Introduction to Gravitational Self-Force

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Abstract

The motion of sufficiently small body in general relativity should be accurately described by a geodesic. However, there should be "gravitational self-force" corrections to geodesic motion, analogous to the "radiation reaction forces" that occur in electrodynamics. It is of considerable importance to be able to calculate these self-force corrections in order to be able to determine such effects as inspiral motion in the extreme mass ratio limit. However, severe difficulties arise if one attempts to consider point particles in the context of general relativity. This article describes these difficulties and how they have been dealt with.

General relativity with suitable forms of matter has a well posed initial value formulation. In principle, therefore, to determine the motion of bodies in general relativity—such as binary neutron stars or black holes—one simply needs to provide appropriate initial data (satisfying the constraint equations) on a spacelike slice and then evolve this data via Einstein's equation. It would be highly desireable to obtain simple analytic descriptions of motion. However, it is clear that, in general, the motion of a body of finite size will depend on the details of its composition as well as the details of its internal states of motion. Therefore, one can hope to get a simple description of motion only in some kind of "point particle limit". Such a limit encompasses many cases of physical interest, such as "extreme mass ratio" inspiral. Of particular interest are the "radiation reaction" or "self-force" effects occuring during inspiral—the radiation reaction being, of course, the cause of the inspiral.

By definition, a "point particle" is an object whose stress-energy tensor is given by a delta-function with support on a worldline. A delta-function makes perfectly good mathematical sense as a distribution. Now, if a "source term" in an equation is distributional in nature, then the solution to this equation can, at best, be distributional in nature. Thus, if one wishes to consider distributional sources, one must generalize the notion of partial differential equations to apply to distributions. In the case of linear equations, this can be done straightforwardly: The notion of differentiation of distributions is well defined, so it makes perfectly good mathematical sense to consider distributional solutions to linear partial differential equations with distributional sources. Indeed, it is very useful to do so, and, for example, for Maxwell's equations even if the notion of a "point charge" did not arise from physical considerations, it would be very convenient for purely mathematical reasons to consider solutions with a delta-function charge-current source.

However, the situation is different in the case of nonlinear partial differential equations. Products of distributions normally can only be defined under special circumstances[1], so it does not usually even make mathematical sense to say that a distribution satisfies a nonlinear equation. Thus, for example, although Maxwell's equations are linear, the coupled system of Maxwell's equations together with the equations of motion of the charged matter sources are nonlinear. Consequently, the complete, "self-consistent" Maxwell/motion equations are

[1] The product of two distributions can be defined if the decay properties of their Fourier transforms are such that the Fourier convolution integral defining their product converges. This will be the case when the wavefront sets of the distributions satisfy an appropriate condition (see [1]).

nonlinear. Hence, *a priori*, these equations are mathematically ill defined for point charge sources. During the past century, there has been considerable discussion and debate as to how to make sense of these equations.

Einstein's equation is nonlinear, so *a priori* it does not make sense to consider this equation with a distributional source. Nevertheless, it has been understood for the past 40 years that it does make mathematical sense to consider Einstein's equation with a shell of matter [2], i.e., an object whose stress-energy tensor is given by a delta-function with support on a timelike hypersurface (the "shell"). Solutions to Einstein's equation with a shell of matter correspond to patching together two smooth solutions along a timelike hypersurface in such a way that the metrics induced by the solutions on the two sides of the shell agree, but the extrinsic curvature is discontinuous. Such a solution corresponds to having a metric that is continuous, but whose first derivative has a jump discontinuity across the shell, and whose second derivative thereby has a delta-function character on the shell. Since the curvature tensor is linear in the second derivatives of the metric and there are no terms containing products of first and second derivatives of the metric, there is no difficulty in making sense of the curvature tensor of such a metric as a distribution.

Unfortunately, however, the situation is much worse for Einstein's equation with a point particle source[2]. An analysis by Geroch and Traschen [4] shows that it does not make mathematical sense to consider solutions of Einstein's equation with a distributional stress-energy tensor supported on a worldline. Mathematically, the expected behavior of the metric near a "point particle" is too singular to make sense of the nonlinear terms in Einstein's equation, even as distributions. Physically, if one tried to compress a body to make it into a point particle, it should collapse to a black hole before a "point particle limit" can be reached.

Since point particles do not make sense in general relativity, it might appear that no simplifications to the description of motion can be achieved. However, the situation is not quite this bad because it does make mathematical sense to consider solutions, h_{ab}, to the linearized Einstein equation off of an arbitrary background solution, g_{ab}, with a distributional stress-energy tensor supported on a world-line. Therefore, one might begin a treatment of

[2] "Strings"—i.e., objects with a distributional stress-energy tensor corresponding to a delta-function with support on a timelike surface of co-dimension two—are a borderline case; see [3]

gravitational self-force by considering considering solutions to

$$G^{(1)}_{ab}[h](t,x^i) = 8\pi M u_a(t) u_b(t) \frac{\delta^{(3)}(x^i - z^i(t))}{\sqrt{-g}} \frac{d\tau}{dt} \ . \tag{1}$$

Here u^a is the unit tangent (i.e., 4-velocity) of the worldline γ defined by $x^i = z^i(t)$, τ is the proper time along γ, and $\delta^{(3)}(x^i - z^i(t))$ denotes the coordinate delta-function, i.e., $\int \delta^{(3)}(x^i - z^i(t)) d^3x^i = 1$. (The right side of this equation could also be written in a manifestly covariant form as $8\pi M \int \delta_4(x, z(\tau)) u_a(\tau) u_b(\tau) d\tau$ where δ_4 denotes the covariant 4-dimensional delta-function.) However, two major difficulties arise in any approach that seeks to derive self-force effects starting with the linearized Einstein equation:

- The linearized Bianchi identity implies that the right side of eq.(1) must be conserved in the background spacetime. For the case of a point-particle stress-energy tensor as occurs here, conservation requires that the worldline γ of the particle is a geodesic of the background spacetime. Therefore, there are no solutions for non-geodesic source curves, making it a hopeless to use the linearized Einstein equation to derive corrections to geodesic motion.

- Even if the first problem were solved, solutions to this equation are singular on the worldine of the particle. Therefore, naive attempts to compute corrections to the motion due to h_{ab}—such as demanding that the particle move on a geodesic of $g_{ab} + h_{ab}$—are virtually certain to encounter severe mathematical difficulties, analogous to the difficulties encountered in treatments of the electromagnetic self-force problem.

The first difficulty has been circumvented by a number of researchers by modifying the linearized Einstein equation as follows: Choose the Lorenz gauge condition, so that the linearized Einstein equation takes the form

$$\nabla^c \nabla_c \tilde{h}_{ab} - 2R^c{}_{ab}{}^d \tilde{h}_{cd} = -16\pi M u_a(t) u_b(t) \frac{\delta^{(3)}(x^i - z^i(t))}{\sqrt{-g}} \frac{d\tau}{dt} \tag{2}$$

$$\nabla^b \tilde{h}_{ab} = 0 \tag{3}$$

where $\tilde{h}_{ab} \equiv h_{ab} - \frac{1}{2} h g_{ab}$ with $h = h_{ab} g^{ab}$. The first equation, by itself, has solutions for any source curve γ; it is only when the Lorenz gauge condition is adjoined that the equations are equivalent to the linearized Einstein equation and geodesic motion is enforced. We will refer to the Lorenz-gauge form of the linearized Einstein equation (2) with the Lorenz gauge condition (3) *not* imposed—as the *relaxed* linearized Einstein equation. If one solves the relaxed linearized Einstein equation while simply *ignoring* the Lorenz gauge condition that was used to derive this equation, one allows for the possibility of non-geodesic motion. Of course, the relaxed linearized Einstein equation is not equivalent to the original linearized Einstein equation. However, because deviations from geodesic motion are expected to be small, the Lorenz gauge violation should likewise be small, and it thus has been argued that solutions to the two systems should agree to sufficient accuracy.

In order to overcome the second difficulty, it is essential to understand the nature of the singular behavior of solutions to the relaxed linearized Einstein equation on the worldline of the particle. In order to do this, we would like to have a short distance expansion for the (retarded) Green's function for a general system of linear wave equations like (2). A formalism for doing this was developed by Hadamard in the 1920's. It is easiest to explain the basic idea of the Hadamard expansion in the Riemannian case rather than the Lorenzian case, i.e., for Laplace equations rather than wave equations. For simplicity, we consider a single equation of the form

$$g^{ab}\nabla_a\nabla_b\phi + A^a\nabla_a\phi + B\phi = 0 \ , \tag{4}$$

where g_{ab} is a Riemannian metric, A^a is a smooth vector field, and B is a smooth function. In the Riemannian case, Green's functions—i.e., distributional solutions to eq.(4) with a delta function source on the right side—are unique up to the addition of a smooth solution[3], so all Green's functions have the same singular behavior[4]. In 4-dimensions, in the case where g_{ab} is flat and both $A^a = 0$ and $V = 0$, a Green's function with source at x' is given explicitly by

$$G(x, x') = \frac{1}{\sigma(x, x')} \tag{5}$$

where $\sigma(x, x')$ denotes the squared geodesic distance between x and x'. This suggests that we seek a solution to the generalized Laplace equation (4) of the form

$$G(x, x') = \frac{U(x, x')}{\sigma(x, x')} + V(x, x')\ln\sigma(x, x') + W(x, x') \tag{6}$$

[3] This follows immediately from "elliptic regularity", since the difference between two Green's functions satisfies the source free Laplace equation (4).

[4] This is not true in the Lorentzian case; the singular behavior of e.g., the retarded, advanced, and Feynman propagators are different from each other.

where V and W are, in turn, expanded as

$$V(x,x') = \sum_{j=0}^{\infty} v_j(x,x')\sigma^j \ , \quad W(x,x') = \sum_{j=0}^{\infty} w_j(x,x')\sigma^j \tag{7}$$

One now proceeds by substituting these expansions into the generalized Laplace equation (4), using the identity $g^{ab}\nabla_a\sigma\nabla_b\sigma = 4\sigma$, and then formally setting the coefficient of each power of σ to zero (see, e.g., [5]). The leading order equation yields a first order ordinary differential equation for U that holds along each geodesic through x'. This equation has a unique solution—the square root of the van Vleck-Morette determinant—that is regular at x'. In a similar manner, setting the coefficient of the higher powers of σ to zero, we get a sequence of "recursion relations" for the quantities v_j and w_j, which uniquely determine them—except for w_0, which can be chosen arbitrarily. In the analytic case, one can then show that the resulting series for V and W have a finite radius of convergence and that the above expansion provides a Green's function. In the C^∞ but non-analytic case, there is no reason to expect the series to converge, but truncated or otherwise suitably modified versions of the series can be used to construct a *parametrix* for eq.(4). i.e., a solution to eq.(4) with source that differs from a δ-function by at most a C^n function. Even in the analytic case, the Hadamard series defines a Green's function only in a sufficiently small neighborhood of x'. Clearly, this neighborhood must be contained within a normal neighborhood of x' in order that σ even be defined.

A similar construction works in the Lorentzian case, i.e., for an equation of the form (4) with g_{ab} of Lorentz signature. The corresponding Hadamard expansion for the retarded Green's function is

$$G_+(x,x') = U(x,x')\delta(\sigma)\Theta(t-t') + V(x,x')\Theta(-\sigma)\Theta(t-t') \tag{8}$$

where V again is given by a series whose coefficients v_j are uniquely determined by recursion relations. The following points should be noted

- In both the Riemannian and Lorentzian cases, $V(x,x')$ satisfies eq.(4) in x. For a self-adjoint equation (as will be the case for eq.(4) if $A^a = 0$), we also have $V(x,x') = V(x',x)$, so—where defined—V is a smooth solution of the homogeneous equation (4) in each variable.

- As already noted above, $v_0(x,x')$ (i.e., the first term in the series (7) for V) is uniquely determined by a recursion relation that can be solved by integrating an ordinary differential equation along geodesics through x'. In particular, in the Lorentzian case, $v_0(x,x')$ can thereby be obtained on the portion, $\mathcal{N}$, of the future lightcone of x' lying within a normal neighborhood of x'. Since $\sigma(x,x') = 0$ for $x \in \mathcal{N}$, we have $V(x,x') = v_0(x,x')$ on $\mathcal{N}$. However, since—as just noted—$V(x,x')$ satisfies the wave equation in x, it is uniquely determined in the domain of dependence of $\mathcal{N}$. Thus, in the Lorentzian case, one obtains the form (8) for the retarded Green's function in a sufficiently small neighborhood of x' in a way that bypasses any convergence issues for the Hadamard series (7).

- For a globally hyperbolic spacetime, the retarded Green's function $G_+(x,x')$ is globally well defined. By contrast, as already emphasized, the Hadamard form (8) of $G_+(x,x')$ can be valid at best within a normal neighborhood of x'. One occasionally sees in the literature Hadamard formulae that are purported to be valid when multiple geodesics connect x and x', wherein a summation is made of contributions of the form (6) or (8) for each geodesic. I do not believe that there is any mathematical justification for these formulae.

- As follows from the "propagation of singularities" theorem [6], it is rigorously true that, globally, $G_+(x,x')$ is singular if and only if there is a future directed null geodesic from x' to x (whether or not this geodesic lies within a normal neighborhood of x').

Using the generalization of the Hadamard form (8) to the retarded Green's function for the relaxed linearized Einstein equation (2), one finds (after a very lengthy calculation) that the solution to eq.(2) in a sufficiently small neighborhood of the point particle source in Fermi normal coordinates is

$$h_{\alpha\beta} = \frac{2M}{r}\delta_{\alpha\beta} - 8Ma_{(\alpha}u_{\beta)}(1 - a_i x^i) + h^{\rm tail}_{\alpha\beta} + M\mathcal{R}_{\alpha\beta} + O(r^2) \ . \tag{9}$$

Here r denotes the distance to the worldline, a^α is the acceleration of the worldline, $\mathcal{R}_{\alpha\beta}$ denotes a term of order r constructed from the curvature of the background spacetime, and

$$h^{\rm tail}_{\alpha\beta} \equiv M \int_{-\infty}^{\tau^-} \left(G_{+\alpha\beta\alpha'\beta'} - \frac{1}{2} g_{\alpha\beta} G_{+\ \ \gamma\alpha'\beta'}^{\ \gamma} \right) u^{\alpha'} u^{\beta'} d\tau' \ . \tag{10}$$

The symbol τ^- means that this integration is to be cut short of $\tau' = \tau$ to avoid the singular behavior of the Green's function there; this instruction is equivalent to using only the "tail" (i.e., interior of the light cone) portion of the Green's function. In the region where the Hadamard form (8) holds (i.e., for x sufficiently close to x'), this corresponds to the contribution to the Green's function arising from $V(x, x')$.

With the above formula (9) for $h_{\alpha\beta}$ as a starting point, the equations of motion of a point particle—accurate enough to take account of self-force corrections—have been obtained by the following 3 approaches:

- One can proceed in parallel with the derivations of Dirac [7] and DeWitt and Brehme [8] for the electromagnetic case and derive the motion from conservation of total stress-energy [9]. This requires an (ad hoc) regularization of the "effective stress energy" associated to $h_{\alpha\beta}$.

- One can derive equations of motion from some simple axioms [10], specifically that: (i) the difference in "gravitational force" between different curves of the same acceleration (in possibly different spacetimes) is given by the (angle average of the) difference in $-\Gamma^{\mu}{}_{\alpha\beta}u^{\alpha}u^{\beta}$ where $\Gamma^{\mu}{}_{\alpha\beta}$ is the Christoffel symbol associated with $h_{\alpha\beta}$ and (ii) the gravitational self-force vanishes for a uniformly accelerating worldline in Minkowski spacetime. This provides a mathematically clean and simple way of obtaining equations of motion, but it is not a true "derivation" since the motion should follow from the assumptions of general relativity without having to make additional postulates.

- One can derive equations of motion via matched asymptotic expansions [9], [11]. The idea here is to postulate a suitable metric form (namely, Schwarzschild plus small perturbations) near the "particle", and then "match" this "near zone" expression to the "far zone" formula (9) for $h_{\alpha\beta}$. Equations of motion then arise from the matching after imposition of a gauge condition. This approach is the closest of the three to a true derivation, but a number of ad hoc and/or not fully justified assumptions have been made, most notably Lorentz gauge relaxation.

All three approaches have led to the following system of equations (in the case where there is no "incoming radiation", i.e., h_{ab} vanishes in the asymptotic past)

$$\nabla^c \nabla_c \tilde{h}_{ab} - 2R^c{}_{ab}{}^d \tilde{h}_{cd} = -16\pi M u_a(t) u_b(t) \frac{\delta^{(3)}(x^i - z^i(t))}{\sqrt{-g}} \frac{d\tau}{dt} \tag{11}$$

$$u^b \nabla_b u^a = -(g^{ab} + u^a u^b)(\nabla_d h^{\rm tail}_{bc} - \frac{1}{2} \nabla_b h^{\rm tail}_{cd}) u^c u^d \tag{12}$$

where it is understood that the retarded solution to the equation for $\tilde{h}_{ab}$ is to be chosen. Equations (11) and (12) are known as the MiSaTaQuWa equations. Note that the equation of motion (12) for the particle formally corresponds to the perturbed geodesic equation in the metric $g_{ab} + h^{\rm tail}_{ab}$. However, it should be emphasized that $h^{\rm tail}_{ab}$ fails to be differentiable on the worldline of the particle and fails to be a (homogeneous) solution to the relaxed linearized Einstein equation. (This lack of differentiablity affects only the spatial derivatives of the spatial components of $h^{\rm tail}_{ab}$, so the right side of eq.(12) is well defined.) Thus, one cannot interpret $h^{\rm tail}_{ab}$ as an effective, regularized, perturbed metric.

An equivalent reformulation of eq.(12) that does admit an interpretation as perturbed geodesic motion in an effective, regularized, perturbed metric has been given by Detweiler and Whiting [12], who proceed as follows. The symmetric Green's function is defined by $G_{\rm sym} = (G_+ + G_-)/2$ where G_- is the advanced Green's function. For the case of a self-adjoint wave equation of the form (4)—i.e., with $A^a = 0$ and g_{ab} Lorentzian—the Hadamard expansion of $G_{\rm sym}$ is given by

$$G_{\rm sym}(x, x') = \frac{1}{2}[U(x, x')\delta(\sigma) + V(x, x')\Theta(-\sigma)] \tag{13}$$

As previously noted $V(x, x')$ is symmetric (i.e., $V(x, x') = V(x', x)$) and is a homogeneous solution of eq.(4) in each variable. However, $V(x, x')$ is, at best, defined only when x lies in a normal neighborhood of x'. In the region where V is defined, Detweiler and Whiting define a new Green's function by

$$G_{\rm DW}(x, x') = \frac{1}{2}[U(x, x')\delta(\sigma) + V(x, x')\Theta(\sigma)] \tag{14}$$

The Detweiler-Whiting Green's function has the very unusual property of having no support in the interior of the future or past light cones. Detweiler and Whiting show that eq.(12) is equivalent to perturbed geodesic motion in the metric $g_{ab} + h^R_{ab}$ where h^R_{ab} is obtained by applying $G_+ - G_{\rm DW}$ for the relaxed linearized Einstein equation to the worldline source. Since h^R_{ab} is a smooth, homogeneous solution to the relaxed linearized Einstein equation, it can be given an interpretation as an effective, regularized, perturbed metric. Of course, an observer making spacetime measurements near the particle would see the metric $g_{ab} + h_{ab}$, not $g_{ab} + h^R_{ab}$.

Although there is a general consensus that eqs.(11) and eq.(12) (or the Detweiler-Whiting version of eq.(12)) should provide a good description of the self-force corrections to the motion of a sufficiently small body, it is important that these equations be put on a firmer foundation both to clarify their range of validity and to potentially enable the systematic calculation of higher order corrections. It is clear that in order to obtain a precise and rigorous derivation of gravitational self-force, it will be necessary to take some kind of "point particle limit", wherein the size, R, of the body goes to zero. However, to avoid difficulties associated with the non-existence of point particles in general relativity, it is essential that one let M go to zero as well. If M goes to zero more slowly than R, the body should collapse to a black hole before the limit $R \to 0$ is achieved. On the other hand, one could consider limits where M goes to 0 more rapidly than R, but finite size effects would then dominate over self-force effects as $R \to 0$. This suggests that we consider a one-parameter family of solutions to Einstein's equation, $g_{ab}(\lambda)$, for which the body scales to zero size and mass in an asymptotically self-similar way as $\lambda \to 0$, so that the ratio R/M approaches a constant in the limit.

Recently, Gralla and I [13] have considered such one-parameter families of bodies (or black holes). In the limit as $\lambda \to 0$—where the body shrinks down to a worldline γ and "disappears"—we proved that γ must be a geodesic. Self-force and finite size effects then arise as perturbative corrections to γ. To first order in λ, these corrections are described by a deviation vector Z^i along γ. In [13], Gralla and I proved that, in the Lorenz gauge, this deviation vector satisfies

$$\frac{d^2 Z^i}{dt^2} = \frac{1}{2M} S^{kl} R_{kl0}{}^i - R_{0j0}{}^i Z^j - \left(h^{\mathrm{tail}\,i}{}_{0,0} - \frac{1}{2} h^{\mathrm{tail},i}_{00} \right) \tag{15}$$

The first term in this equation corresponds to the usual "spin force" [14], i.e., the leading order finite size correction to the motion. The second term is the usual right side of the geodesic deviation equation. (This term must appear since the the corrections to motion must allow for the possibility of a perturbation to a nearby geodesic.) The last term corresponds to the self-force term appearing on right side of eq.(12). It should be emphasized that eq.(15) arises as the perturbative correction to geodesic motion for any one-parameter family satisfying our assumptions, and holds for black holes as well as ordinary bodies.

Although the self-force term in eq.(15) corresponds to the right side of eq.(12), these equations have different meanings. Equation (15) is a first order perturbative correction to geodesic motion, and no Lorenz gauge relaxation is involved in this equation since h_{ab} is sourced by a geodesic γ. By contrast, eq.(12) is supposed to hold even when the cumulative deviations from geodesic motion are large, and Lorenz gauge relaxation is thereby essential. Given that eq.(15) holds rigorously as a perturbative result, what is the status of the MiSaTaQuWa equations (11), (12)? In [13], we argued that the MiSaTaQuWa equations arise as "self-consistent perturbative equations" associated with the perturbative result (15). If the deviations from geodesic motion are *locally* small—even though cumulative effects may yield large deviations from any individual geodesic over long periods of time—then the MiSaTaQuWa equations should provide an accurate description of motion.

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