SOLITONS

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1. Introduction

1.1. Historical Account

A solitary wave is defined as a spatially confined (localized), non-dispersive and non-singular solution of a non-linear field theory, i.e. one without superposition principle (possible for example in shallow water but not on a simple string). Therefore they have been thought impossible because a dispersive and non-linear medium has been expected to alter any shape of any wave over time. Dispersion alone leads to shock fronts of the propagating wave. That the non-linearity can compensate for the dispersion leading to a propagating and stable wave having constant velocity and shape came as a surprise. A solitary wave was firstly discussed in 1845 by J. Scott Russell in the “Report of the British Association for the Advancement of Science”. He observed a solitary wave traveling along a water channel. The existence and importance was disputed until D.J. Korteweg and G. de Vries gave a complete account of solutions to the non-linear hydro dynamical equation in 1895. In 1955, research into stable and non-dispersive but localized solutions in non-linear media was taken up again when the equipartition of energy between 64 weakly and non-linearly coupled harmonic oscillators was modeled numerically [1]. Starting with only one oscillator excited the energy distributed itself over the whole mode system but returned almost completely to the first excited one. Thermodynamic equilibrium was not reached and the excitation was stable in that sense. From then onwards solitary waves or solitons (their definitions change from author to author) have become more and more important.

Although for quite some time only classical solutions (i.e. not quantized) in low dimensional spaces have been considered their importance was recognized in quite different areas of physics. Information technology, struggling with signal broadening along transmission lines, would certainly gain from the use of non-dispersive pulses. For particle physicists a localized and stable wave might be a good model for elementary particles opening up in a non-linear field theory the possibility of what would have to be a wave packet in a linear one (newer fundamental gauge theories are non-Abelian and therefore non-linear). Notably among others is the Skyrme model that aims to describe nucleons and nucleon-nucleon interactions. Topological solitons give rise to pre-quantum mechanical quantization of charges.

For any non-linear theory the soliton is at least as fundamental a solution as the sine wave. Recently, there have been profound advances in finding solitons in higher dimensional theories and in quantizing them. Doing quantum mechanics one finds relations between solitons that go very deep and are entirely unexpected from a classical viewpoint.

1.2. Shallow Water Solitary Waves

A “swell” or German “Seegang” is due to the fact that the superposition principle does not apply. Bigger waves gain energy from smaller ones - they do not go through each other and reappear again undisturbed. Concentrating on waves in a straight channel with shallow water one is effectively left with a 1+1 dimensional problem described by the Korteweg de Vries equation (non-linear hydro dynamical equation) [2]. One models the non-linearity by a factor proportional to the displacement and the dispersion by a factor proportional to its third derivative. The wave equation can be written as:

$$\frac{\partial \Psi}{\partial t} + c(1 + b\Psi)\frac{\partial \Psi}{\partial x} + d\frac{\partial^3 \Psi}{\partial x^3} = 0 \quad (1.2.1)$$

b is the non-linearity and d the dispersion constant. A solitary wave is shape stable and of constant velocity, i.e. it obeys $\Psi = \Psi_{(z)}$ and z = (vt-x) with v being the velocity of the wave, and it is localized, i.e. $\Psi = 0$ in the limit of $x \rightarrow \pm\infty$. The wave equation, being a partial differential one, simplifies to an ordinary differential equation for $\Psi_{(z)}$,

$$(\Psi')^2 = \Psi^2 (v - c - ⅓\, b\, c\, \Psi)/d \quad (1.2.2)$$

, showing that with increasing b or decreasing d the shape becomes more peaked. Applying the condition for the wave crest (maximum) which is $\Psi'_{max} = 0$, we obtain

$$v = c\,(1 + ⅓\, b\, \Psi_{max}) \quad (1.2.3)$$

, implying that with increasing b or Ψ_{max} the speed increases. A solution to the wave equation is

$$\Psi = 3(v - c)/(b - c) \quad \mathrm{sech}^2[\, z\, ((v\text{-}c)/4d)^{1/2}\,], \quad (1.2.4)$$

where sech = 1/cosh avoids the ambiguous $\cosh^{-1}$. This solitary wave looks similar to a Gaussian bell in its smoothness and localization and in that it does not go below zero displacement before or after the very wave body.

2. Solitons in Field Theory

2.1. Definitions and Classification

The definitions for “solitary wave” and “soliton” are not standardized. Both are loosely called soliton. For a *solitary wave* holds either for the wave Ψ itself (displacement, shape or field value) or for its energy density w, that it is a function f with the following four properties:

A) $f_{(x,\,t)} = f_{(vt-x)}$ (2.1.1)

B) It is non-singular.

C) It is integrable (has only finite energy).

D) It is localized, i.e. it holds

$$\Psi = \text{const}_{\text{vac}} \quad \text{or} \quad w = 0 \qquad (2.1.2),\ (2.1.3)$$

both in the limit of $x \rightarrow \pm\infty$, where $\text{const}_{\text{vac}}$ is any vacuum value. The definition using the energy density w is more appropriate for particle physics but cannot describe objects in theories without conserved energy.

Solitons or “*indestructible solitons*” are solitary waves that can pass through each other or bounce back from each other, emerging asymptotically (as t goes to infinity) unaltered from the collision. They may be displaced but not accelerated, i.e. N incident solitons with energy density

$$\lim(t \rightarrow -\infty)\ w = \Sigma^{N}_{n=1}\, w_N \qquad ; w_n := w_n\,(\mathbf{x} - \mathbf{a}_n - \mathbf{v}_n\, t) \qquad (2.1.4)$$

interact in such a way that holds (and I think unnecessary restrictive on the velocity vector):

$$\lim(t \rightarrow +\infty)\ w = \Sigma^{N}_{n=1}\, w_N \qquad ; w_n := w_n\,(\mathbf{x} - (\mathbf{a}_n + \boldsymbol{\delta}_n) - \mathbf{v}_n\, t) \qquad (2.1.5)$$

The displacement vector $\boldsymbol{\delta}$ will often be interpreted as a time gain or lag due to mutual attraction or repulsive forces between the solitons.

Even when *solitary waves* are discovered in the spectrum of a theory, it is extremely complicated to find out whether a theory allows for *solitons* or to prove that *solitons* are possible or impossible in a particular theory. They are found only in one spatial dimension and if Lorentz invariance is demanded, only for the sine-Gordon equation (see next section). Please note that sometimes I will mingle the terms “soliton” and “solitary wave” just like it is done in every literature I know of.

In renormalizable relativistic local (i.e. the action is dependent on local field) field theories all solitary waves are either non-topological or topological. An example for the former kind is the water canal solitary solution to the Korteweg de Vries equation. Non-topological means that the boundary conditions at infinity are topologically the same for the vacuum as for the soliton. The vacuum can be non-degenerate but an additive conservation law is required. For more on non-topological solitary waves please see [3].

We will be more interested in the topological solitary wave solutions of renormalizable, relativistic and local field theories. They need a degenerate vacuum. The boundary conditions at infinity are topologically different for the solitary wave than for a physical vacuum state. Take for example the twist on an infinitely long rope. Defining one "end" as the true vacuum, the other one (at $+\infty$) can have any angle (e.g. -2π due to a right turn once around the axis along the rope). This angle is a conserved charge; the twist - having localized energy due to torsion - is stable since one would have to turn an infinitely long end in order to untwist the rope which needs a semi-infinite amount of energy. This solitary wave is turned into a truly topological one if it has clothes pegs along the rope with the physical vacuum being the untwisted rope having all clothes pegs pointing with the heavy ends downwards in a gravitational field. Now there are distinct classes of vacuums corresponding to the angles $n2\pi$ with n being an integer. In general: The stability of topological solitons is due to the distinct classes of vacuums at the boundaries where "these boundary conditions are characterized by a particular correspondence (mapping) between the group space and co-ordinate space, and because these mappings are not continuously deformable into one another they are topologically distinct." [4].

2.2. Topological Solitary Waves

2.2.1. Kink Solutions of 1+1 Dimensions

The kink is basically just like the twist on a rope with clothes pegs. It is not important for particle physics but still an excellent and well known example. One starts with the free Klein-Gordon field and adds a potential with a degenerate vacuum:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - V_{(\phi)}$$, with Hamiltonian $$\mathcal{H} = \frac{1}{2}\left(\frac{\partial \phi}{\partial t}\right)^2 + \frac{1}{2}\left(\frac{\partial \phi}{\partial x}\right)^2 + V_{(\phi)} \qquad (2.2.1.1)$$

Applying the Euler-Lagrange equation (i.e. by variation of the action) one obtains the equation of motion:

$$\partial_\mu \partial^\mu \phi - \left(\frac{\partial V}{\partial \phi}\right) = 0 \qquad (2.2.1.2)$$

Since a moving solution is easily found by boosting (here Lorentz transforming) a stationary solution we may concentrate on the latter. For a stationary (static) solution holds

$$(\partial\Phi/\partial t) = 0 \quad \Rightarrow \quad (\partial^2\Phi/\partial x^2) = \partial V/\partial\Phi \qquad (2.2.1.3)$$

, which shows that there is no static solitary wave solution if V has only one minimum (vacuum) $\partial V/\partial\Phi = 0$. (2.2.1.3) gives integrated

$$\int\Phi'\Phi''dx = \int\Phi''\,(dV/d\Phi)\,dx \quad \Rightarrow \quad \tfrac{1}{2}(\partial\Phi/\partial x)^2 = V_{(\Phi)}. \qquad (2.2.1.4)$$

Together with the boundary conditions required for solitary waves this gives us solutions only for neighboring vacuums. Enumerating the vacuums by integer n's there are kinks joining a vacuum n to the vacuum n+1 and anti-kinks joining n to n-1 with the conserved charge $N := (n_2 - n_1) = \pm 1$. (2.2.1.5)

The name "kink" describes the shape of the field Φ where it goes over from one vacuum value to another one. We can use (2.2.1.4) to integrate:

$$\frac{d\phi}{dx} = \pm\sqrt{2V} \quad \Rightarrow \quad (x - x_0) = \pm\int_{\phi_{(x_0)}}^{\phi_{(x)}} \frac{d\tilde{\phi}}{\sqrt{2V_{\tilde{\phi}}}} \qquad (2.2.1.6)$$

One example allowing for kinks is the Φ^4 -theory used for symmetry breaking. The potential

$$V = \tfrac{1}{4}\lambda\,[(m^2/\lambda) - \Phi^2]^2 \qquad ; m^2, \lambda \in \mathbb{R}^+ \qquad (2.2.1.7)$$

, with λ being a self interaction coupling constant has the vacuums at $\Phi = \pm(m/\sqrt{\lambda})$. The equation of motion follows as $\Box\Phi - m^2\,\Phi + \lambda\Phi^3 = 0$. (2.2.1.8)

Using (2.2.1.6) with the choice $\Phi|x_0 = 0$ and via integration over one obtains:

$$\phi_{(x)} = \pm\frac{1}{g}\tanh\left[\frac{m}{\sqrt{2}}(x - x_0)\right] \; ; \; g = \frac{\sqrt{\lambda}}{m} \qquad (2.2.1.9)$$

, with the (−)-sign for the anti-kink. These are solitary waves because the energy density is localized

$$w_{(x)} \propto \mathrm{sech}^4[(x - x_0)\,m/\sqrt{2}], \qquad (2.2.1.10)$$

and the energy, called the classical kink mass M_{cl}, is finite:

$$M_{cl} = W = \int w\,dx = \tfrac{1}{3}\,m\,g^{-2}\sqrt{8} \qquad (2.2.1.11)$$

These kinks are not solitons because there are no field configurations with more than one solitary kink. One kink joins two vacuums and another one after that would have to join to the next vacuums but the potential has only two vacuums. A moving kink

$$\Phi_{(x)} = 1/g \ \tanh[m\, z\, \gamma/\sqrt{2}] \ ; z = (x - x_0 - vt) \text{ and } \gamma = (1-v^2)^{-1/2} \qquad (2.2.1.12)$$

can be shown to collide with an anti-kink in a not shape conserving way. Although not a soliton the width and the mass of the kink behave like the width and mass of a particle. The width suffers length contraction and for the mass holds $W = \gamma\, M_{cl}$. (2.2.1.13)

Very important and characteristic for solitary waves is the fact that the solutions cannot be obtained perturbatively starting from the linear expression having $g,\lambda = 0$. The existence of solitary solutions does not depend on the strength of the non-linear coupling. To proof this, we define $\sigma := g\,\Phi$ and $L_{(\sigma)} := g^2\, L_{(\Phi)}$ (2.2.1.14)

In order to write

$$V_{(\Phi)} \longrightarrow g^{-2}\, V_{(\Phi)} \qquad \Rightarrow \qquad L_{(\sigma)} = -\tfrac{1}{2}\, (\partial\sigma/\partial x^{\mu})^2 - V_{(\sigma)} \qquad (2.2.1.15)$$

, which is g-independent. As long as $g,\lambda \neq 0$ the solitary wave solutions cannot be neglected. In fact, the solitary waves become singular (go to infinity) as $\lambda \to 0$. The non-perturbative character is evident in case of topological solitons because a perturbation around one vacuum will not produce a solution (excitation) close to a vacuum of a different homotopy class. The whole of non-Abelian gauge theories is enriched since soliton related methods open up all the aspects which have been previously inaccessible by the perturbation series. Quark confinement for example might be best described by solitons. Since quarks are asymptotically free inside the hadrons they can be described by plane waves if inside but at distances comparable to the hadron radius the dynamics of quark systems is better described with bound solutions of solitary waves

2.2.2. Sine-Gordon Kink

There are many possible relativistic kinks in 1+1 dimensions, but ones that are solitons have only been found for the potential

$$V = \left(\frac{m}{g}\right)^2 \left[1 - \cos(g\phi)\right] = \frac{1}{2!} m^2 \phi^2 - \frac{\lambda}{4!}\phi^4 + \frac{\lambda^2}{m^2 6!}\phi^6 - \ldots\ldots \qquad (2.2.2.1)$$

This potential has been applied to several problems in condensed state physics like for instance the propagation of dislocations in crystals. The vacuums are at $\Phi = n\, 2\pi/g$ with $n \in \mathbb{Z}$, although one better thinks of them as being one vacuum with the internal Φ-space being compactified modulo $2\pi/g$. The equation of motion is called the Sine-Gordon equation:

$$\Box\Phi + (m^2/g)\sin(g\Phi) = 0 \qquad (2.2.2.2)$$

For the static kink we may apply everything outlined before, i.e.: (2.2.1.3) to (2.2.1.6). A short calculation shows that the kinks energy is finite and inverse to the self interaction coupling:

$$E = \int_{-\infty}^{\infty} \mathcal{H} dx = \int_{-\infty}^{\infty} \left[\frac{1}{2} \left(\frac{\partial \phi}{\partial x} \right)^2 + V \right] dx = \int_{-\infty}^{\infty} (2V) dx = \int_{0}^{2\pi/g} \sqrt{2V} d\phi \qquad (2.2.2.3)$$

$$E = \frac{\sqrt{2}m}{g} \int_{0}^{2\pi/g} \sqrt{1 - \cos(g\phi)} d\phi = \frac{\sqrt{2}m}{g} \int_{0}^{2\pi} \sqrt{1 - \cos(\alpha)} d\alpha = \frac{8m}{g^2} \qquad (2.2.2.4)$$

A divergenceless current can be defined with the (non-Noether) current:

$$J^\mu = \frac{g}{2\pi} \varepsilon^{\mu\nu} \partial_\nu \phi \quad ; \quad \varepsilon^{01} = 1 \qquad (2.2.2.5)$$

, where ε stands for the totally anti-symmetric matrix. It holds $\partial_\mu J^\mu = 0$ (2.2.2.6)

and $$N = \int_{-\infty}^{\infty} J^0 dx = \frac{g}{2\pi} \int_{-\infty}^{\infty} \left(\frac{\partial \phi}{\partial x} \right) dx = \frac{g}{2\pi} \left(\phi_{(\infty)} - \phi_{(-\infty)} \right) = (n_2 - n_1)$$ indeed. (2.2.2.7)

Using (2.2.1.6) to get to the static kink and then boosting the static solution gives the moving sine-Gordon kink Φ^+ and anti-kink Φ^- [using z and γ as in (2.2.1.12)]:

$$\Phi^{\pm}_{(x)} = \pm 4 \arctan[\exp(z\,\gamma/\sqrt{g})]. \qquad (2.2.2.8)$$

These solitary waves are indestructible solitons since there are for example quite simple expressions Φ that obey $\Phi_{(t=\pm\infty)} = \Phi^+ + \Phi^-$ (Forward scattering, the solitons pass through each other and the displacement shows mutual attraction), $\Phi_{(t=\pm\infty)} = \Phi^+ + \Phi^+$ (Backward scattering, the solitons bounce back from one another and the displacement shows mutual repulsion) or $\Phi_{(t=\pm\infty)} = \Phi^- + \Phi^-$ (Backward scattering of anti-kinks). Moreover there are bound solutions called doublets or breather solutions. Here a kink and an anti-kink pass through each other again and again, never separating entirely due to their mutual attraction. One speaks here of solitons although they are not functions of z = (x-vt). For the mathematical expressions please see [5]. That any number of such kinks and anti-kinks are solitons (and not just solitary) is proven with help of the so called inverse scattering method. The proof is archived by expressing the Hamiltonian of the whole system of n solitons with action-angle variables in such a way that it is fully separable into single particle Hamiltonians of plane waves, solitons, anti-solitons and doublets. For the sine-Gordon system one can say that the colliding solitons preserve their shape because of an infinite number of conservation laws.

Please note that the doublets are bound solutions although we have not entered the realm of quantum mechanics yet. Since the discovery that atoms are only stable because of quantum mechanical orbits one thought that quantum mechanics is vital in order to have stable bound states in a relativistic field theory. Moreover, even QED, as successful as it is, does not permit stable and bound solutions (the positronium decays). The doublet solutions show the possibility of stable bound states in classical relativistic field theories that are non-linear.

2.2.3. Vortices

A quantized vortex in two spatial or a quantized vortex line in three spatial dimensions is stable because of their quantization being due to a topological charge, the winding number. Objects with different winding numbers are not continuously deformable into one another just like a rubber band going twice around S^1 cannot be continuously deformed (in S^1)into one that goes around only once. Starting with the Higgs Lagrangian of the BSC-theory one obtains a degenerate vacuum via symmetry breaking. This leads to a quantized magnetic flux line describing the Abricosov flux line in type II superconductors.

For the static situation holds: $\Psi = \Psi\, e^{i\Phi}$ (2.2.3.1)

and $$\mathcal{L} = -\left(\overset{\mu}{\nabla} - iq\overset{\mu}{A}\right)\Psi\left(\overset{\mu}{\nabla} + iq\overset{\mu}{A}\right)\Psi^* - m^2|\Psi|^2 - \lambda|\Psi|^4 \quad ; \quad \lambda > 0$$ (2.2.3.2)

By setting $m^2 = a\,(T-T_c)$, one has spontaneous symmetry breaking if

$T < T_c$. (2.2.3.3)

For the degenerate vacuum follows: $\Psi_{min} = |m^2/(2\lambda)|^{1/2}\, e^{i\alpha}$. (2.2.3.4)

In order to describe superconductivity one takes the low energy behavior, i.e. the Higgs Lagrangian around Ψ_{min}. Single valuedness of the wave function Ψ as it goes around a vortex centre leads to the topological charge of the winding number n and with electrical charge quantization added to flux quantization to

$\Phi = n\,(2\pi/q)$ (2.2.3.5)

in natural units, where q = 2e for the Cooper pair.

Please note that we treat q as being some fixed factor of the theory (coupling constant) and that the quantization we are interested in is due to the topological charge n. With r, z and φ being cylindrical co-ordinates and n the winding number there are solutions specified by Ψ_z = constant and the limit $\lim(r\rightarrow\infty)\, \Psi = |\,\Psi_{min}\,|\, e^{in\varphi}$, (2.2.3.6)

that have the boundary conditions at infinity different than the conditions for the vacuum which is (2.2.3.4) with a chosen and fixed (symmetry is broken) α. The flux quantization is treated in books on superconductivity and super fluidity. An approximate solution giving $|\Psi|_{(r)}$ and $A^\mu_{(r)}$ for a static vertex has been found by Nielsen and Olesen [6]. A good introduction covering this is [7]. The phase of Ψ defines a mapping of the boundary of the co-ordinate space (here S^1) onto the group space of the gauge symmetry U(1) (being

S^1 as well). This map $\Psi: S^1 \rightarrow S^1$ depends on the winding number n. Mappings of different n cannot be continuously deformed into one another. One says that the first homotopy group of S^1 is not trivial:

$$\pi_n(S^n) = \mathbb{Z} \qquad \Rightarrow \qquad \pi_1(S^1) = \mathbb{Z}. \qquad (2.2.3.7)$$

If the gauge group were SU(2) instead then the group space would be S^3 since SU(2) is the group of 2x2 matrices

$$U = u_0 + i\sum_{j=1}^{3} u_j \sigma_j \; ; \quad \sum_{\mu=0}^{3} u_\mu^{\;2} = 1 \qquad (2.2.3.8)$$

the latter of (2.2.3.8) being the equation of a unit sphere in E^4. $\pi_1(S^3)$ is trivial, i.e.

$$\pi_m(S^m) = 0 \quad \text{for all } m<n, \qquad \Rightarrow \qquad \pi_1(S^3) = 0. \qquad (2.2.3.9)$$

The same holds for generalized "vortices" due to the Abelian U(1) symmetry in three (or more) +1 dimensions. They cannot exist, because $\pi_m(S^1) = 0$ for all $m>1$. (2.2.3.10)

To summarize: The topologies of co-ordinate and group spaces yield arguments for the possible existence or non-existence of topological solitary waves. All what remains to be established is whether those topological charged objects have finite energy. If so, they will pop up given sufficient energy, say in the Big Bang.

For example: Because of $\pi_2(S^1) = 0$, there are no point singularities in He II, but for isotropic ferromagnetic systems where the magnetization can point in any direction and the boundary of E^3 is S^2 point singularities are topologically possible [$\pi_2(S^2) = \mathbb{Z}$] and of finite energy; hence they exist. On the other hand, the ferromagnet will not have line singularities [$\pi_1(S^2) = 0$].

2.2.4. 't Hooft-Polyakov Monopoles

Dirac quantization condition:

$$e\Phi_{total} = n(2\pi) \text{ or } q_{el}\, q_{mag} = 2\pi \text{ for the minimal case} \qquad (2.2.4.2)$$

For a derivation see [8]. We will be interested into the 't Hooft-Polyakov monopole [9] which is more likely to exist because it is topologically possible if the group space is that of SO(3) which is doubly connected (recall that SU(2) has no solitons in 2+1 dimensions because its group space is singly connected.) and this group is important for some theories in 3+1 dimensions. Furthermore, this kind of monopole is more likely to exist because we do not have to introduce the magnetic charge by hand - for "beauty reasons" - but the charge comes from a theory with only electrical charges at the outset. The magnetic ones arise topologically. Finally: This monopole solution has been shown by t' Hooft to be energy finite indeed.

The Lagrangian one uses and which leads to a non-Abelian theory with an O(3) symmetry that will be broken is as follows:

$$\mathcal{L} = -\frac{1}{4}F^{a}_{\mu\nu}F^{\mu\nu a} + \frac{1}{2}\left(D_{\mu}\phi^{a}\right)\left(D^{\mu}\phi^{a}\right) - \frac{m^{2}}{2}\phi^{a}\phi^{a} - \lambda\left(\phi^{a}\phi^{a}\right)^{2} \qquad (2.2.4.3)$$

, where the gauge field F is

$$F^{a}_{\mu\nu} = \partial_{\mu}A^{a}_{\nu} - \partial_{\nu}A^{a}_{\mu} + e\varepsilon^{abc}A^{b}_{\mu}A^{c}_{\mu}\text{; a is the group index} \qquad (2.2.4.4)$$

, and the isovector Higgs field ϕ^{a} comes with the covariant derivative

$$\left(D_{\mu}\phi^{a}\right) = \partial_{\mu}\phi^{a} + e\varepsilon^{abc}A^{b}_{\mu}\phi^{c} \, . \qquad (2.2.4.5)$$

The equation of motion following from this is

$$\left(D_{\mu}\left(D^{\mu}\phi^{a}\right)\right) + \left(m^{2} + 4\lambda\phi^{b}\phi^{b}\right)\phi^{a} = 0 \qquad (2.2.4.6)$$

, which allows for solutions

$$\lim_{r\to\infty} A^{a}_{i} = -\varepsilon_{iab}\frac{r^{b}}{er^{2}} \;,\; A^{a}_{0} = 0 \;\text{ and }\; \lim_{r\to\infty}\phi^{a} = F\frac{r^{a}}{r} \qquad (2.2.4.7)$$

, that have, if viewed from spatial infinity, a radial magnetic field just like the Dirac monopole only that the magnetic flux is $\Phi = 4\pi/e$. This is twice as much as the Dirac quantization condition gives as the minimum. A magnetic current K_{μ} can easily be defined (for this and derivations see [10]) in such a way that $\partial_{\mu}K^{\mu} = 0$ without it being a Noether current. The topological nature of this current can be seen again by expressing the magnetic charge by integration $M \propto \int K^{0}d^{3}x$. It turns out that this can be rewritten as integration over the sphere S^{2} at infinity - that is the boundary of co-ordinate space. It follows that

$$M = d/e \qquad \text{for all } d \in \mathbb{Z} \qquad (2.2.4.8)$$

(d is the so called "Brouwer degree") due to the fact that the single valued vector Φ will be covered an integral number of times while the integration covers S^{2} once. Again we meet the concept of a mapping of group space - here the vacuum manifold S^{2} after symmetry breaking describing the orientations of the isovector Φ^{a} - onto the boundary of the co-ordinate space.

However, the non-Abelian electroweak group is given by the Weinberg-Salam model and is SU(2)xU(1) which does not have the topology of SO(3). t' Hooft-Polyakov monopoles exist in case that $\pi_{2}(G/H)$ is non-trivial, where G denotes the gauge group and H the unbroken subgroup (here always U(1)). Therefore the Weinberg-Salam model is monopole free. They may exist in GUTs where SU(2)xU(1) is embedded in SU(5) for example. Magnetic monopoles and even dyons, which are objects with magnetic and electric charge, have been found in a variety of models [11].

2.2.5. Instantons

Instantons are localized in time and space. They arise in Euclidean theories, that is in theories where one substitutes the Minkowskian metric η by the Euclidean metric δ and

$$\mu \in \{0, 1, 2, 3\} \rightarrow \mu \in \{1, 2, 3, 4\} \qquad (2.2.5.1)$$

so that the theory is not LT but O(4) invariant ($\tau^2 = x^\mu x^\mu$). This can be thought of as an analytical continuation of (real) time to the imaginary. The 3+1 space-time becomes E^4 whose boundary is S^3. Therefore one may find topological objects (solitons) to SU(2) gauge field theories since there are non-homotopic mappings ϕ :S3→ S3 because it holds $\pi_3(S^3) = \mathbb{Z}$. Hence one does not need spontaneous symmetry breaking for instantons. Instantons are localized in time and therefore one speaks of finite action solutions rather than of finite energy ones. One defines the Euclidean action as follows:

$$S_{Euclid} = -i\, S_{Minkowski} \qquad (2.2.5.2)$$

Given for instance the Klein-Gordon system:

$$S_{Min} = \int dx^0 \int d\vec{x} \left[\frac{1}{2}\left(\frac{\partial\phi}{\partial x^0}\right)^2 - \frac{1}{2}(\nabla\phi)^2 - m^2\phi 2 \right] \qquad (2.2.5.3)$$

$$\Rightarrow \left(\frac{\partial^2}{\partial (x^0)^2} - \nabla^2 \right)\phi + m^2\phi = 0 \qquad (2.2.5.4)$$

$$S_{Euc} = -\int dx^4 \int d\vec{x} \left[-\frac{1}{2}\left(\frac{\partial\phi}{\partial x^0}\right)^2 - \frac{1}{2}(\nabla\phi)^2 - m^2\phi 2 \right] \qquad (2.2.5.5)$$

$$\Rightarrow \left(-\frac{\partial^2}{\partial (x^4)^2} - \nabla^2 \right)\phi + m^2\phi = 0 \qquad (2.2.5.6)$$

the latter allowing for different solutions than (2.2.5.4) of course. At the classical level, instantons in En are just the static solutions of the Minkowskian n+1 space-time because every Minkowskian space-time has the spatial dimensions purely Euclidean. Some of the solitary objects we met can be used as instantons - only that the finite energy requirement is substituted by a finite action, although this is trivial from the "(n+1)-space-time-point-of-view" [compare the following with (2.2.5.3 and 5)]:

$S_{Min} = \int dt \int d^3x$ [T-V] is an action in a 3+1 space-time.

$S_{Euc} = \int d^4x$ [T+V] is an energy in a 4+1 space-time.

Some aspects of Minkowskian quantum field theories can be studied by starting from classical Euclidean versions. Just like the topological solitary wave has different vacuums at different spatial co-ordinate infinities so has the instanton the same change occurring along the x4-axis. Hence, an instanton is a tunneling from one vacuum to another of another homotopy class. The space-time in between has positive field energy and is therefore a potential barrier. The instanton makes only sense if quantum mechanical because it needs the quantum tunneling through that barrier which is forbidden classically. The tunneling (barrier penetration-) amplitude is e-S thus only objects of finite action contribute.

Example for an instanton is one in a system with no matter fields (ϕ a) and only self interacting Yang-Mills fields (SU(2) gauge fields A^a_μ ; a=1,2,3).

A space-time dependent gauge transformation at the E4- boundary S3 is performed. The boundary becomes a pure gauge vacuum. The field tensor $F_{\mu\nu}$ vanishes there but has to be non-zero in between.

Instantons lead to T and therefore CPT-symmetry violations and can lead to baryon and lepton number (these are topological charges (!)) changes. Those transitions (e.g.: p + n → e^+ + anti(ν_e) come with very small amplitude. Very large lifetimes like 10218 years for the deuteron are a characteristic of instanton calculations.

3. Quantization

Not having a superposition principle renders the quantization of solitary objects complicated. One cannot think of the shape of the solitary wave as being the shape of the wave function for the reason alone that a quantum soliton cannot be localized in space all the time. The uncertainty principle will cause a spreading. Thus the definition of the quantum soliton is bound to the corresponding classical one. There are a lot of ways of quantizing the solitary waves (e.g. functional, canonical etc) and both, solitons and instantons are quantized semi classically, notably among the important methods is the WKB approximation which is an expansion in terms of η. Considering that the crucial action for quantum mechanics is S/η, and with (2.2.1.15) this becomes

$$(1/\eta g^2) \int L_{(\sigma)} \, d^4x \tag{3.1}$$

, such that g and η play much the same role in an expansion, a very recent result will not be surprising: In the weak coupling limit any classical boson field solitary wave implies the existence of a corresponding quantum solution. The quantization is only possible if the non-linear coupling constants are small although the maybe most promising feature of solitons is that they are approached in non-perturbative ways. Only the quantum corrections are perturbative. For more on quantization see [3] or the book “Solitons and Instantons”.

4. References

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