A Covariant Information-Density Cutoff in Curved Space-Time

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In information theory, the link between continuous information and discrete information is established through well-known sampling theorems. Sampling theory explains, for example, how frequency-filtered music signals are reconstructible perfectly from discrete samples. In this Letter, sampling theory is generalized to pseudo-Riemannian manifolds. This provides a new set of mathematical tools for the study of space-time at the Planck scale: theories formulated on a differentiable space-time manifold can be completely equivalent to lattice theories. There is a close connection to generalized uncertainty relations which have appeared in string theory and other studies of quantum gravity.

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It is generally assumed that the notion of distance loses operational meaning at the Planck scale, $l_P \approx 10^{-35}m$ (assuming 3+1 dimensions), due to the combined effects of general relativity and quantum theory. Namely, if one tried to resolve a spatial region with an uncertainty of less than a Planck length, then the corresponding momentum uncertainty should randomly curve and thereby significantly disturb the very region in space that was meant to be resolved. It is expected, therefore, that the existence of a smallest possible length, area or volume, at the Planck scale or above, plays a central role in the yet-to-be-found theory of quantum gravity.

In the literature, no consensus has been reached as to whether this implies that space-time is discrete. On the one hand, quantization literally means discretization, and space-time discreteness is indeed naturally accommodated within the functional analytic framework of quantum theory, see, e.g., [1]. Also, most interacting quantum field theories are mathematically well-defined only on lattices. On the other hand, within the mathematical framework of general relativity, space-time is naturally described as a differentiable manifold and deep principles such as local Lorentz invariance would appear to be violated if space-time were discrete.

There is the possibility that the cardinality of space-time is between discrete and continuous, but it is strongly restricted by results of Gödel and Cohen. In [2], they proved that both are consistent with conventional (ZF) set theory: to adopt an axiom claiming the existence of sets with intermediate cardinality or to adopt an axiom claiming their non-existence. Therefore, it is not possible to explicitly construct any set of cardinality between discrete and continuous infinity from the axioms of conventional set theory. If space-time is describable as a set and if this set is of intermediate cardinality, then its description cannot be constructive and requires mathematics beyond conventional set theory.

In this Letter, we consider a simpler possibility. In a concrete sense, space-time could be simultaneously discrete *and* continuous. Namely, in the simplest case, physical fields could be differentiable functions which possess merely a finite density of degrees of freedom. If such a field's amplitude is sampled at discrete points of the space-time manifold then the field's amplitude at all points in the manifold are reconstructible from those samples - if the sample points are spaced densely enough. The minimum average sample density which allows the reconstruction of fields could be, for example, on the order of the Planck scale. All physical entities such as fields, Lagrangians and actions can then be written either as living on a differentiable manifold, thereby displaying external symmetries or, equivalently, as living on any one of the sampling lattices of sufficiently small average spacing, thereby displaying ultraviolet finiteness. Such theories need not break any symmetries of the manifold because among all sufficiently tightly spaced lattices no particular lattice is preferred.

In the information theory community, the mathematics of classes of functions which can be reconstructed from discrete samples is well-known, namely as sampling theory. Shannon, in his seminal work [3], introduced sampling theory as the link between continuous information and discrete information. Our aim here is to extend this link between discrete and continuous information to curved space-times.

The use of sampling theory in quantum mechanics was first suggested in [4], where a close connection was shown to generalized uncertainty relations that had appeared in studies of quantum gravity and string theory, see [5]. In the simplest case, these uncertainty relations are of the form $\Delta x \Delta p \geq \frac{\hbar}{2}(1+\beta(\Delta p)^2)$ and imply the existence of a finite minimum uncertainty in position $\Delta x_{min} = \hbar\sqrt{\beta}$, as is easily verified. In [4], it was shown that a finite lower bound to the position uncertainty implies that the wave functions possess the sampling property, i.e. that they can be reconstructed everywhere from discrete samples if those samples are taken at a spacing that is at least as small as the minimum position uncertainty. In [6], it was suggested that the freedom of choice of sampling lattice may be related to gauge symmetries.

In recent proceedings, see [7], sampling theory for physical theories on curved spaces was outlined. Building on [7], this Letter introduces sampling theory on Riemannian and pseudo-Riemannian manifolds. We will obtain a covariant information density cutoff together with a new sampling theoretic principle that is consistent with the Lorentz contraction of sampling lattices.

The basic sampling theorem goes back to Cauchy in the early 19th century, see [8]. Consider the set of square integrable functions f whose frequency content is bounded by ω_{max}, i.e., which can be written as $f(x) = \int_{-\omega_{max}}^{\omega_{max}} \tilde{f}(\omega) e^{i\omega x} d\omega$. These f are called bandlimited functions with bandwidth ω_{max}. If the amplitudes $\{f(x_n)\}$ of such a function are known at equidistantly-spaced discrete points $\{x_n\}$ whose spacing is π/ω_{max}, then the function's amplitudes $f(x)$ can be reconstructed for all x. The reconstruction formula is:

$$f(x) \ = \ \sum_{n=-\infty}^{\infty} f(x_n) \, \frac{\sin[(x-x_n)\omega_{max}]}{(x-x_n)\omega_{max}} \qquad (1)$$

This sampling theorem is in ubiquitous use, e.g., in digital audio and video as well as in scientific data taking. Sampling theory, see [8], studies generalizations of the theorem for various classes of functions, for non-equidistant sampling, for multi-variable functions and it investigates the stability of the reconstruction in the presence of noise.

Following [7], we now define a general framework for sampling on Riemannian manifolds. The key assumption in the basic sampling theorem is a frequency cutoff, more precisely a cutoff of the spectrum of the self-adjoint differential operator $-id/dx$. On a multi-dimensional curved space, a covariant analog of a bandlimit is the cutoff of the spectrum of a scalar self-adjoint differential operator. As an explicit example one may choose the Laplace-Beltrami operator $\Delta = |g|^{-1/2}\partial_i g^{ij} |g|^{1/2} \partial_j$ where $|g|$ is the determinant of the metric tensor g.

Consider then the Hilbert space $\mathcal{H}$ of square integrable, say scalar, functions over the manifold and the dense domain $\mathcal{D} \subset \mathcal{H}$ on which the considered operator, say the Laplacian, is essentially self-adjoint. Using sloppy but convenient terminology we will speak of all points of the spectrum as eigenvalues, λ, with corresponding "eigenvectors" $|\lambda)$, keeping in mind that the manifold will generally be noncompact and its spectrum therefore not discrete. We use the notation $|\ )$ in analogy to Dirac's bra-ket notation, but with round brackets to distinguish from quantum states. The $|\phi)$ that we consider here could be, for example, the scalar fields that are being integrated over in a quantum field theoretical path integral.

The operator $-\Delta$ is positive and its spectrum is an invariant of the manifold. A spatially covariant "bandwidth" cutoff in nature then means that physical fields are elements of $\mathcal{D}_{ph} = P.\mathcal{D}$, where P projects onto the subspace of $\mathcal{D}$ which is spanned by the eigenspaces of $-\Delta$ with eigenvalues smaller than some fixed maximum value λ_{max}, which could be, e.g., on the order of $1/l_P^2$.

For example, in quantum field theoretical actions this type of cutoff arises if $-\Delta$ is the lowest order term in a power series in $-\Delta$ whose radius of convergence is finite, say $1/l_P^2$. Examples are the geometric series $l_P^{-2}\phi^* \sum_{n=1}^{\infty}(-l_P^2\Delta)^n\phi$ and $l_P^{-2}\sum_{n=1}^{\infty}(-l_P^2\phi^*\Delta\phi)^n$. Such series correspond to Planck-scale modified dispersion relations, a concept that has recently attracted considerable attention in the context of the transplanckian problems in black hole radiation and inflationary fluctuations, see, e.g., [9, 10]. Interestingly, also the Dirac-Born-Infeld action may be viewed as providing a minimum length cutoff through this mechanism, namely when expanding the square root in its action as power series with finite radius of convergence. See, in particular, [11].

If, by this or another mechanism, the yet-to-be-found theory of quantum gravity does yield a bandwidth cutoff, how do the fields in the physical domain $\mathcal{D}_{ph}$ acquire the sampling property? For simplicity, assume that one chart covers the N-dimensional manifold. The coordinates $\hat{x}_j$, for $j = 1, ..., N$ act as multiplication operators that map scalar functions to scalar functions: $\hat{x}_j : \phi(x) \to x_j\phi(x)$. On their domain within the Hilbert space $\mathcal{H}$ these operators are essentially self-adjoint, with an "Hilbert basis" of non-normalizable joint eigenvectors $\{|x)\}$ with continuum normalization $\mathbf{1} = \int d^N x \ |g|^{1/2} \ |x)(x|$. We have $(x|\phi) = \phi(x)$. Since Δ, being a differential operator, cannot commute with the position operators $\hat{x}_j$ we obtain the situation $P|x) \neq |x)$. Thus, on the restricted domain $\mathcal{D}_{ph}$, the multiplication operators $\hat{x}_j$ are merely symmetric but not self-adjoint (intuitively, for lack of eigenvectors). Correspondingly, the uncertainty relations are modified, similar to the toy cases discussed in [12].

Consider now a physical field, i.e., a vector $|\phi) \in \mathcal{D}_{ph}$. Assume that the field's amplitudes $\phi(x_n) = (x_n|\phi)$ are known at discrete points $\{x_n\}$ of the manifold. While all position eigenvectors $|x)$ are needed to span $\mathcal{H}$, sufficiently dense discrete subsets $\{|x_n)\}$ of the set of vectors $\{P|x)\}$ can span D_{ph}. A field's coefficients $\{\phi(x_n)\}$ then fully determine the Hilbert space vector $|\phi) \in D_{ph}$ and they determine, therefore, also $(x|\phi)$ for all x. Namely, defining $K_{n\lambda} = (x_n|\lambda)$, the set of sampling points $\{x_n\}$ is sufficiently dense for reconstruction iff K is invertible. To see this, insert the resolution of the identity in terms of the eigenbasis $\{|\lambda)\}$ of $-\Delta$ into $(x|\phi)$:

$$(x|\phi) = \sum\!\!\!\!\!\!\!\int_{|\lambda|<\lambda_{max}} (x|\lambda)(\lambda|\phi) \ d\lambda \qquad (2)$$

We use the combined sum and integral notation since the spectrum of $-\Delta$ may be discrete and/or continuous (the manifold $\mathcal{M}$ need not be compact) and it is understood that eigenvalues can be degenerate. With K invertible, one obtains $(\lambda|\phi) = \sum_n K^{-1}_{\lambda,n}\phi(x_n)$ which, when substituted back into Eq.2, yields

$$\phi(x) = \sum_n G(x, x_n)\ \phi(x_n), \tag{3}$$

with the reconstruction kernel:

$$G(x, x_n) \ = \ \sum\!\!\!\!\!\!\int_{|\lambda|<\lambda_{max}} (x|\lambda) K^{-1}_{\lambda,n}\ d\lambda \tag{4}$$

In conventional applications of information theory, the sample points $\{x_n\}$ are required to be dense enough to allow stable reconstruction in the presence of noise: functions reconstructed from small samples must have small norm, in the sense that there exists a $C > 0$ such that

$$(\phi|\phi) \ \leq \ C \sum_n |\phi(x_n)|^2 \quad \text{for all } |\phi) \in D_{ph}. \tag{5}$$

The minimum sample density for stable reconstruction in flat Euclidean space equals the bandwidth volume in Fourier space, up to a constant, as was shown by H.J. Landau in [13]. It is also the density of degrees of freedom, defined as the dimension of the space of bandlimited functions which possess essential support in a given volume. Reconstruction stability is highly nontrivial already for classical signals in flat space: a bandwidth cutoff does not prevent bandlimited functions from oscillating arbitrarily fast in an arbitrarily large region. These superoscillations recruit degrees of freedom from outside the considered region at the expense of reconstruction stability. As was shown in [14], superoscillations in quantum mechanical wave functions produce effects that raise thermodynamic and measurement theoretic issues. It should be most interesting, therefore, to explore for physical fields in curved space-time the role of quantum fluctuations as noise and to use our new approach to generalize Landau's theorem to Riemannian manifolds. For fixed noise, the density of degrees of freedom then yields the maximum Shannon information density as usual, see [3].

We close the case of Riemannian manifolds with several remarks. In the case of one dimension and equidistant samples, Eqs.2-4 reduce to the basic sampling theorem of Eq.1. Another simple case is that of compact manifolds. Their Laplacian possesses a discrete spectrum whose cutoff renders $\mathcal{D}_{ph}$ finite-dimensional. A corresponding finite number of sampling points suffices for reconstruction. Since sampling theory has its origins and most of its applications in communication engineering, sampling theory on Riemannian manifolds has been little studied so far. An exception is the $SU(2)$ group manifold, for which the spectral cutoff yields the much-discussed fuzzy sphere, see, e.g., [15]. Very interesting results were obtained by Pesenson, see, e.g., [16], who considered, in particular, the case of homogeneous manifolds. In [16], reconstruction works differently, however, namely by approaching the solution iteratively in a Sobolev space setting. Useful methods should also be available from the field of spectral geometry, see, e.g., [17], which studies the close relationship between the properties of a manifold and the spectrum of its Laplacian and, in particular, from the field of noncommutative geometry and the techniques based on the spectral triple, see [18].

Let us now turn to the entirely new case of sampling in pseudo-Riemannian manifolds. We define a (N+1) dimensional covariant "bandlimit" as a cutoff of the spectrum of a scalar self-adjoint differential operator such as the Dirac or the d'Alembert operator $\Box$ (as opposed to, e.g., regularization through a generalized ζ-function). While these operators are self-adjoint, they are not elliptic and their spectrum needs to be cut off from above and below. This will lead us to a new sampling-theoretic principle that accounts for the Lorentz contraction of lattice spacings: Each temporal frequency component $\phi(\omega, \underline{x})$ of a field possesses its own finite spatial bandwidth and can be reconstructed from discrete spatial samples of corresponding density. Equivalently, each spatial mode $\phi(t, \underline{k})$ possesses a finite temporal bandwidth and can be reconstructed from samples taken at correspondingly dense discrete sampling times.

To see this, consider first the case of flat (N+1)-dimensional Minkowski space-time. Fourier theory is applicable and cutting off the spectrum of, e.g., the d'Alembertian amounts to requiring $|p_0^2 - \vec{p}^2| < l_P^{-2}$, as is implementable in the action, e.g., through a power series in $p_0^2 - \vec{p}^2$ with radius of convergence l_P^{-2}. Thus, for each fixed p_0 there is a finite bandwidth volume, $B(p_0)$, in the N-dimensional space of $\vec{p}$ values. The bandwidth volume is of ball shape if $|p_0| \leq l_P^{-1}$ and is of spherical shell shape if $|p_0| > l_P^{-1}$. Each frequency component p_0 of a field possesses its own finite spatial bandwidth and can be reconstructed from discrete samples:

$$\phi(p_0, \vec{x}) = \sum_{\vec{n}} \phi(p_0, \vec{x}_{\vec{n}}(p_0)) G(p_0, \vec{x}, \vec{x}_{\vec{n}}(p_0)) \tag{6}$$

For example, a kernel, G, for stable reconstruction from equidistant samples follows from the sampling theorem of Eq.1 by viewing each of the bandwidth volumes as contained in a rectangular bandwidth box. By Landau's theorem, [13], the precise minimum sample density for stable reconstruction of a temporal frequency mode, p_0, is given by the bandwidth volume $B(p_0)$ in momentum space. For sub-Planckian frequency modes, $|p_0| \leq l_P^{-1}$, the maximal bandwidth volume $B(p_0)$ is the volume of the D-dimensional sphere with radius $\sqrt{2} l_P^{-1}$. Thus, all these modes can be reconstructed stably from samples spaced at around the Planckian density. For trans-Planckian frequency modes, $|p_0| \gg l_P^{-1}$, the scaling of the bandwidth volume depends on D, namely $B(p_0) = O(|p_0|^{D-2})$ for $|p_0| \to \infty$. Thus, by Landau's theorem, the minimum sample density for stable reconstruction as $|p_0| \to \infty$ is decreasing for $D = 1$, while constant for $D = 2$ and increasing for $D \geq 3$. Therefore, in one and two spatial dimensions, Planck scale sample density suffices to stably reconstruct all temporal frequency modes, i.e. the entire field. In the curved space setting this might apply, e.g., to strings/string bits and to the holographic principle for horizons respectively, see, e.g., [19]. For $D \geq 3$ no single sample density suffices to stably reconstruct simultaneously all temporal modes. Similar to the discussion above, it is clear that each spatial mode $\vec{p}$ possesses a finite temporal bandwidth and that, independently of D, all spatial modes can be stably reconstructed from times series of Planck scale spacing. Generally, if the reconstruction is not required to be stable, significantly sparser sample densities may suffice.

The analysis of Minkowski space generalizes straightforwardly to static space-times. These possess coordinates in which the d'Alembertian consists of a simple temporal part and an elliptic self-adjoint spatial part. In this case, each temporal frequency mode possesses the equivalent of a spatial bandwidth, and vice versa. It should be most interesting to carry through a corresponding analysis of the sampling theory in generic space-times with singularities and with horizons, where space-like and time-like coordinates can switch their roles. Recall that while the reconstruction formulas of fields depend on the choice of coordinates, the physics of the bandwidth cutoff is, of course, invariant. For those pseudo-Riemannian manifolds which allow Wick rotation the situation reduces to the simpler case of Riemannian manifolds and Wick rotation of the reconstruction formulas.

Of interest to phenomenology is that, in typical inflationary scenarios, modes which are today of cosmological size grew from the Planck scale merely about five orders of magnitude before their dynamics froze upon crossing the Hubble horizon. Therefore, Planck scale physics could have a small but potentially measurable effect on the inflationary predictions for the CMB amplitude- and polarization spectra, see [21]. A small effect is predicted, in particular, if the above-mentioned generalized uncertainty relations hold, see [10], which would give fields the sampling property. So far, these models introduce Planck scale physics in ways that are tied to the preferred foliation of space-time into the essentially flat space-like hypersurfaces defined by the cosmic microwave background rest frames, and these models are, therefore, breaking general covariance. The new approach here can be used to predict possible effects of a generally covariant sampling theoretic cutoff on CMB predictions. While free field theory suffices for this purpose, note that field theoretic interaction terms, i.e., higher than second powers of fields (second powers occur as scalar products in the Hilbert space of fields) would have to be nontrivial in order to yield a result within the cut-off Hilbert space.

Interestingly, regularization of field theories in curved space is known to induce in the action a series in the curvature tensor, see [20]. We now see that if this series possesses a finite radius of convergence it induces a bandwidth cutoff and sampling theorem for the metric itself, i.e., a cutoff on the curvature of space-time.

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