $-d\tau^2$. The other coordinate symbols have their customary denotations. $f(r_s)$ is a shaping function defined by Alcubierre to be:

$$f(r_s) = \frac{\tanh(\sigma(r_s + R)) - \tanh(\sigma(r_s - R))}{2\tanh(\sigma R)} \tag{2}$$

In this shaping function, σ is a parameter that governs the wall thickness of the warp sphere that forms around the spacecraft generating the warp field. R is the variable that is the actual physical radius of the warp sphere. For arbitrarily large σ, the shaping function $f(r_s)$ approaches a "top hat" function of radius R centered at the field's frame origin. The input parameter, r_s, in the shaping function is:

$$r_s(t) = [(x - x_s(t))^2 + y^2 + z^2]^{1/2} \tag{3}$$

The variable $x_s(t)$ can be considered the x coordinate of the spacecraft with respect to, say, the Local Inertial Frame (LIF) of an observer located here on the earth. Correspondingly, $r_s(t)$ (and hence the warp field) is dependent upon the x location of the spacecraft with respect to the earth. Mathematically, this just says that the warp sphere is "attached" to the ship's moving frame - which would make intuitive sense as the ship is hypothetically generating the warp field (even if only in theory).

3. YORK TIME AND THE MECHANISM OF THE METRIC

Alcubierre claims that the driving mechanism behind the warp field effect is the expansion of space behind the spacecraft and a corresponding contraction of space in front of the spacecraft. This effect is mathematically derived from the metric described in Equation 1 by means of the York Time. Alcubierre uses the variable θ as the York Time in [1], so we shall use the same nomenclature here to maintain consistency with the original paper. The York Time θ derived in [1] is:

$$\theta = v_s \frac{x_s}{r_s} \frac{df}{dr_s} \tag{4}$$

The derivative of the shaping function $f(r_s)$ in Equation (4) is:

$$\frac{d}{dr_s} f(r_s) = \frac{\sigma \sec h^2(\sigma(r_s + R)) - \sigma \sec h^2(\sigma(r_s - R))}{2\tanh(\sigma R)} \tag{5}$$

The derivative of the shaping function can be envisioned as an upside down circular wastepaper basket shoved down into a flat sheet of very pliable rubber. The end result is a thin walled shell (with an unaffected center where the spacecraft would be located) that is always negative. When this is put into Equation 4 and combined with the fact that x_s varies from positive to negative, obviously centered in the spacecraft's frame, the derivative gets linearly scaled from a negative value in front of the ship for $+x_s$, to zero at $x_s = 0$, scaled back up to an equivalent positive value aft of the ship for $-x_s$. This works in concert with the r_s variable which defines the radial distance from the ship's local frame origin. A plot of the York Time of the warp drive metric is shown here in Figure 1.

Alcubierre pointed out that travelers on a hypothetical spacecraft utilizing a warp field to travel, say, to Alpha Centari quickly would travel to some "safe" distance away from the Earth, reduce its velocity there to zero, and then turn on the warp field. Depending on the details of the warp field, the spacecraft can then be made to acquire any arbitrary velocity with respect to an observer here on Earth. And an observer on the spacecraft "sees" the Earth recede with a corresponding

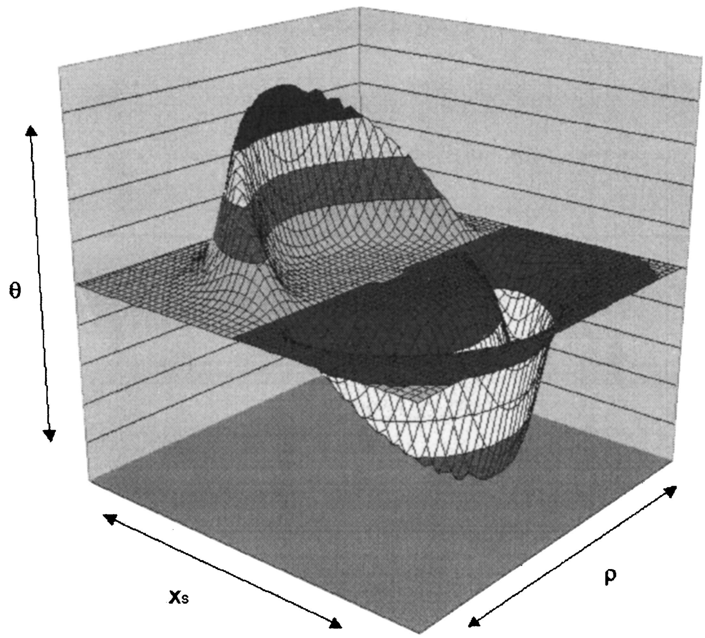

Figure 1.

velocity. Recall however, that locally, the spacecraft never exceeds the speed of light and it always proceeds into its own local future light cone.

This seems a quite reasonable physical interpretation of the mathematics of the Alcubierre metric. However, there is a consideration that suggests an alternative view to this explanation. That consideration involves the energy density component of the stress tensor, in particular:

$$T^{00} = -\frac{1}{8\pi}\frac{v_s^2\rho^2}{4r_s^2}\left(\frac{df}{dr_s}\right)^2 \tag{6}$$

A plot of this energy density is displayed in Figure 2. Note that the field is axisymmetrical about the x-axis (toroidal), and that the energy density is symmetric about the $x_s = 0$ surface. This means that the energy density is an unbiased energy field along the $+x$ and $-x$ axis of the spacecraft generating the field. Also note that the energy density directly along the x-axis is exactly zero.

Using the above information, let us reconsider the journey of a spacecraft going to Alpha Centari. The spacecraft uses some conventional means of propulsion to travel a “safe” distance from the Earth. At that point the spacecraft reduces its velocity to zero (relative to the Earth). The crew then turns on its field generator to produce the desired energy densities creating a warp sphere around the ship

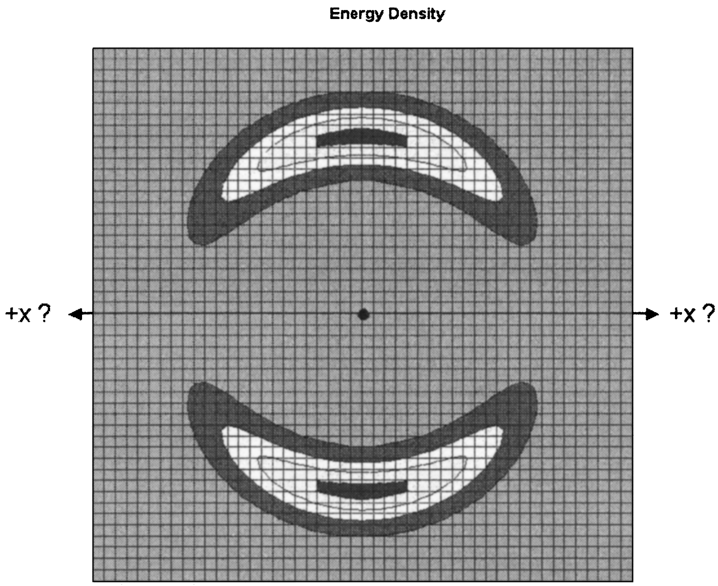

Figure 2.

proper. The choice of direction for the positive x-axis for the ship's LIF, however, as seen by the stress energy tensor $T^{\mu\nu}$ is completely arbitrary since it is symmetric about the $x_s = 0$ surface. So how does the ship know which way to go? The energy density curves local space-time, but since it has no bias along the x-axis, how does space know which way to contract and expand?

Since the choice of x positive is arbitrary (in a mathematical sense) without changing the physical manifestation of the energy density component, perhaps the York Time is not the driving mechanism of the metric. Rather it may be a side effect of the underlying driving mechanism–some sort of Doppler effect, or stress/strain on the space metric perhaps. Evidently, for space to generate the York Time effect, an initial condition must be provided so that a bias fixes the x direction. It is this bias that will control on which side of the craft space contracts, and on which side it expands. To better understand this issue we will first need to consider an alternative derivation of the warp field metric that enables us to put Alcubierre's Warp Drive Metric into its canonical form by performing a gauge transformation on the metric.

4. ANOTHER VIEW OF THE ALCUBIERRE METRIC

Imagine that we could develop a "NASA Golf Ball" spacecraft with the distinctive feature that its local clock rate can be increased to an arbitrarily fast rate with respect to our clock rates here on earth. Suppose that we decided to send this "NASA Golf Ball" to nearby Alpha Centari. Before turning up the clock rate on board, we would first give it an initial velocity, say 0.1c. After achieving the desired velocity, we would then boost the clock rate of the "NASA Golf Ball" by a factor of 100 with respect to clock rates here on earth. Viewed from Earth we would then see the "NASA Golf Ball" run in fast forward, much like watching a video tape on fast forward. This would mean that the spacecraft would *appear* to have a final velocity of 10c (100×0.1c). Of course, the "NASA Golf Ball" would never *locally* break the speed of light and always be traveling locally at 0.1c. The clocks on board the "NASA Golf Ball" would even see 43+ years elapse before arriving at Alpha Centari. We here on earth however, would see the probe make it to its destination in only $\sim$0.43 years because the high local clock rate on the spacecraft would, if you will, "suck in" and "spit out" the space through which it travels at a much higher than "normal" rate.

A spacecraft of this sort would be beneficial for sending unmanned probes to arbitrarily distant stars. But what if we wanted to put an astronaut on-board the "NASA Golf Ball" (obviously a big golf ball)? Since the on board clocks of the "NASA Golf Ball" still see 43 plus years go by before arriving at their destination, any passengers aboard the probe would be quite old when the probe arrived at its destination. What if we could control which parts of the probe had higher clock rates? What if we could locally boost clock rates on the surface of the spacecraft in such a fashion as to create a hollow sphere where the clock rates on the surface could be made to be arbitrarily fast, while the clock rates within the sphere could be made to maintain a commensurate rate with clocks here on earth. Nature has no objection to variable clock rates existing throughout any given region of space-time as long as the gravitational fields present therein can be made to vary appropriately. The obvious example of such behavior is the varying clock rates that are a function of radial distance (altitude) from a massive spherical object.

Let us reconsider a voyage of the "NASA Golf Ball" to Alpha Centari employing the ability to locally control clock rates throughout its volume. The "NASA Golf Ball" sets off on its journey from earth, complete with astronaut, accelerating to 0.1c. Upon reaching 0.1c, we cause the local clock rates only on the surface (or just outside the surface) of the "NASA Golf Ball" to speed up by a factor of 100. We will make the clock rates inside this spherical shell keep time with clocks located here on earth. As before, we see the probe obtain an apparent velocity of 10c (100×0.1c). But this time, the astronaut sees the same thing that we do since s/he has the same clock rates. Moreover, since the astronaut is at rest with respect to the "NASA Golf Ball", s/he has the same local speed of 0.1c and never locally breaks the speed of light. In this fashion, both observers here on earth, and passengers aboard the "NASA Golf Ball" can be made to see a probe travel time of $\sim$0.43 years to Alpha Centauri. Can such a field be mathematically modeled?

We now put the Alcubierre Warp Drive Metric into its canonical form by performing a gauge transformation. A gauge transformation of the type needed is:

$$t \to k(t + \mathrm{L}(x_i)) \tag{7}$$

With it we shall restate the Alcubierre Warp Drive Metric:

$$ds^2 = -dt^2 + (dx - v_s f(r_s)dt)^2 + dy^2 + dz^2 \tag{1}$$

After some effort, the canonical form can be shown to be:

$$ds^2 = \left(v_s^2 f(r_s)^2 - 1\right)\left(dt - \frac{v_s f(r_s)}{v_s^2 f(r_s)^2 - 1}dx\right)^2 - dx^2 + dy^2 + dz^2 \tag{8}$$

In this equation, our new dt' is:

$$dt'^2 = \left(dt - \frac{v_s f(r_s)}{v_s^2 f(r_s)^2 - 1}dx\right)^2 \tag{9}$$

Note that at the origin of the spacecraft's frame, $dt' = dt$ and hence $d\tau$ as shown in [1]. Since we now have the equation in canonical form, we can extract the gravitational potential Φ: . . .

$$\Phi = \frac{1}{2}\ln\left|1 - v_s^2 f(r_s)^2\right| \tag{10}$$

Using Equation (10), we can derive the boost as:

$$\gamma = \cosh\left(\frac{1}{2}\ln\left|1 - v_s^2 f(r_s)^2\right|\right) \tag{11}$$

Having completed the process of putting the Alcubierre Warp Drive Metric into its canonical form in Equation (8), we now turn to some of its interesting mathematical aspects. Notice that the x component of the space-like portion of the metric seems to display the sort of behavior found interior to a black hole's event horizon since it is of opposite sign with respect to the rest of the space-like metric components. (Sign reversal of the time and radial coordinates as an in-falling body passes inside the event horizon is a well-known feature of the Schwarzschild solution.) Next, consider the boost for the field contained in Equation (11). This is the equation of most immediate interest and most relevant to the previous "NASA Golf Ball" discussion. A plot of the boost is shown here in Figure 3.

The plot shows that the boost for the field is a sphere of increasing magnitude with a steady value through the middle. This, in turn, shows that the spacecraft buried inside the field is isolated from the outside in that the boost appears to have a constant value throughout. Alcubierre has already proven in [1] (as we discussed earlier) that proper time, $d\tau$, is equal to coordinate time, dt, at the origin of the spacecraft's frame. Basically, what we have mathematically is a sphere whose outside surface appears to external observers to have an arbitrarily fast clock rate.

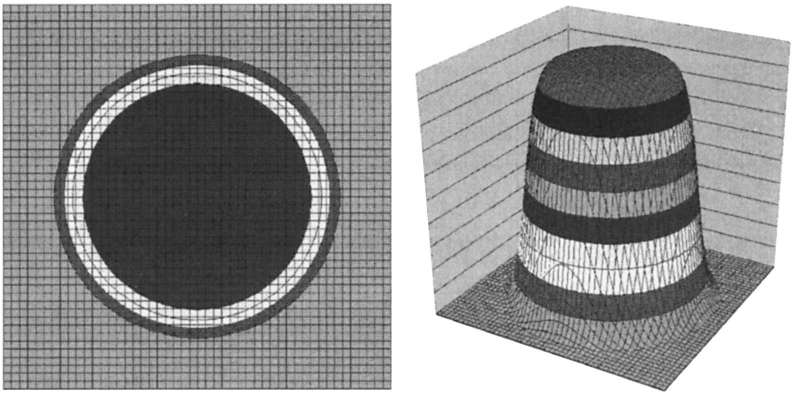

Figure 3. Boost for the Alcubierre Warp Drive Metric.

Any observers on the inside of this sphere will see the same thing that external observers see since it has already been shown that proper time and coordinate time are the same at the spacecraft's origin.

It might be argued that the field at the surface of the spacecraft that produces the "superluminal" velocity of the surrounding "warp bubble" would create titanic tidal forces, destroying any material used to make the vehicle. The field does not need to exist directly in the surface of the spacecraft and could be made to have a radius large enough to protect the ship proper from permanent damage.

Of course, all of these arguments may be completely moot since the effect requires the presence of negative energy densities, which may be a physical impossibility. However, it is always beneficial to discuss the mathematical principles of theories and exchange ideas on what may be happening, even if only from a non-tangible perspective, as it is this process that sometimes gives others insights on how to interpret real physical data or even generate more elegant mathematical representations of real physical phenomenon. I also find it very satisfying that two completely different approaches to generate a hypothetical warp drive within the context of the General Relativity Theory would end up with the same equation. This corroborates the inherent stability and robustness of the GRT and the beauty of pseudo-Riemannian geometry.

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