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DISCRETE SPACETIME AND LORENTZ INVARIANCE

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The idea of discrete spacetime originates from the fundamental length of Heisenberg and the elementary domain of Yukawa. The concept of discrete spacetime is explained in brief. To transfer from continuous spacetime to discrete spacetime the differentials appearing in a theory should be replaced by finite differences. The quantum field theory on discrete spacetime is briefly reviewed. Finally it becomes clear how to conform the discrete spacetime to the special theory of relativity.

1. INTRODUCTION

It is widely recognized that if we assume the continuous spacetime and the microcausality we necessarily come up against the divergence difficulty in the quantum field theory. To eliminate this divergence many prescriptions are proposed, for example, renormalization, regularization, cutoff and so on. Of course they succeeded in some respects, but still contain unsatisfactory points. A few physicists then began to consider in a different manner. Heisenberg introduced a fundamental constant 'ℓ' besides $\hbar$ and c, which has dimension of length and is about 10^{-13}cm. Then all physical quantities are written in dimensionless form. He said in his lecture at the Cavendish Laboratory in 1949[1].

"*If the future theory contains such a constant in whatever form, it is natural to assume that the usual correspondence between the classical wave description and its quantum-theoretical analogue only holds for distances very much greater than ℓ, but fails in the region of smaller distances.*"

Later on he formulated his famous non-linear spinor theory of elementary particles, where ℓ appears as a coupling constant[2]. Yukawa considered that the spacetime cannot be divided infinitely but consists of finite domains, which he called elementary domains[3]. He did not mention about the size of the elementary domain clearly, but he considered it probably about a size of an elementary particle.

As we easily understand, the spacetime is defined and recognized by the distribution of matter or physical objects. Therefore, if there exists a smallest object in nature, it has no meaning to consider a distance much smaller than this size of the object, because we have no means to measure the distance. Thus, the existence of fundamental length may be naturally accepted. Of course, when we consider a very large object compared to this size, we may assume that the spacetime is continuous.

If we accept the fundamental length λ, all derivatives with respect to spacetime are replaced by finite differences:

$$\partial_\mu\phi(x) \rightarrow \Delta_\mu\phi(x) = \frac{1}{\lambda}[\phi(x+\lambda\hat{\mu})-\phi(x)],$$

where $\hat{\mu}$ dentes the unit vector of μ-direction and μ=0,1,2,3. Hereafter we assume λ=1. This replacement should be applied to all physical theories. However, the theories (including the classical theories) that are already established should not be changed. The fundamental length is assumed to be invariant under Lorentz transformation, that is, it has the same value in any Lorentz frame. The introduction of such a length seems at once to contradict the special theory of relativity. This is in fact our main theme of this report.

2. REVIEW OF QFT ON DISCRETE SPACETIME

Before we discuss it, we briefly review the quantum field theory on the discrete spacetime by using a very simple model. We begin with a Lagrangian density of massless scalar field $\phi(t,x)$ in two-dimensional spacetime:

$$L_c = \frac{1}{2}[(\partial\phi/\partial t)^2 - (\partial\phi/\partial x)^2].$$

This density is clearly invariant under Lorentz transformation. We replace the derivatives by finite differences and get

$$L_d = \frac{1}{2}[(\dot{\Delta}\phi)^2 - (\Delta'\phi)^2],$$

where $\dot{\Delta}$ and Δ' are difference operators with respect to t and x respectively. We call this replacement "*quantization of spacetime*".

The action sum is now

$$S = \sum_{t,x} L_d \,.$$

By the principle of least action we have the field equation:

$$\dot{\Delta}^2\phi(t-1,x) - \Delta'^2\phi(t,x-1) = 0$$

or explicitly

$$\phi(t+1,x)+\phi(t-1,x)-\phi(t,x+1)-\phi(t,x-1) = 0.$$

The equation is easily solved to give

$$\phi(t,x)=\int_{-\pi}^{\pi} d\theta[A(\theta)e^{-i(|\theta|t-\theta x)}+A^*(\theta)e^{i(|\theta|t-\theta x)}].$$

We define the canonical momentum conjugate to $\phi(t,x)$ by

$$\pi(t,x) \equiv \frac{\partial}{\partial\dot{\Delta}\phi(t,x)}\sum_x L_d = \frac{1}{2}[\phi(t+1,x)-\phi(t-1,x)],$$

where we used the field equation. To quantize the field $\phi(t,x)$ we assume the equal time commutation relations:

$$[\phi(t,x),\pi(t,x')] = i\delta_{x,x'}\,,$$

$$[\phi(t,x),\phi(t,x')] = [\pi(t,x),\pi(t,x')] = 0.$$

Substituting the expression for $\pi(t,x)$, we have the commutation relations:

$$[\phi(t,x),\phi(t+1,x')] = i\delta_{x,x'}\,,$$

$$[\phi(t,x),\phi(t,x')] = 0.$$

It must be noticed that these ϕ's are at nearest neighboring times.

We write $\phi(t,x)$ in a normalized form:

$$\phi(t,x) = \frac{1}{(2\pi)^{1/2}}\int_{-\pi}^{\pi}\frac{d\theta}{(2\sin|\theta|)^{1/2}}\cdot[a(\theta)e^{-i(|\theta|t-\theta x)}+a^*(\theta)e^{i(|\theta|t-\theta x)}].$$

The commutation relations are

$$[a(\theta),a^*(\theta')] = \delta(\theta-\theta'),$$

$$[a(\theta),a(\theta')] = [a^*(\theta),a^*(\theta')] = 0.$$

Then we have the commutation relation of $\phi(t,x)$'s at arbitrarily separated spacetime points:

$$[\phi(t,x),\phi(t',x')] \equiv iD(t-t',x-x'),$$

$$D(t,x) = \frac{1}{2\pi i}\int_{-\pi}^{\pi}\frac{d\theta}{2\sin|\theta|}[e^{-i(|\theta|t-\theta x)}-e^{i(|\theta|t-\theta x)}]$$

$$= \begin{cases} -1 & \text{for } t+x=\text{odd, timelike and } t>0,\\ +1 & \text{for } t+x=\text{odd, timelike and } t<0,\\ 0 & \text{for otherwise.}\end{cases}$$

We notice that there exist many points inside the light cone (t+x=even), where D is equal to zero. It means, $\phi(t,x)$ at these points commute with $\phi(0,0)$ at the origin.

If we define the vacuum $|0\rangle$ by $a(\theta)|0\rangle=0$, the propagator defined by

$$D_F(t-t',x-x') \equiv \langle 0|T\phi(t,x)\phi(t',x')|0\rangle$$

is calculated to give

$$D_F(t,x) = \frac{i}{16\pi^2}\int_{-\pi}^{\pi}d\theta\int_{-\pi}^{\pi}d\phi\,\frac{e^{-i(\phi t-\theta x)}}{\sin^2\frac{\phi}{2}-\sin^2\frac{\theta}{2}+i\varepsilon}$$

$$= \frac{1}{4\pi}\int_{-\pi}^{\pi}\frac{d\theta}{\sin|\theta|}[\theta(t)e^{-i(|\theta|t-\theta x)} + \theta(-t)e^{i(|\theta|t-\theta x)}].$$

To see the convergence of the theory this $D_F(t,x)$ in 1+1 dimensional spacetime is not suitable, because it contains a divergent integral at $\theta=0$. By simple extension we see that, while the corresponding quantity in 1+2 dimensional spacetime is still divergent, the propagator of 1+3 dimension is completely convergent[4]. From this fact we might understand that the space in which we live is three dimensional.

The direct results of discrete spacetime are:
1. The field theory includes no divergence. Any physical quantity is calculated to give a finite value without recourse to the renormalization.
2. One elementary particle cannot carry arbitrarily high energy-momentum. There exists an upper limit.

3. LORENTZ INVARIANCE

Now we return to the problem of Lorentz invariance. The above quantization procedure is summarized as follows:

1) Assume a Lorentz invariant Lagrangian.
2) Fix a frame of coordinates in spacetime.
3) Quantize the spacetime referring to the coordinates.
4) Quantize the fields included in the Lagrangian.

Of course, if we have two quantized spacetimes referring to different frames of coordinates, they can never be made to coincide by Lorentz transformation. However, since the original Lagrangian in continuous spacetime is Lorentz invariant, every frame of coordinates is equivalent and thus every quantized spacetime should be equivalent. This means that we should have the same consequences from the theory independently of the frame of quantization. We may call this "*Lorentz equivalence*".

The situation is very similar to the case of quantization of angular momentum. The original Lagrangian is invariant under space-rotation. We first fix the frame of coordinates in space and quantize the angular momentum. Especially the quantization of z-component means that the wave functions are eigenstates of the z-component, that is, the z-component is diagonal. Therefore, if we rotate the z-axis in a different direction, this z-component of angular momentum is no longer diagonal. In other words, two kinds of angular momentum which are quantized referring to different z-directions cannot be transformed to each other by space rotation. Notwithstanding, we may choose the z-axis arbitrarily but obtain the same results from the theory independently of the choice. In practice we know the most suitable direction of z-axis, for example, we choose it in the direction of a magnetic field.

In our case of discrete spacetime it is of course arbitrary which frame of coordinates we choose, but there may exist a suitable frame of coordinates. The frame of coordinates we consider may probably be the rest frame of the observer.

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