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SUPER-GAUGE TRANSFORMATIONS

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Abstract A systematic method for constructing Wess–Zumino supergauge transformations is exhibited

In a recent article Wess and Zumino [1] have invented an interesting new symmetry. Generalizing from the dual model super-gauge symmetry [2] these authors succeeded in defining an analogous transformation group in four-dimensional space-time This invention is quite remarkable in at least two respects (i) the irreducible representations of this symmetry combine fermions with bosons and (ii) the strictures of O'Raifeartaigh's theorem are circumvented – we seem to have here a relativistic spin-containing symmetry which is consistent with unitarity[‡] Moreover, in a simple Lagrangian model involving two scalars and a Majorana spinor, Wess and Zumino found that, in the one-loop approximation, there is only one (logarithmic) divergence [3]

The purpose of this paper is to present a rather simple method which can be used for the construction of at least some of the representations of this symmetry.

We shall confine our considerations to the 14-parameter subalgebra of the Wess–

‡ The group of Wess and Zumino can be looked upon as a sort of quasi U(2, 3) the set of unitary 5×5 matrices, g, whose elements g_α^β, $\alpha, \beta = 1, 2, 3, 4$, and g_5^5 are ordinary complex numbers while g_α^5 and g_5^α are anti-commuting c-numbers. The subgroup $SU(2, 2) \times U(1)$ of the ordinary sort is identified with the product of the 15-parameter conformal group of space-time and a 1-parameter group of γ_5 transformations. The anticommuting parts are identified with super-gauge transformations. Looked at in this way an immediate generalization to U(2,4) (or U(2,5)) is suggested The ordinary subgroup $SU(2, 2) \times U(2)$ (or $SU(2,2) \times U(3)$) might then be said to include a strictly internal SU(2) (or SU(3)) symmetry Contrasted with this marriage of internal symmetries with space-time symmetries, one may also consider a rather trivial generalization of Wess and Zumino's work where each one of their fields is considered as (for example) the adjoint representation of an internal symmetry U(n).

Zumino system which is generated by the Poincare operators $J_{\mu\nu}$, P_μ and the Majorana spinor S_α. In addition to the usual commutation rules involving $J_{\mu\nu}$ and P_μ only, we have

$$[S_\alpha, P_\mu] = 0 ,$$

$$[S_\alpha, J_{\mu\nu}] = \tfrac{1}{2}(\sigma_{\mu\nu})_\alpha^\beta S_\beta , \tag{1}$$

$$\{S_\alpha, S_\beta\} = (\gamma_\mu C)_{\alpha\beta} P_\mu ,$$

where C denotes the charge conjugation matrix†. The last of these rules can be expressed in the alternative version

$$[\bar\epsilon_1 S, \bar S \epsilon_2] = -\bar\epsilon_1 \gamma_\mu \epsilon_2 P_\mu , \tag{2}$$

where ϵ_1 and ϵ_2 are two arbitrary Majorana spinors which anticommute with one another and with S (Notice that $\bar\epsilon S = \bar S \epsilon$ is a hermitian operator and that $\bar\epsilon_1\gamma_\mu\epsilon_2 = -\bar\epsilon_2\gamma_\mu\epsilon_1$ is an imaginary 4-vector†)

Our basic approach is to work out the group action on the space of left cosets with respect to the subgroup of homogeneous Lorentz transformations This "space" is essentially eight-dimensional, being parametrized by the 4-vector x_μ and the (anticommuting c-number) Majorana spinor θ_α A simple way to obtain the group action on this homogeneous space is to define the unitary operators

$$L(x, \theta) = \exp[i x_\mu P_\mu] \exp[i \bar\theta^\alpha S_\alpha] , \tag{3}$$

and consider what happens to them when any one of the operators representing, respectively, a translation, a homogeneous Lorentz transformation or a super-gauge transformation is applied on the left One finds,

$$\begin{aligned}
\exp[i c_\mu P_\mu]\, L(x, \theta) &= L(x + c, \theta) , \\
\exp[\tfrac{1}{2} i \omega_{\mu\nu} J_{\mu\nu}]\, L(x, \theta) &= L(\Lambda x, a(\Lambda)\theta) \exp[\tfrac{1}{2} i\omega_{\mu\nu} J_{\mu\nu}] , \\
\exp[i\bar\epsilon S]\, L(x, \theta) &= L[x_\mu - \tfrac{1}{2} i \bar\epsilon\gamma_\mu\theta, \theta + \epsilon] ,
\end{aligned} \tag{4}$$

where $a(\Lambda) = \exp(\tfrac{1}{4} i)\omega_{\mu\nu}\sigma_{\mu\nu}$ denotes the usual spinor representation of the homogeneous Lorentz group††. (Notice that, because of the Majorana constraints on ϵ

† Our notational conventions are as follows The Dirac matrices satisfy $(\tfrac{1}{2})\ \{\gamma_\mu, \gamma_\nu\} = \eta_{\mu\nu} = \mathrm{diag}\,(+ - - -)$ and adjoint spinors are defined by $\bar\psi = \psi^\dagger\gamma_0$. The matrices γ_0, $\gamma_0\gamma_\mu$, $\gamma_0\sigma_{\mu\nu}$, $\gamma_0 i\gamma_\mu\gamma_5$, $\gamma_0\gamma_5$ are hermitian The charge conjugate of ψ is defined by $\psi^c = C\bar\psi^T$ where $C^T = -C$ and $C^{-1}\gamma_\mu C = -\gamma_\mu^T$ By a Majorana spinor we mean $\psi^c = \psi$. It is useful to remember that the matrices $\gamma_\mu C$ and $\sigma_{\mu\nu}C = (\tfrac{1}{2}i)[\gamma_\mu, \gamma_\nu]C$ are symmetric while C, $\gamma_5 C$ and $i\gamma_\mu\gamma_5 C$ are antisymmetric In particular, it follows that $\bar\psi_1\psi_2 = \bar\psi_2\psi_1$, $\bar\psi_1\gamma_\mu\psi_2 = -\bar\psi_2\gamma_\mu\psi_1$, $\bar\psi_1\sigma_{\mu\nu}\psi_2 = -\bar\psi_2\sigma_{\mu\nu}\psi_1$, $\bar\psi_1 i\gamma_\mu\gamma_5\psi_2 = \bar\psi_2 i\gamma_\mu\gamma_5\psi_1$, $\bar\psi_1\gamma_5\psi_2 = \bar\psi_2\gamma_5\psi_1$ if ψ_1 and ψ_2 are anticommuting Majorana spinors.

†† Space reflections are incorporated by requiring that θ transform according to the rule $\theta \to i\gamma_0\theta$ Likewise for dilations, $x \to \lambda x$, $\theta \to \lambda^{\frac{1}{2}}\theta$ and γ_5 transformations, $\theta \to (\cos\alpha + \gamma_5 \sin\alpha)\theta$ with real α

and θ, the displacement in x_μ caused by a super-gauge transformation is real) Eqs (4) serve to define the action of the group on the space of the parameters x and θ and indicate how any field defined over this space should transform. Thus, for example, the scalar super-field $\Phi(x, \theta)$ should satisfy

$$\exp[i\bar{\epsilon}S]\,\Phi(x,\theta)\exp[-i\bar{\epsilon}S] = \Phi[x - \tfrac{1}{2}i\bar{\epsilon}\gamma_\mu\theta, \theta + \epsilon] \tag{5}$$

By appending a Lorentz index[‡] one could define a vector super-field $\Phi_\mu(x, \theta)$, a spinor super-field $\Psi_\alpha(x, \theta)$, etc

If we were dealing with an arbitrary group then we should not be very pleased with fields defined on an eight-dimensional space-time It was this aspect of the old attempts at combining internal symmetries in a non-trivial way with the Poincare group which hindered their development. The truly remarkable and exciting feature of the Wess-Zumino group is that the superfield $\Phi(x, \theta)$ in eight dimensions is exactly equivalent to a 16-component set of ordinary fields in four dimensions One simply has to expend Φ in powers of θ_α and observe that the series must terminate in the fourth order This is due, of course, to the fact that the monomials

$$\theta_{\alpha_1}\theta_{\alpha_2}\cdots\theta_{\alpha_n}$$

must be completely antisymmetric and therefore vanish for $n > 4$ Therefore we can write

$$\Phi(x,\theta) = \bar{\phi}(x) + \bar{\phi}^\alpha(x)\,\theta_\alpha + \tfrac{1}{2}\bar{\phi}^{[\alpha\beta]}(x)\,\theta_\beta\theta_\alpha + \tfrac{1}{6}\bar{\phi}^{[\alpha\beta\gamma]}(x)\theta_\gamma\theta_\beta\theta_\alpha + \tfrac{1}{24}\bar{\phi}^{[\alpha\beta\gamma\delta]}\theta_\delta\theta_\gamma\theta_\beta\theta_\alpha\,. \tag{6}$$

The number of independent real components involved here can be halved by imposing the reality condition

$$\Phi(x,\theta)^* = \Phi(x,\theta)\,, \tag{7}$$

which reads, in terms of the component fields,

$$\begin{aligned} \phi(x) &= \bar{\phi}(x)\,, \\ \phi_\alpha(x) &= C_{\alpha\alpha'}\bar{\phi}^{\alpha'}(x)\,, \\ \phi_{[\alpha\beta]}(x) &= -C_{\alpha\alpha'}C_{\beta\beta'}\bar{\phi}^{[\alpha'\beta']}(x)\,, \\ \phi_{[\alpha\beta\gamma]}(x) &= -C_{\alpha\alpha'}C_{\beta\beta'}C_{\gamma\gamma'}\bar{\phi}^{[\alpha'\beta'\gamma']}(x)\,, \\ \phi_{[\alpha\beta\gamma\delta]}(x) &= C_{\alpha\alpha'}C_{\beta\beta'}C_{\gamma\gamma'}C_{\delta\delta'}\bar{\phi}^{[\alpha'\beta'\gamma'\delta']}(x)\,, \end{aligned} \tag{8}$$

[‡] The super-gauge transformations induce no Lorentz transformation

where the barred quantities are defined in the usual way, $\bar{\phi}(x) = \phi(x)^*$, $\bar{\phi}^\alpha(x) = \phi_\beta(x)^*(\gamma_0)^\alpha_\beta$, $\bar{\phi}^{[\alpha\beta]}(x) = \phi_{[\alpha'\beta']}(x)^*(\gamma_0)^\alpha_{\alpha'}(\gamma_0)^\beta_{\beta'}$, etc

The behaviour of these components under an infinitesimal super-gauge transformation can be extracted from (5). One finds

$$\begin{aligned}
\delta\bar{\phi} &= \bar{\phi}^\alpha \epsilon_\alpha\,,\\
\delta\bar{\phi}^\beta &= -\bar{\phi}^{[\alpha\beta]}\epsilon_\alpha - \tfrac{1}{2}i(\bar{\epsilon}\gamma_\mu)^\beta\partial_\mu\bar{\phi}\,,\\
\delta\bar{\phi}^{[\beta\gamma]} &= \bar{\phi}^{[\alpha\beta\gamma]}\epsilon_\alpha + \tfrac{1}{2}i(\bar{\epsilon}\gamma_\mu)^\gamma\partial_\mu\bar{\phi}^\beta - \tfrac{1}{2}i(\bar{\epsilon}\gamma_\mu)^\beta\partial_\mu\bar{\phi}^\gamma\,,\\
\delta\bar{\phi}^{[\beta\gamma\delta]} &= -\bar{\phi}^{[\alpha\beta\gamma\delta]}\epsilon_\alpha - \tfrac{1}{2}i(\bar{\epsilon}\gamma_\mu)^\delta\partial_\mu\bar{\phi}^{[\beta\gamma]} - \tfrac{1}{2}i(\bar{\epsilon}\gamma_\mu)^\gamma\partial_\mu\bar{\phi}^{[\delta\beta]} - \tfrac{1}{2}i(\bar{\epsilon}\gamma_\mu)^\beta\partial_\mu\bar{\phi}^{[\gamma\delta]}\,,\\
\delta\bar{\phi}^{[\alpha\beta\gamma\delta]} &= \tfrac{1}{2}i(\bar{\epsilon}\gamma_\mu)^\delta\partial_\mu\bar{\phi}^{[\alpha\beta\gamma]} - \tfrac{1}{2}i(\bar{\epsilon}\gamma_\mu)^\gamma\partial_\mu\bar{\phi}^{[\delta\alpha\beta]} +\\
&\quad + \tfrac{1}{2}i(\bar{\epsilon}\gamma_\mu)^\beta\partial_\mu\bar{\phi}^{[\gamma\delta\alpha]} - \tfrac{1}{2}(\bar{\epsilon}\gamma_\mu)^\alpha\partial_\mu\bar{\phi}^{[\beta\gamma\delta]}\,.
\end{aligned}\tag{9}$$

(These rules imply, in particular, that the space-time integral of the component $\phi_{[\alpha\beta\gamma\delta]}$ should be an invariant if surface effects can be neglected) The representation (9) turns out to be reducible (although not fully reducible) To see this, it is convenient to introduce a new notation for the components Write

$$\begin{aligned}
\bar{\phi}(x) &= A(x)\,,\\
\bar{\phi}^\alpha(x) &= \bar{\psi}^\alpha(x)\,,\\
\bar{\phi}^{[\alpha\beta]}(x) &= (C^{-1})^{\alpha\beta}F(x) + (C^{-1}\gamma_5)^{\alpha\beta}G(x) + (C^{-1}i\gamma_\mu\gamma_5)^{\alpha\beta}[a_\mu(x) + \tfrac{1}{2}\partial_\mu B(x)]\,,\\
\bar{\phi}^{[\alpha\beta\gamma]}(x) &= \epsilon^{\alpha\beta\gamma\delta}[\lambda_\delta(x) + \tfrac{1}{2}i(\gamma_\mu)^{\delta'}_\delta\,\partial_\mu\psi_{\delta'}(x)]\,,\\
\bar{\phi}^{[\alpha\beta\gamma\delta]}(x) &= \epsilon^{\alpha\beta\gamma\delta}[D(x) - \tfrac{1}{4}\partial^2 A(x)]
\end{aligned}\tag{10}$$

If the reality conditions (8) are imposed then all boson components are real and the fermion components ψ and λ are Majorana spinors. The axial-vector field a_μ is constrained to be transverse, $\partial_\mu a_\mu = 0$ The transformation rules (9) now take the form†

$$\begin{aligned}
\delta A &= \bar{\epsilon}\psi\,,\\
\delta B &= \bar{\epsilon}\gamma_5\psi - \frac{i}{\partial^2}\bar{\epsilon}\gamma_5\not{\partial}\lambda\,,
\end{aligned}$$

† Some of the details in these rules are affected by conventions in the definition of C Our C is defined such that $(\tfrac{1}{2})\epsilon^{\alpha\beta\gamma\delta}C_{\gamma\delta} = -(C^{-1})^{\alpha\beta}$, $(\tfrac{1}{2})\epsilon^{\alpha\beta\gamma\delta}(\gamma_5 C)_{\gamma\delta} = +(C^{-1}\gamma_5)^{\alpha\beta}$ and $(\tfrac{1}{2})\epsilon^{\alpha\beta\gamma\delta}(i\gamma_\mu\gamma_5 C)_{\gamma\delta} = +(C^{-1}i\gamma_\mu\gamma_5)^{\alpha\beta}$

$$\delta\psi = \tfrac{1}{2}i\partial_\mu A\gamma_\mu\epsilon + \tfrac{1}{2}i\partial_\mu B\gamma_\mu\gamma_5\epsilon + F\epsilon + G\gamma_5\epsilon + a_\mu i\gamma_\mu\gamma_5\epsilon\,,$$

$$\delta F = \tfrac{1}{2}i\bar{\epsilon}\,\partial\!\!\!/\psi + \tfrac{1}{2}\bar{\epsilon}\lambda\,,$$

$$\delta G = \tfrac{1}{2}i\bar{\epsilon}\gamma_5\partial\!\!\!/\,\psi + \tfrac{1}{2}\bar{\epsilon}\gamma_5\lambda\,, \tag{11}$$

$$\delta\lambda = D\epsilon + \tfrac{1}{4}i\,[\partial_\mu a_\nu - \partial_\nu a_\mu]\,\sigma_{\mu\nu}\gamma_5\epsilon\,,$$

$$\delta a_\mu = \tfrac{1}{2}\bar{\epsilon}\gamma_\mu\gamma_5\lambda - \tfrac{1}{2}i\,\frac{\partial_\mu\partial_\nu}{\partial^2}\,\bar{\epsilon}\gamma_\nu\gamma_5\lambda\,,$$

$$\delta D = \tfrac{1}{2}i\bar{\epsilon}\,\partial\!\!\!/\,\lambda\,, \tag{12}$$

and it appears that we are dealing with one of those curious representations that does not reduce in the usual way The eight independent components in the set D, λ and $a_\mu(\partial_\mu a_\mu = 0)$ clearly transform irreducibly However, unless they are set equal to zero, they involve themselves in the transformations of the other eight components, A, B, ψ, F and G[†]

The setting to zero of D, λ and a_μ can be viewed as a covariant constraint. In fact one can construct an axial vector which generalizes the well-known Pauli-Lubanski operator,

$$K_\mu = \tfrac{1}{2}\epsilon_{\mu\nu\kappa\rho}P_\nu J_{\kappa\rho} - \tfrac{1}{4}\bar{S}i\gamma_\mu\gamma_5 S\,, \tag{13}$$

whose transverse part is super-gauge invariant Thus, the antisymmetric tensor

$$K_{\mu\nu} = P_\mu K_\nu - P_\nu K_\mu \tag{14}$$

commutes with both translations and super-gauge transformations This operator is realized by the following differential expression

$$K_{\mu\nu}\Phi = (X_{\mu\nu}C)_{\alpha\beta}\,\frac{\partial}{\partial\bar{\theta}_\alpha}\,\frac{\partial\Phi}{\partial\bar{\theta}_\beta}$$

$$+\,i(X_{\mu\nu}\gamma_\rho\theta)_\alpha\,\frac{\partial}{\partial x_\rho}\,\frac{\partial\Phi}{\partial\bar{\theta}_\alpha} - \tfrac{1}{4}\bar{\theta}X_{\mu\nu}\theta\,\frac{\partial}{\partial x_\rho}\,\frac{\partial\Phi}{\partial x_\rho}\,, \tag{15}$$

where $X_{\mu\nu} = -(\tfrac{1}{4})\,(\gamma_\mu\gamma_5\partial_\nu - \gamma_\nu\gamma_5\partial_\mu)$. The equations $K_{\mu\nu}\Phi = 0$ are solved by $D = \lambda = a = 0$

The combining of representations into products is, at least in some cases, quite easy One simply multiplies the super-fields The detailed combinations of compo-

[†] We have adapted our notation here to that of Wess and Zumino, ref [1] Thus, our eq (11) with λ, a_μ and D set equal to zero corresponds to their eq (8) with $\partial_\mu\alpha = 0$ Similarly, our (12) corresponds to their formulae on p 48

nents will be revealed by expanding the result in powers of θ For example if

$$\Phi_3(x,\theta)=\Phi_1(x,\theta)\,\Phi_2(x,\theta)\,,$$

then, using the components defined by (6),

$$\begin{aligned}
\bar\phi^{3} &= \bar\phi_1\bar\phi_2\,,\\
\phi_3^{\alpha} &= \bar\phi_1\bar\phi_2^{\alpha}+\bar\phi_1^{\alpha}\bar\phi_2\,,\\
\bar\phi_3^{\alpha\beta} &= \bar\phi_1\bar\phi_2^{\alpha\beta}+\bar\phi_1^{\alpha}\bar\phi_2^{\beta}-\bar\phi_1^{\beta}\bar\phi_2^{\alpha}+\bar\phi_1^{\alpha\beta}\bar\phi_2\,,\\
\bar\phi_3^{\alpha\beta\gamma} &= \phi_1\bar\phi_2^{\alpha\beta\gamma}+\sum_{\text{cyc.}}\left[\bar\phi_1^{\alpha}\bar\phi_2^{\beta\gamma}+\bar\phi_1^{\alpha\beta}\bar\phi_2^{\gamma}\right]+\bar\phi_1^{\alpha\beta\gamma}\bar\phi_2\,,\\
\bar\phi_3^{\alpha\beta\gamma\delta} &= \bar\phi_1\bar\phi_2^{\alpha\beta\gamma\delta}+\sum\bar\phi_1^{\alpha}\bar\phi_2^{\beta\gamma\delta}+\sum\bar\phi_1^{\alpha\beta}\bar\phi_2^{\gamma\delta}+\sum\bar\phi_1^{\alpha\beta\gamma}\bar\phi_2^{\delta}+\bar\phi_1^{\alpha\beta\gamma\delta}\bar\phi_2
\end{aligned}\tag{16}$$

In particular, with $\Phi_1=\Phi_2$ satisfying $K_{\mu\nu}\Phi=0$, it is a simple matter to show that

$$\begin{aligned}
\bar\phi_3^{\alpha\beta\gamma\delta} &= \epsilon^{\alpha\beta\gamma\delta}\left[-\tfrac{1}{4}A\,\partial^2A+\tfrac{1}{2}\bar\psi\partial\!\!\!/\,\psi+\tfrac{1}{4}(\partial_\mu B)^2+F^2+G^2\right]\\
&= \tfrac{1}{2}\epsilon^{\alpha\beta\gamma\delta}\left[\tfrac{1}{2}(\partial_\mu A)^2+\tfrac{1}{2}(\partial_\mu B)^2+\bar\psi\, i\partial\!\!\!/\,\psi+2F^2+2G^2-\tfrac{1}{2}\partial_\mu(A\,\partial_\mu A)\right].
\end{aligned}\tag{17}$$

According to the rules (9) this object must transform by a gradient Its space-time integral is invariant. Wess and Zumino have proposed to use it as a Lagrangian density [1, 3]

The approach discussed in this paper may not prove to be the most serviceable one available but, with the present hazy understanding of this curious and potentially important symmetry, it seems worthwhile to examine every avenue†.

References

[1] J Wess and B Zumino, Nucl Phys B70 (1974) 39

[2] A Neveu and J H Schwarz, Nucl Phys B31 (1971) 86,
P Ramond, Phys Rev D3 (1971) 2415,
Y Aharonov, A Casher and L Susskind, Phys Letters 35B (1971) 512,
J-L Gervais and B. Sakita, Nucl Phys B34 (1971) 632,
Y Iwasaki and K Kikkawa, City College (NY) preprint (1973).

[3] J Wess and B Zumino, Phys Letters 49B (1974) 52

[4] R Penrose and M.A H MacCullum, Phys. Reports 6 (1973) 241

† From relations (1) and (2), it appears that the Wess–Zumino formalism may have close connections with the Twistor formalism of Penrose (ref [4]), though the motivations of the two formalisms are apparently completely different.