

Introduction to Seiberg-Witten Theory and its Stringy Origin¹

W. Lerche²

CERN, Geneva, Switzerland

We give an elementary introduction to the recent solution of $N = 2$ supersymmetric Yang-Mills theory. In addition, we review how it can be re-derived from string duality.

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²e-mail: lerche@nxth04.cern.ch

Introduction to Seiberg-Witten Theory and its Stringy Origin

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We give an elementary introduction to the recent solution of $N = 2$ supersymmetric Yang-Mills theory. In addition, we review how it can be re-derived from string duality.

1. Introduction

In the last two years, there has been a remarkable progress in understanding non-perturbative properties of supersymmetric field and string theories. This dramatic development was initiated by the work of Seiberg and Witten on $N=2$ supersymmetric Yang-Mills theory [1], and by Hull and Townsend on heterotic-type II string equivalence [2]. By now, many non-perturbatively exact statements can be made about various types of supersymmetric Yang-Mills theories with and without matter, and even more drastic statements about superstring theories in various dimensions.

It has become evident that the main insight is of conceptual nature and goes far beyond original expectations. The picture that seems to emerge is that the various known, perturbatively defined string theories represent non-perturbatively equivalent, or dual, descriptions of one and the same fundamental theory. Moreover, strings do not appear to play a very privileged role in this theory, besides higher dimensional p -branes [3]. It may well turn out, ultimately, that there is just one theory that is fully consistent at the non-perturbative level, or a just small number of such theories. Though the number of free parameters (“moduli”) may a priori be very large –which would hamper predictive power– it is clear that investigating this kind of issues is important and will shape our understanding of the very nature of grand unification.

A full treatment of these matters is surely out-

side the scope of these lecture notes, and would be premature anyway. We therefore limit ourselves to discussing some of the basic concepts, and since some of these arise already in supersymmetric Yang-Mills theory, we think it is a good idea to start in section 2 with a very basic introduction to the original work of Seiberg and Witten (for gauge group $SU(2)$). We will emphasize the idea of analytic continuation and the underlying monodromy problem, and show how the effective action can be explicitly computed. For other reviews on this subject, see [4].

In section 3 we will then explain the generalization to other gauge groups, emphasizing the role of the “simple singularities” that are canonically associated with the simply laced Lie groups of type ADE [5]. The simple singularities will turn out to be the key to understand how the SW theory arises in string theory. Indeed, as we will explain in section 4, the SW theory can actually be *derived* from Hull-Townsend string duality. This string duality will also allow to interpret the geometrical structure of the SW theory, in particular the “auxiliary” Riemann surface, in concrete physical terms. In section 5, it will turn out [6] that the SW geometry has indeed a natural interpretation in terms of a very peculiar string theory !

2. $N = 2$ Yang-Mills Theory

2.1. Overview

So, in a nutshell, what is all the excitement about that has made furor even in the mass me-

dia ? As one of the main results one may state the exact non-perturbative low energy effective Lagrangian of $N=2$ supersymmetric Yang-Mills theory with gauge group $SU(2)$; it contains, in particular, the effective, renormalized gauge coupling, g_{eff} , and theta-angle, θ_{eff} :

$$\left(\frac{\theta_{\text{eff}}(a)}{\pi} + \frac{8\pi i}{g_{\text{eff}}^2(a)} \right) = \quad (2.1)$$

$$\underbrace{\frac{8\pi i}{g_0^2}}_{\text{bare}} + \underbrace{\frac{2i}{\pi} \log \left[\frac{a^2}{\Lambda^2} \right]}_{\text{one-loop}} - \underbrace{\frac{i}{\pi} \sum_{\ell=1}^{\infty} c_{\ell} \left(\frac{\Lambda}{a} \right)^{4\ell}}_{\text{instanton corrections}}$$

Here, Λ is the dynamically generated scale at which the gauge coupling becomes strong, and a is the Higgs field. This effective, field dependent coupling arises by setting the renormalization scale, μ , equal to the characteristic scale of the theory, which is given by the Higgs VEV: $g_{\text{eff}}(\mu) \rightarrow g_{\text{eff}}(a)$. The running of the perturbative coupling constant thus looks as follows [7]:

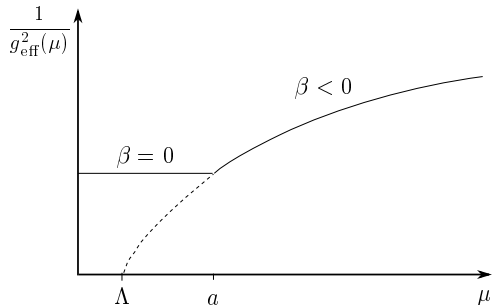


Figure 1. At scales above the Higgs VEV a , the masses of the non-abelian gauge bosons, W^{\pm} , are negligible, and we can see the ordinary running of the coupling constant of an asymptotically free theory. At scales below a , W^{\pm} freeze out, and we are left with just an effective $U(1)$ gauge theory with vanishing β -function.

The general form of the full non-perturbatively corrected coupling (2.1) has been known for some time [8]. One knows in particular that all what can come from perturbation theory arises up to one loop order only [9], and the amount of R -charge violation (given by 8ℓ) of the ℓ -instanton process; the latter gives rise to the powers 4ℓ in

(2.1). What is a priori not known in (2.1) are the precise values of the instanton coefficients c_{ℓ} , and it is the achievement of Seiberg and Witten to determine all of these coefficients explicitly. These coefficients give infinitely many predictions for zero momentum correlators involving a and gauginos in non-trivial instanton backgrounds. Such correlators are topological and also have an interpretation [10] in terms of Donaldson theory, which deals with topological invariants of four-manifolds. It is the ease of determination of such topological quantities that has been one of the main reasons for excitement on the mathematician's side. The fact that highly non-trivial mathematical results can be reproduced gives striking evidence that S&W's approach for solving the Yang-Mills theory is indeed correct, even though some details, like a rigorous field theoretic definition of the theory, may not yet be completely settled. Furthermore, explicit computations [11] of some of the instanton coefficients by more conventional field theoretical methods have shown complete agreement with the predicted c_{ℓ} .

It is, however, presently not clear what lessons can ultimately be drawn for non-supersymmetric theories, like ordinary QCD. The hope is, of course, that even though supersymmetry is an essential ingredient in the construction, it is only a technical device that facilitates computations,³ and that nevertheless the supersymmetric toy model displays the physically relevant features. See [12] for an analysis in this direction.

Let us list some typical features of supersymmetric field theories:

- *Non-renormalization properties:* perturbative quantum corrections are less violent; this is related to a

- *Holomorphic structure,* which leads to vacuum degeneracies, and allows to use powerful methods of complex analysis.

³This may also apply to the role of space-time supersymmetry in string theory; there is no intrinsic relation between string theory and (low scale) space-time supersymmetry.

•*Duality symmetries* between electric and magnetic, or weak and strong coupling sectors, are more or less manifest, depending on the number of supersymmetries.

The maximum number of supersymmetries is four in a globally supersymmetric theory:

• $N=4$ supersymmetric Yang-Mills theory is conjectured to be self-dual [13], i.e., completely invariant under the exchange of electric and magnetic sectors. However, though interesting, this theory is too simple for the present purpose of investigating non-trivial quantum corrections, since there aren't any in this theory.

• $N=1$ supersymmetric Yang-Mills theory, on the other hand, is presumably not exactly solvable, since the quantum corrections are not under full control; only certain sub-sectors of the theory are governed by holomorphic objects (like the chiral superpotential), and thus are protected from perturbative quantum corrections. Indeed many interesting results on exact effective superpotentials have been obtained recently [14].

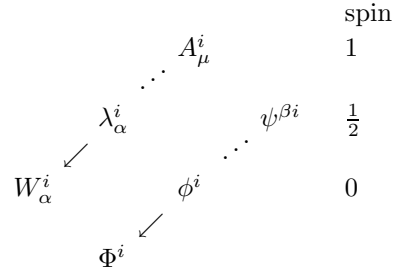
• $N=2$ supersymmetric Yang-Mills theory is at the border between “trivial” and “not fully solvable”, in that it is in the low-energy limit exactly solvable. It is governed by a holomorphic function, the “prepotential” \mathcal{F} , for which the perturbative quantum corrections are under complete control, i.e., occur just to one loop order.

Having motivated why it is particularly fruitful to study $N=2$ Yang-Mills theory, we now turn to discuss it in more detail.

2.2. The Semi-Classical Theory for $G = SU(2)$

The fields of pure $N=2$ Yang-Mills theory are vector supermultiplets in the adjoint representa-

tion of the gauge group. For convenience, one often rewrites such multiplets in terms of $N=1$ chiral multiplets, W_α^i, Φ^i , as follows:



The bottom component, the scalar field $\phi \equiv \phi_i \sigma_i$, has the following potential:

$$V(\phi) = \text{Tr}[\phi, \phi^\dagger]^2 . \quad (2.2)$$

This potential displays a typical feature of supersymmetric theories, namely flat directions along which $V(\phi) \equiv 0$. That is, field configurations

$$\phi = a \sigma_3 \quad (2.3)$$

do not cost any energy. Of course, if $a \neq 0$, there is a spontaneous symmetry breakdown: $SU(2) \hookrightarrow U(1)$. A more suitable “order” parameter is given by the gauge invariant Casimir

$$u(a) = \text{Tr} \phi^2 = 2a^2 . \quad (2.4)$$

It is in particular invariant under the Weyl group of $SU(2)$, which acts as $a \rightarrow -a$ and is, physically, the discrete remnant of the gauge transformations that act within the Cartan subalgebra. The quantity u represents a good coordinate of the manifold \mathcal{M}_c of inequivalent vacua, which one usually calls “moduli space”. Since u can be any complex number, the moduli space is given by the complex plane, which may be compactified to the Riemann sphere by adding a point at infinity.

In the bulk of \mathcal{M}_c one has an unbroken $U(1)$ gauge symmetry, which is enhanced to $SU(2)$ just at the origin. What we are after is a “Wilsonian” effective lagrangian description of the theory, for any given value of u . Such an effective lagrangian can in principle be obtained by integrating out all fluctuations above some scale μ

(that, as we have indicated earlier, is chosen to be equal to a). In particular, we would integrate out the massive non-abelian gauge bosons W^\pm , to obtain an effective action that involves only the neutral gauge multiplet, $W^0 = (A \equiv \Phi^0, W_\lambda^0)$. It is clear that, semi-classically, this theory can possibly be meaningful only outside a neighborhood of $u = 0$, since at $u = 0$ the non-abelian gauge bosons W^\pm become massless, and the effective description in terms of only W^0 cannot be accurate – actually, it would become meaningless. This tells that $u = 0$ will be a singular point on \mathcal{M}_c (besides the point of infinity). In order to have a well-defined theory near $u = 0$, one would need to include the charged W -bosons in the effective theory; one then says that the gauge bosons W^\pm “resolve” the singularity.

It is clear from Fig.1 that, because of asymptotic freedom, the region near $u = \infty$ will correspond to weak coupling, so that only in this “semi-classical” region reliable computations can be done in perturbation theory. On the other hand, the theory will be strongly coupled near the classical $SU(2)$ -enhancement point $u = 0$, so that a priori no reliable quantum statements about the theory can be made here.

It is known (just from supersymmetry) that the low energy effective lagrangian⁴ is completely determined by a holomorphic prepotential \mathcal{F} and must be of the form:

$$\mathcal{L} = \frac{1}{4\pi} \text{Im} \left[\int d^4\theta K(A, \bar{A}) + \int d^2\theta \left(\frac{1}{2} \sum \tau(A) W^\alpha W_\alpha \right) \right]. \quad (2.5)$$

Here, $\Phi \equiv: A \sigma_3$, and

$$K(A, \bar{A}) = \frac{\partial \mathcal{F}(A)}{\partial A} \bar{A} \quad (2.6)$$

⁴By this we mean the piece of the effective lagrangian that is leading for vanishing momenta, i.e., that contains at most two derivatives. There are of course infinitely many higher derivative terms in the full effective action. These are not governed by holomorphic quantities, and thus we do not have much control of them. See [15] for some results in this direction.

is the “Kähler potential” which gives a supersymmetric non-linear σ -model for the field A , and

$$\tau(A) = \frac{\partial^2 \mathcal{F}(A)}{\partial^2 A}. \quad (2.7)$$

That is, the bosonic piece of (2.5) is, schematically,

$$\mathcal{L} = \text{Im}(\tau) \left\{ \partial a \partial \bar{a} + F \cdot F \right\} + \text{Re}(\tau) F \cdot \tilde{F} + \dots, \quad (2.8)$$

from which we see that

$$\tau(a) \equiv \frac{\theta(a)}{\pi} + \frac{8\pi i}{g^2(a)} \quad (2.9)$$

represents the complexified effective gauge coupling, and $\text{Im}(\tau)$ is the σ -model metric on \mathcal{M}_c . Classically, $\mathcal{F}(A) = \frac{1}{2} \tau_0 A^2$, where τ_0 is the bare coupling constant. However, the full quantum prepotential will receive [9] perturbative (one-loop) and non-perturbative corrections, and must be of the form [8]:

$$\mathcal{F}(A) = \frac{1}{2} \tau_0 A^2 + \frac{i}{\pi} A^2 \log \left[\frac{A^2}{\Lambda^2} \right] + \frac{1}{2\pi i} A^2 \sum_{\ell=1}^{\infty} c_\ell \left(\frac{\Lambda}{A} \right)^{4\ell}. \quad (2.10)$$

By taking two derivatives, \mathcal{F} gives rise to the effective coupling (2.1) mentioned in the introduction. Note that indeed for large $a \equiv A|_{\theta=0}$, the instanton sum converges well, and the theory is dominated by semi-classical, one-loop physics.

A crucial insight [1] is that the global properties of the effective gauge coupling $\tau(a)$ are very important. Specifically, we know that near $u = \infty$:

$$\tau = \text{const} + \frac{2i}{\pi} \log \left[\frac{u}{\Lambda^2} \right] + \text{single-valued}. \quad (2.11)$$

This implies that if we loop around $u = \infty$ in the moduli space, the logarithm will produce an extra shift of $2\pi i$ because of its branch cut, and thus:

$$\tau \longrightarrow \tau - 4. \quad (2.12)$$

From (2.9) it is clear that this monodromy just corresponds to an irrelevant shift of the θ -angle, but what we learn is that τ , as well as \mathcal{F} , are not functions but rather multi-valued sections. Actually, the full story is more complicated than that, in that also the imaginary part, $\text{Im}\tau = \frac{8\pi}{g^2}$, will be globally non-trivial.

More specifically, we see from (2.8) that $\text{Im}(\tau)$ represents a metric on the moduli space, and the physical requirement of unitarity implies that it must be positive throughout the moduli space:

$$\text{Im}(\tau(u)) > 0. \quad (2.13)$$

It is now a simple mathematical fact that since $\text{Im}(\tau)$ is a harmonic function (ie., $\partial\bar{\partial}\text{Im}(\tau) = 0$), it cannot have a minimum if it is globally defined. Thus, in order not to conflict with unitarity, we learn that $\text{Im}(\tau)$ can only be locally defined – a priori, it is defined only in the semi-classical coordinate patch near infinity, cf., (2.11). We thus conclude that the global structure of the true "quantum" moduli space, \mathcal{M}_q , must be very different as compared to the classical moduli space, \mathcal{M}_c . In particular, any situation with just two singularities must be excluded.

2.3. The exact quantum moduli space

The question thus arises, how many and what kind of singularities the exact quantum moduli space should have, and what the physical significance of these singularities might be. Seiberg and Witten proposed that there should be two singularities at $u = \pm\Lambda^2$, where Λ is the dynamically generated quantum scale, and that the classical singularity at the origin disappears – see Fig.2.

Though this proposal will prove to be a physically motivated and self-consistent assumption about the strong coupling behavior, it is very difficult, at least for for now, to derive it rigorously. But there is a whole bunch of arguments, with varying degree of rigor, why precisely the situation depicted in Fig.2 must be the correct one. For example, the absence of a singularity at $u = 0$ (which implies that there are, in the full quantum theory, no extra massless gauge fields W^\pm) is mo-

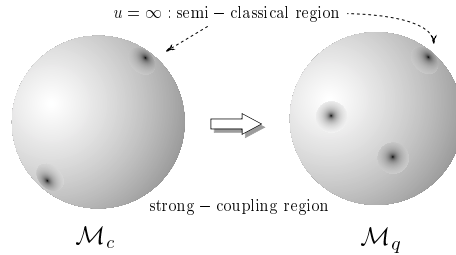


Figure 2. *The transition from the classical to the exact quantum theory involves splitting and shifting of the strong coupling singularity away from $u = 0$ to $u = \pm\Lambda^2$.*

tivated by the absence of an R -current that a superconformal theory with massless gauge bosons would otherwise have [1]. Furthermore, the appearance of just two, and not⁵ $2n$ strong coupling singularities reflects that the corresponding $N=1$ theory (obtained by explicitly breaking the $N=2$ theory by a mass term for Φ) has precisely two vacua (from Witten's index, $\text{Tr}(-1)^F = n$ for $SU(n)$). More mathematically speaking, the singularity structure poses, as will be explained later, a particular non-abelian monodromy problem, and it can be shown that there is no solution for this problem for any other arrangement of singularities (under mild assumptions about the form of these singularities) [16].

The most interesting question is clearly what the physical significance of the extra strong coupling singularities is. One expects in analogy to the classical theory, where the singularity at $u = 0$ is due to the extra massless gauge bosons W^\pm , that the strong coupling singularities in the quantum moduli space should be attributed to certain excitations becoming massless as well. Guided by the early ideas of 't Hooft about confinement [17], Seiberg and Witten postulated that near these singularities certain 't Hooft-Polyakov monopoles must become arbitrarily light.

There is a powerful tool to get a handle on soliton masses in theories with extended supersymmetry, namely the BPS-formula [13]:

$$m^2 \geq |Z|^2, \quad (2.14)$$

⁵The number of singularities must be consistent with global R -symmetry, which acts as $u \rightarrow -u$.

where Z is the central charge of the superalgebra in question. For $N = 2$ supersymmetry, this formula immediately follows from unitarity ($\bar{Q}Q > 0$), in combination with the anti-commutator

$$\{Q_{\alpha i}, \bar{Q}_{\beta j}\} = \delta_{ij} \gamma_{\alpha\beta}^{\mu} P_{\mu} + \delta_{\alpha\beta} \epsilon_{ij} U + (\gamma_5)_{\alpha\beta} \epsilon_{ij} V,$$

where $|Z|^2 \equiv U^2 + V^2$. The important point is that the BPS bound (2.14) is saturated by a certain class of excitations, namely by the ‘‘BPS-states’’ that obey $Q|\psi\rangle = 0$. The idea is that if a state obeys this condition semi-classically, it obeys it also in the exact quantum theory. This is because the number of degrees of freedom of a ‘‘short’’ (or ‘‘chiral’’) multiplet that obeys $Q|\psi\rangle = 0$ is smaller as compared to those of a generic supersymmetry multiplet, and the number of degrees of freedom is supposed not to jump when switching on quantum corrections. In particular, since ’t Hooft-Polyakov monopoles do satisfy the BPS bound semi-classically, they must obey it in the exact theory as well. From semi-classical considerations we can also learn that the monopoles lie in $N = 2$ hypermultiplets, which have maximum spin $\frac{1}{2}$.

For $N = 2$ supersymmetric Yang-Mills theories, the central charge takes the form

$$Z = qa + ga_D, \quad (2.15)$$

where (g, q) are the (magnetic, electric) quantum numbers of the BPS state under consideration. Above, a_D is the ‘‘magnetic dual’’ of the electric Higgs field a and belongs to the $N = 2$ vector multiplet $(A_D, W_{\alpha, D})$ that contains the dual, magnetic photon, A_D^{μ} . By studying the electric-magnetic duality transformation, under which the ordinary electric gauge potential A^{μ} transforms into A_D^{μ} , it turns out [1] that in the $N = 2$ Yang-Mills theory the dual variable a_D is simply given by:

$$a_D = \frac{\partial}{\partial a} \mathcal{F}(a). \quad (2.16)$$

The general idea is that at the singularity at $u = \Lambda^2$, one would have $a \neq 0$ but $a_D = 0$, such

that (by (2.15)) a monopole hypermultiplet with charges $(g, q) = (\pm 1, 0)$ would be massless. On the other hand, one would have that in the exact theory $u = 0$ does not imply $a = 0$, so that in contrast to the classical theory, no gauge bosons (with charges $(0, \pm 2)$) become massless. This in particular would imply that the classical relation $u = 2a^2$ can hold only asymptotically in the weak-coupling region.

The point is to view $a_D(u)$ as a variable that is on an equal footing as $a(u)$; it just belongs to a dual gauge multiplet that couples locally to magnetically charged excitations, in the same way that a couples locally to electric excitations (such as W^{\pm}). A priori, it would not matter which variable we use to describe the theory, and which variable we actually use will rather depend on the region of \mathcal{M}_q that we are looking at. More specifically, in the original semi-classical, ‘‘electric’’ region near $u = \infty$, the preferred local variable is a , and an appropriate lagrangian is given by (2.11). As mentioned above, the instanton sum converges well for large $a \simeq \sqrt{u/2}$.

However, if we try to extend $\mathcal{F}(a)$ to a region far enough away from $u = \infty$, we will leave the domain of convergence of the instanton sum, and we cannot really make any more much sense of \mathcal{F} . That is, in attempting to *globally extend* the effective lagrangian description outside the semi-classical coordinate patch, we face the problem of suitably analytically continuing \mathcal{F} . The point is that even though we cannot have a choice of \mathcal{F} that would be globally valid anywhere on \mathcal{M}_q (it would be in conflict with positivity, cf., (2.13)), we can resum the instanton terms in \mathcal{F} in terms of other variables, to yield another form of the lagrangian that converges well in another region of \mathcal{M}_q .

The reader might already have guessed that while a is the preferred variable near $u = \infty$, it is a_D that is the preferred variable in the ‘‘magnetic’’ strong coupling coordinate patch centered at $u = \Lambda^2$. More precisely, near $u = \Lambda^2$ we expect to have the following, dual form of the effective lagrangian:

$$\mathcal{F}_D(a_D) = \frac{1}{2}\tau_0^D a_D^2 - \frac{i}{4\pi} a_D^2 \log \left[\frac{a_D}{\Lambda} \right] - \frac{1}{2\pi i} \Lambda^2 \sum_{\ell=1}^{\infty} c_\ell^D \left(\frac{ia_D}{\Lambda} \right)^\ell. \quad (2.17)$$

The infinite sum indeed converges well, because at this singularity $a_D \rightarrow 0$.

From the coefficient of the logarithm we see that the theory is non-asymptotically free (positive β -function), and thus weakly coupled for $a_D \rightarrow 0$ (though strongly coupled in terms of the original variable, a). Indeed the dual theory is simply given by an abelian $U(1)$ gauge theory (contributing zero to the β -function), coupled to charged matter that is integrated out (and that would be massless at $a_D = 0$). The magnitude of the coefficient shows that there should be a single matter field with unit charge coupling to the (dual) photon, which belongs to a $N=2$ hypermultiplet. This extra matter hypermultiplet is just the dual representative of the massless magnetic monopole. To the dual magnetic photon related to a_D , the monopole looks like an ordinary, elementary (local) field, in spite of that it couples to the original electric photon in a non-local way. It is this dual, abelian reformulation of the original non-abelian instanton problem what leads to substantial simplifications, especially to the mathematician's profit.

Note that the infinite sum of correction terms in (2.17) reflects the effect of integrating out infinitely many massive BPS states, and though its physical meaning is completely different, has the same information content as the instanton sum in the original lagrangian, (2.11). Note also that the situation at the other singularity, $u = -\Lambda^2$, does not present anything new, in that (by $u \rightarrow -u$ symmetry) it is isomorphic to the the situation at $u = \Lambda^2$ and related to it by simply replacing a_D in $\mathcal{F}_D(a_D)$ by $a_D - 2a$. The whole scheme can therefore be depicted as in Fig.3.

The alert reader might have noticed that so far nothing concrete was achieved yet – instead, we have introduced another set of infinitely many un-

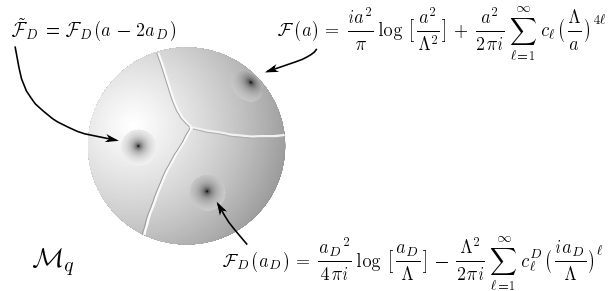


Figure 3. The exact quantum moduli space is covered by three distinct regions, in the center of each of which the theory is weakly coupled when choosing suitable local variables. A local effective lagrangian exists in each coordinate patch, representing a particular perturbative approximation. None of such lagrangians is more fundamental than the other ones, and no local lagrangian exists that would be globally valid throughout the moduli space.

knowns, c_ℓ^D – and also that we have just guessed the coefficient of the logarithm in (2.17). Indeed, this specific coefficient cannot be derived at this point, but rather is part of the assumption that a single monopole with unit charge becomes massless at $u = \Lambda^2$.

The issue is now to determine the values of all the unknown coefficients in \mathcal{F} , \mathcal{F}_D (2.11),(2.17) from the assumptions that govern the *local*, i.e., perturbative behavior of the theory in each of the three coordinate patches in Fig.3. The local behavior is determined by the coefficients of the logarithms, which can reliably be computed in one-loop perturbation theory and directly reflect the charge quantum numbers of the fields that are supposed to be light near a given singularity.

The key idea is that it is the *patching together of the known local data in a globally consistent way* that will completely fix the theory (up to irrelevant ambiguities like θ -shifts). More precisely, the one-loop term determines the local monodromy M around a given singularity, and this acts on the section $\begin{pmatrix} a_D \\ a \end{pmatrix}$ as follows:

$$\begin{pmatrix} a_D(u) \\ a(u) \end{pmatrix} \longrightarrow M \begin{pmatrix} a_D(u) \\ a(u) \end{pmatrix}. \quad (2.18)$$

In particular, from our knowledge of the asymptotic behavior of $a_D(u)$, $a(u)$ at semi-classical in-

finitly,

$$\begin{pmatrix} a_D(u) \\ a(u) \end{pmatrix} \simeq \begin{pmatrix} \frac{i}{\pi} \sqrt{2u} \log(u/\Lambda^2) \\ \sqrt{u/2} \end{pmatrix} \quad (2.19)$$

we infer that for a loop around $u = \infty$:

$$M_\infty = \begin{pmatrix} -1 & 4 \\ 0 & -1 \end{pmatrix}. \quad (2.20)$$

As for the strong coupling singularities at $u = \pm\Lambda^2$, we choose a different strategy: we know on general grounds that the monodromy of a dyon with charges (g, q) that becomes massless at a given singularity is given by:

$$M^{(g,q)} = \begin{pmatrix} 1 + qg & q^2 \\ -g^2 & 1 - gq \end{pmatrix} \quad (2.21)$$

This can be seen in various ways, one of which will be explained further below.

The global consistency condition on how to patch together the local, perturbative data is then simply

$$M_{+\Lambda^2} \cdot M_{-\Lambda^2} = M_\infty, \quad (2.22)$$

since we can smoothly pull the monodromy paths γ around the Riemann sphere (u_0 is an arbitrary base point):

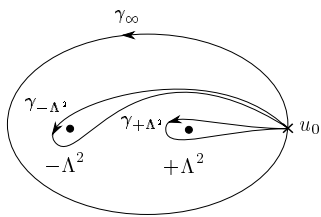


Figure 4. Monodromy paths in the u -plane.

One may view equation (2.22) as a condition on the possible massless spectra at $u = \pm\Lambda^2$. For matrices of the restricted form (2.21), its solution is:

$$M_{+\Lambda^2} = M^{(1,0)}$$

$$M_{-\Lambda^2} = M^{(1,-2)}, \quad (2.23)$$

which is unique up to irrelevant conjugacy. From this we can read off the allowed (magnetic, electric) quantum numbers of the massless monopoles/dyons. They indeed give back the coefficient of the logarithmic term of \mathcal{F}_D that we had anticipated in eq. (2.17).

If we would consider a situation with more than two strong coupling singularities, we would have to solve an equation like (2.22) with the corresponding product of matrices (2.21). However, it can be deduced [16] that such equations for more than two such matrices do not have any solution.

2.4. Solving the monodromy problem

The physics problem has now become a mathematical one, namely simply to find multi-valued functions $a(u), a_D(u)$ that display the required monodromies $M_{\pm\Lambda^2, \infty}$ around the singularities (and that in addition lead to a coupling $\tau \equiv \partial_a a_D$ with $\text{Im}\tau > 0$). This is a classical mathematical problem, the ‘‘Riemann-Hilbert’’ problem, which is known to have a unique⁶ solution.

The RH problem can be accessed from two complementary point of views: either by considering a, a_D as solutions of a differential equation with regular singular points, or from considering a, a_D as certain period integrals related to some auxiliary ‘‘spectral surface’’ X . The latter approach, to be discussed momentarily, allows an easy geometric implementation of the right monodromy properties, while the differential equation approach, to be considered later, is more useful for obtaining explicit expressions for $a(u)$ and $a_D(u)$.

Any two of the monodromy matrices $M_{\pm\Lambda^2, \infty}$ generate the monodromy group Γ_M , which constitutes the subgroup $\Gamma_0(4)$ of the modular group $SL(2, \mathbb{Z})$ and consists of matrices of the form

$$\Gamma_0(4) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}), b = 0 \pmod{4} \right\}.$$

⁶Unique up to multiplication of $\begin{pmatrix} a_D \\ a \end{pmatrix}(u)$ by an entire function; this can however be fixed by imposing the correct, semi-classical asymptotic behavior.

Mathematically speaking, the quantum moduli space can thus be viewed as the upper half-plane modulo the monodromy group:

$$\mathcal{M}_q \cong H^+ / \Gamma_0(4). \quad (2.24)$$

This group represents the quantum symmetries of the theory, and acts (because of (2.18)) on the gauge coupling $\tau = \frac{\partial a_D(u)}{\partial a(u)}$ via $\tau \rightarrow \frac{a\tau+b}{c\tau+d}$. In particular, we see that $S: \tau \rightarrow -\frac{1}{\tau}$ is *not* part of Γ_M , and this means that the theory is not weak-string coupling duality invariant (in contrast to $N=4$ Yang-Mills theory).

Now, motivated by the appearance of a subgroup of the modular group (which is the group of the discontinuous reparametrizations of a torus), the basic idea is that the monodromy problem can be formulated in terms of a toroidal Riemann surface, whose moduli space is precisely \mathcal{M}_q [1]. Such an elliptic curve indeed exists and can be algebraically characterized by:⁷

$$\begin{aligned} X_1: y^2(x, u) &= (x^2 - u)^2 - \Lambda^4 \\ &\equiv : \prod_{i=1}^4 (x - e_i(u, \Lambda)). \end{aligned} \quad (2.25)$$

The point is to interpret the gauge coupling $\tau(a)$ as the period ‘‘matrix’’ of this torus, and this has the added bonus that manifestly $\text{Im}(\tau) > 0$ is guaranteed, by virtue of a mathematical theorem called ‘‘Riemann’s second relation’’. As such is τ defined by a ratio of period integrals:

$$\tau(u) = \frac{\varpi_D(u)}{\varpi(u)}, \quad (2.26)$$

where

$$\varpi_D(u) = \oint_{\beta} \omega, \quad \varpi(u) = \oint_{\alpha} \omega \quad (2.27)$$

with the holomorphic differential $\omega \equiv \frac{1}{\sqrt{2\pi}} \frac{dx}{y(x, u)}$. Here, α, β are canonical basis homology cycles of the torus, like shown as in Fig.5.

⁷There are various physically equivalent forms of this curve.

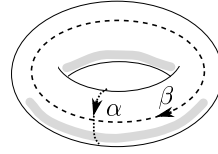


Figure 5. Basis of one-cycles on the torus.

From the relation $\tau = \partial_a a_D$ we thus infer that

$$\varpi_D(u) = \frac{\partial a_D(u)}{\partial u}, \quad \varpi(u) = \frac{\partial a(u)}{\partial u}. \quad (2.28)$$

That is, the yet unknown functions $a_D(u), a(u)$, and consequently the prepotential $\mathcal{F} = \int_a a_D(a)$, are supposed to be obtained by integrations of torus periods. Note that (2.28) implies that we can also write

$$a_D(u) = \oint_{\beta} \lambda_{SW}, \quad a(u) = \oint_{\alpha} \lambda_{SW}, \quad (2.29)$$

where

$$\lambda_{SW} = \frac{1}{\sqrt{2\pi}} x^2 \frac{dx}{y(x, u)} \quad (2.30)$$

(up to exact pieces) is a particular meromorphic one-form (for $z \equiv \frac{1}{x} \rightarrow 0$ it has a second order pole: $\lambda_{SW} \sim \frac{dz}{z^2}$).

What needs to be shown is that the periods, derived from the specific choice of elliptic curve given in (2.25), indeed enjoy the correct monodromy properties. The periods (2.27) and (2.29) are actually largely fixed by their monodromy properties around the singularities of \mathcal{M}_q , and obviously just reflect the monodromy properties of the basis homology cycles, α and β . It therefore suffices to study how the basis cycles α, β of the torus transform when we loop around a given singularity.

For this, we represent the above torus in a convenient way that is well-known in the mathematical literature: we will represent it in terms of a two-sheeted cover of the branched x -plane. More precisely, denoting the four zeroes of $y^2(x, u) = 0$ by

$$e_1 = -\sqrt{u + \Lambda^2}, \quad e_2 = -\sqrt{u - \Lambda^2}$$

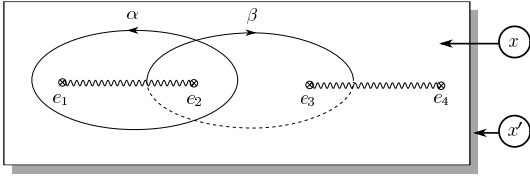


Figure 6. Representation of the auxiliary elliptic curve X_1 in terms of a two-sheeted covering of the branched x -plane. The two sheets are meant to be glued together along the cuts that run between the branch points $e_i(u)$. Shown is our choice of homology basis, given by the cycles α, β . This picture corresponds to the choice of the basepoint $u_0 > \Lambda^2$ real.

$$e_3 = \sqrt{u - \Lambda^2}, \quad e_4 = \sqrt{u + \Lambda^2}, \quad (2.31)$$

we can specify the torus in the way depicted in Fig.6.

The singularities in the quantum moduli space arise when the torus degenerates, and this obviously happens when any two of the zeros e_i coincide. This can be expressed as the vanishing of the “discriminant”

$$\Delta_\Lambda = \prod_{i < j}^4 (e_i - e_j)^2 = (2\Lambda)^8 (u^2 - \Lambda^4). \quad (2.32)$$

The zeroes of Δ_Λ describe the following degenerations of the elliptic curve:

i_+) $u \rightarrow +\Lambda^2$, for which $(e_2 \rightarrow e_3)$, i.e., the cycle $\nu_{+\Lambda^2} \equiv \beta$ degenerates,

i_-) $u \rightarrow -\Lambda^2$, for which $(e_1 \rightarrow e_4)$, i.e., the cycle $\nu_{-\Lambda^2} \equiv \beta - 2\alpha$ degenerates,

ii) $\Lambda^2/u \rightarrow 0$, for which $(e_1 \rightarrow e_2)$ and $(e_3 \rightarrow e_4)$.

It is now easy to see that a loop $\gamma_{+\Lambda^2}$ around the singularity at $u = \Lambda^2$ makes e_2 and e_3 rotate around each other, so that the cycle α gets transformed into $\alpha - \beta$, as can be seen from Fig.7. This means that on the basis vector $\begin{pmatrix} \beta \\ \alpha \end{pmatrix}$, the monodromy action looks

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \equiv M^{(1,0)} = M_{+\Lambda^2}. \quad (2.33)$$

Similarly, from Fig.7 one can see that the mon-

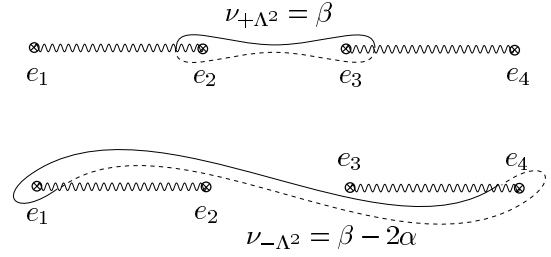


Figure 7. Vanishing cycles on the torus that shrink to zero as one moves towards a degeneration point.

odromy around $u = -\Lambda^2$ is given by

$$\begin{pmatrix} -1 & 4 \\ -1 & 3 \end{pmatrix} \equiv M^{(1,-2)} = M_{-\Lambda^2}. \quad (2.34)$$

To obtain the monodromy around $\Lambda^2/u \rightarrow 0$, one can compactify the u -plane to \mathbb{P}^1 , as we did before, and get the monodromy at infinity from the global relation $M_\infty = M_{+\Lambda^2} M_{-\Lambda^2}$ (cf., Fig.4.).

We thus have reproduced the monodromy matrices associated with the exact quantum moduli space directly from the elliptic curve (2.25), and what this means is that the integrated torus periods $a_D(u), a(u)$ defined by (2.28) must indeed have the requisite monodromy properties. However, before we are going to explicitly determine these functions in the next section, let us say some more words on the general logic of what we have just been doing.

We have seen in Fig.7 that when we loop around a singularity in \mathcal{M}_q , the branch points $e_i(u)$ exchange along certain paths, ν , which shrink to zero as $e_i \rightarrow e_j$. Such paths are called “vanishing cycles” and play, as we will see, an important role for the properties of BPS states. Indeed, in a quite general context, many features of a BPS spectrum can be encoded in the singular homology of an appropriate auxiliary surface X .

Concretely, assume that a path vanishes at a singularity that has the following expansion in terms of given basis cycles:

$$\nu = g\beta + q\alpha. \quad (2.35)$$

Then obviously, assuming that λ does not blow

up, we have

$$0 = \oint_{\nu} \lambda = g \oint_{\beta} \lambda + q \oint_{\alpha} \lambda = g a_D + q a \equiv Z,$$

so that we have at the singularity a massless BPS state with (magnetic, electric) charges equal to (g, q) . That is, we can simply read off the quantum numbers of massless states from the coordinates of the vanishing cycle! Obviously, under a change of homology basis, the charges change as well, but this is nothing but a duality rotation. What remains invariant is the intersection number

$$\nu_i \circ \nu_j = \nu^t \cdot \Omega \cdot \nu = g_i q_j - g_j q_i \in \mathbb{Z}, \quad (2.36)$$

where \circ is the intersection product of one-cycles and Ω is the symplectic (skew-symmetric) intersection metric for the basis cycles. Note that this represents the well-known Dirac-Zwanziger quantization condition for the possible electric and magnetic charges, and we see that it is satisfied by construction. The vanishing of the r.h.s. of (2.36) is required for two states to be local with respect to each other [17,18]. This means that only states that are related to non-intersecting vanishing cycles are mutually local. In our case, the monopole with charges $(1, 0)$, the dyon with charges $(1, -2)$ and the (massive) gauge boson W^+ with charges $(0, 2)$ are all mutually non-local, and thus cannot be simultaneously represented in a local lagrangian.

Furthermore, there is a closed formula for the monodromy around a given singularity associated with a vanishing cycle: the monodromy action on any given cycle, $\gamma \in H_1(X, \mathbb{Z})$, is directly determined in terms of this vanishing cycle ν by means of the ‘‘Picard-Lefschetz’’ formula [5]:

$$M_{\nu}: \quad \gamma \longrightarrow \gamma - (\gamma \circ \nu) \nu. \quad (2.37)$$

This implies that for a vanishing cycle of the form (2.35), the monodromy matrix is precisely the one given in (2.21), as promised.

2.5. The BPS Spectrum

We noted above that the global consistency relation (2.22) is solved by monodromy matrices that correspond to a monopole with charges $(g, q) = \pm(1, 0)$ and to a dyon with charges $\pm(1, -2)$. These excitations are massless at $u = \Lambda^2$ and $u = -\Lambda^2$, respectively. We now like to ask about other BPS states that may exist, though these must be massive throughout the moduli space.

For this, remember that the charge labels (g, q) are highly ambiguous, because they are defined only up to symplectic transformations; this reflects the choice of homology basis. The charges can thus be changed by conjugation by any monodromy transformation belonging to $\Gamma_0(4)$. In particular, looping around $u = \infty$ acts as

$$M_{\infty} \cdot M^{(g, q)} \cdot M_{\infty}^{-1} = M^{(-g, -q-4g)}, \quad (2.38)$$

and thus will shift the electric charge, $q \rightarrow -q - 4g$. This corresponds to $\tau \rightarrow \tau - 4$ and to $\theta \rightarrow \theta - 4\pi$, and hence is a manifestation of the fact [19] that the electric charge of a dyon changes if the θ -angle is changed – there is no absolute definition of the electric charge of a dyon.

It also means that the weak coupling spectrum of the theory must be invariant under shifts $\theta \rightarrow \theta - 4\pi n$, $n \in \mathbb{Z}$. That is, under ‘‘spectral flow’’ induced by smoothly changing θ by 4π , the BPS spectrum must map back to itself, though individual states need not map back to themselves. More precisely, since the above monodromy conjugation can be induced by arbitrarily small loops around $u = \infty$, we know that the BPS spectrum should consist in the weak coupling patch at least of dyons with charges $\pm(1, 2\ell)$, $\ell \in \mathbb{Z}$, besides the massive gauge bosons $W^{\pm} \sim (0, \pm 2)$.

A very important point made in [1] is that the stable BPS spectrum in the strong coupling region is, in fact, different and consists only of a subset of the above semi-classical BPS spectrum. This is because the moduli space \mathcal{M}_q decomposes into two regions, $\mathcal{M}_q^{\text{weak}}$ and $\mathcal{M}_q^{\text{strong}}$, with different physics. They are separated by a line \mathcal{C} , on which most of the semi-classical BPS states decay.

This line is defined by

$$\mathcal{C} = \left\{ u : \frac{a_D(u)}{a(u)} \in \mathbb{R} \right\}, \quad (2.39)$$

and turns out to be almost an ellipse passing through the singular points at $u = \pm\Lambda^2$; see Fig.8. Indeed all possible singularities associated with massless BPS states must lie on \mathcal{C} , since if $Z = ga_D + qa = 0$ for $g, q \in \mathbb{Z}$, then $a_D/a \in \mathbb{R}$.

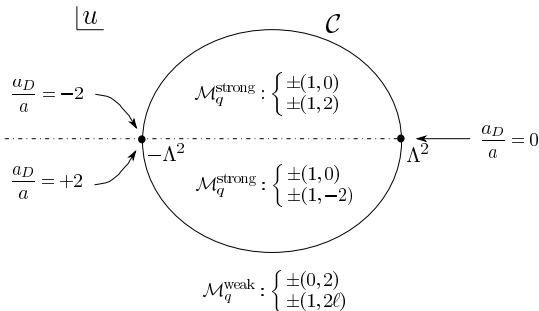


Figure 8. The line \mathcal{C} of marginal stability separates the strong coupling BPS spectrum from the semi-classical BPS spectrum. Both spectra are indicated here by the charges of the stable states. The dashed line represents the logarithmic branch cut.

One can check [20] that if one traces \mathcal{C} clockwise starting from $u = -\Lambda^2$, $(a_D/a)(u)$ varies monotonically from -2 to $+2$, with $(a_D/a)(\Lambda^2) = 0$. That we do not map back to $(a_D/a) = -2$ is due to the branch cut of the logarithm in $a(u)$. Thus there is really an ambiguity in the electric charge of the dyon: if we approach $u = -\Lambda^2$ from the upper-half u -plane, the dyon has charges $\pm(1, 2)$, which is M_∞ -conjugate to $\pm(1, -2)$ that we had before.

The physical significance of the marginal line of stability \mathcal{C} is that when $(a_D/a)(u) \in \mathbb{R}$, the lattice (or “Jacobian”) of the central charges $Z = ga_D + qa$ degenerates to a line. Then mass and charge conservation do not any more prohibit BPS states to decay into monopoles and dyons, because the triangle inequality $|Z_{g_1+g_2, q_1+q_2}| \leq |Z_{g_1, q_1}| + |Z_{g_2, q_2}|$ becomes saturated. For example, if $a_D = \xi a$, $\xi \in [0, 2]$, then the gauge field with $(g, q) = (0, 2)$ and $m_{(0,2)} = 2|a|$ is unstable

against decay into a monopole-dyon pair, with $m_{(-1,2)} = (2 - \xi)|a|$ and $m_{(1,0)} = \xi|a|$.

These purely kinematical considerations do not, a priori, prove that such decays actually take place, but we will see later in section 5, from an entirely different perspective, that the quantum BPS states indeed do decay (or rather degenerate) precisely in this manner.

With a more detailed analysis [20], employing the global symmetry $u \rightarrow -u$, one can show that the only stable BPS states in $\mathcal{M}_q^{\text{strong}}$ are indeed precisely the monopole and the dyon, and no other states. Furthermore, one can show that the semi-classical, stable BPS spectrum in $\mathcal{M}_q^{\text{weak}}$ consists precisely of the above-mentioned states $\pm(0, 2)$ and $\pm(1, 2\ell)$, $\ell \in \mathbb{Z}$, and of no other states.

2.6. Picard-Fuchs equations

In order to obtain the effective action explicitly, one needs to evaluate the period integrals (2.27). However, instead of directly computing the integrals, one may use the fact that the periods form a system of solutions of the Picard-Fuchs equation associated with the curve (2.25). One then has to evaluate the integrals only in leading order, just to determine the correct linear combinations of the solutions.

Concretely, in order to derive the PF equations (see also refs. [21]), let us first write the defining relation of the curve (2.25) in homogenous form, by introducing another coordinate z :

$$W(x, y, z, u) \equiv (x^2 - uz^2)^2 - z^4 - y^2 = 0 \quad (2.40)$$

(here we have set $\Lambda = 1$). We also introduce the following integrals over certain globally defined one-forms:

$$\Omega_1 = \oint_\gamma \frac{1}{W} \bar{\omega}, \quad \Omega_2 = \oint_\gamma \frac{x^2 z^2}{W^2} \bar{\omega}, \quad (2.41)$$

where γ is a one-cycle that winds around the surface $W = 0$, and $\bar{\omega}$ is an appropriate volume form on \mathbb{P}^3 . The point is that we do not need to evaluate these periods by explicitly performing the

integrations. Rather, the integrands should be considered here as dummy variables, introduced only to conveniently derive the PF equations that will then be solved by other means. By elementary algebra one easily finds:

$$\frac{\partial}{\partial u} \Omega_1 = \oint_{\gamma} \frac{2z^2(x^2 - uz^2)}{W^2} \omega \quad (2.42)$$

$$= \frac{2}{(u^2 - 1)} \Omega_2 - \oint_{\gamma} \frac{uz}{2(u^2 - 1)} \frac{\partial_z W}{W^2} \omega, \quad (2.43)$$

where we have used in the second line the following expansion into ‘‘ring elements and vanishing relations’’:

$$2z^2(x^2 - uz^2) \equiv -\frac{2}{(u^2 - 1)} x^2 z^2 - \frac{u}{2(u^2 - 1)} z \partial_z W.$$

Integrating by parts, we can cancel W in the second term to get

$$\frac{\partial}{\partial u} \Omega_1 = -\frac{2}{(u^2 - 1)} \Omega_2 - \frac{u}{2(u^2 - 1)} \Omega_1.$$

We can repeat a similar game for Ω_2 , and obtain, after multiple partial integrations, the following differential identity:

$$\begin{aligned} \frac{\partial}{\partial u} \Omega_2 &= \oint_{\gamma} x z^4 \frac{\partial_x W}{W^3} \omega \\ &= \frac{1}{8(u^2 - 1)} \Omega_1 + \frac{u}{2(u^2 - 1)} \Omega_2. \end{aligned} \quad (2.44)$$

We now can eliminate Ω_2 from (2.43) and (2.44) to obtain a differential equation for the fundamental period: $\mathcal{L}\Omega_1 = 0$, with $\mathcal{L} = (\Lambda^4 - u^2)\partial_u^2 - 2u\partial_u - \frac{1}{4}$. This Picard-Fuchs equation is supposed to be satisfied by all the periods, in particular by $(\varpi_D(u), \varpi(u)) \equiv (\partial_u a_D, \partial_u a)$. In terms of the variable $\alpha = \frac{u^2}{\Lambda^4}$, the PF differential operator turns into $(\theta_\alpha = \alpha\partial_\alpha)$

$$\mathcal{L} = \theta_\alpha(\theta_\alpha - \frac{1}{2}) - \alpha(\theta_\alpha + \frac{1}{4})^2, \quad (2.45)$$

which constitutes a hypergeometric system of type $(a, b, c) = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$.

It is also possible to derive a second order differential equation for the section (a_D, a) directly

[22]. In fact, one easily verifies that $\mathcal{L}\partial_u = \partial_u \tilde{\mathcal{L}}$ with

$$\tilde{\mathcal{L}} = \theta_\alpha(\theta_\alpha - \frac{1}{2}) - \alpha(\theta_\alpha - \frac{1}{4})^2, \quad (2.46)$$

and this forms a hypergeometric system of type $(-\frac{1}{4}, -\frac{1}{4}; \frac{1}{2})$. One may also check directly that $\tilde{\mathcal{L}} \cdot \oint \lambda = 0$.

The solutions of $\tilde{\mathcal{L}}(a_D(u), a(u)) = 0$ in terms of hypergeometric functions, and their analytic continuation over the complex plane, are of course well known. For $|u| > |\Lambda|$ a system of solutions to the Picard-Fuchs equations is given by w_0 and w_1 with

$$w_0(u) = \frac{\sqrt{u}}{\Lambda} \sum c(n) \left(\frac{\Lambda^4}{u^2}\right)^n, \quad c(n) = \frac{(\frac{1}{4})_n (-\frac{1}{4})_n}{(1)_n^2}$$

$$\text{and } w_1(u) = w_0(u) \log\left(\frac{\Lambda^4}{u^2}\right) + \frac{\sqrt{u}}{\Lambda} \sum d(n) \left(\frac{\Lambda^4}{u^2}\right)^n,$$

$$\begin{aligned} \text{where } d(n) &\equiv c(n) [2(\psi(1) - \psi(n+1)) \\ &+ \psi(n + \frac{1}{4}) - \psi(\frac{1}{4}) + \psi(n - \frac{1}{4}) - \psi(-\frac{1}{4})] \end{aligned}$$

and where $(a)_m \equiv \Gamma(a+m)/\Gamma(a)$ is the Pochhammer symbol and ψ the digamma function. Matching the asymptotic expansions of the period integrals one finds

$$\begin{aligned} a(u) &= \frac{\Lambda}{\sqrt{2}} w_0(u) \quad (2.47) \\ a_D(u) &= -\frac{i\Lambda}{\sqrt{2}\pi} \left[w_1(u) + (4 - 6 \log(2)) w_0(u) \right], \end{aligned}$$

which transform under counter-clockwise continuation of u along γ_∞ (c.f., Fig.4) precisely as in (2.20). These expansions correspond to particular linear combinations of hypergeometric functions, the most concise form of which are

$$\begin{aligned} a_D(\alpha) &= \frac{i}{4} \Lambda (\alpha - 1) {}_2F_1\left(\frac{3}{4}, \frac{3}{4}; 2; 1 - \alpha\right) \\ a(\alpha) &= \frac{1}{\sqrt{2}} \Lambda \alpha^{1/4} {}_2F_1\left(-\frac{1}{4}, \frac{1}{4}; 1; \frac{1}{\alpha}\right). \end{aligned}$$

From these expressions the prepotential in the semi-classical regime near infinity in the moduli

space can readily be computed to any given order. Inverting $a(u)$ as series for large a/Λ yields for the first few terms $\frac{u(a)}{\Lambda^2} = 2\left(\frac{a}{\Lambda}\right)^2 + \frac{1}{16}\left(\frac{a}{\Lambda}\right)^2 + \frac{5}{4096}\left(\frac{a}{\Lambda}\right)^6 + O\left(\left(\frac{a}{\Lambda}\right)^{10}\right)$. After inserting this into $a_D(u)$, one obtains \mathcal{F} by integration as follows:

$$\mathcal{F}(a) = \frac{ia^2}{2\pi} \left(2 \log \frac{a^2}{\Lambda^2} - 6 + 8 \log 2 - \sum_{\ell=1}^{\infty} c_{\ell} \left(\frac{\Lambda}{a} \right)^{4\ell} \right).$$

It has indeed the form advertised in (2.11). Specifically, the first few terms of the instanton expansion are:

ℓ	1	2	3	4	5	6
c_{ℓ}	$\frac{1}{2^5}$	$\frac{5}{2^{14}}$	$\frac{3}{2^{18}}$	$\frac{1469}{2^{31}}$	$\frac{4471}{2^{34} \cdot 5}$	$\frac{40397}{2^{43}}$

One can treat the dual magnetic semi-classical regime in an analogous way. Near the point $u = \Lambda^2$ where the monopole becomes massless, we introduce $z = (u - \Lambda^2)/(2\Lambda^2)$ and rewrite the Picard-Fuchs operator as

$$\mathcal{L} = z(\theta_z - \frac{1}{2})^2 + \theta_z(\theta_z - 1). \quad (2.48)$$

At $z = 0$, the indices are 0 and 1, and we have again one power series

$$w_0(z) = \Lambda^2 \sum c(n) z^{n+1}, \quad c(n) = (-1)^n \frac{(\frac{1}{2})_n^2}{(1)_n(2)_n}$$

and a logarithmic solution

$$w_1(z) = w_0(z) \log(z) + \sum d(n) z^{n+1} - 4,$$

with

$$d(n) \equiv c(n) \left[2\left(\psi\left(n + \frac{1}{2}\right) - \psi\left(\frac{1}{2}\right)\right) + \psi\left(n + \frac{1}{4}\right) - \psi\left(\frac{1}{4}\right) + \psi(1) - \psi(n+1) + \psi(2) - \psi(n+2) \right].$$

For small z one can easily evaluate the lowest order expansion of the period integrals and thereby determine the analytic continuation of the solutions from the weak coupling to the strong coupling domain:

$$a_D = 2 \int_{e_2}^{e_3} \lambda = i\Lambda w_0(z)$$

$$a = 2 \int_{e_1}^{e_2} \lambda = -\frac{\Lambda}{2\pi} (w_1(z) - (1 + \log(2))w_0(z)).$$

This exhibits the monodromy of (2.33) along the path $\gamma_{+\Lambda^2}$. Inverting $a_D(z)$ yields $z(a_D) = -2\tilde{a}_D + \frac{1}{4}\tilde{a}_D^2 + \frac{1}{32}\tilde{a}_D^3 + \mathcal{O}(\tilde{a}_D^4)$, with $\tilde{a}_D \equiv ia_D/\Lambda$. After inserting this into $a(z)$ we integrate w.r.t. a_D and obtain the dual prepotential \mathcal{F}_D as follows:

$$\mathcal{F}_D(a_D) = \frac{i\Lambda^2}{2\pi} \left(\tilde{a}_D^2 \log \left[-\frac{i}{4} \sqrt{\tilde{a}_D} \right] + \sum_{\ell=1}^{\infty} c_{\ell}^D \tilde{a}_D^{\ell} \right),$$

where the lowest threshold correction coefficients c_{ℓ}^D are

ℓ	1	2	3	4	5	6
c_{ℓ}^D	4	$-\frac{3}{4}$	$\frac{1}{2^4}$	$\frac{5}{2^9}$	$\frac{11}{2^{12}}$	$\frac{63}{2^{16}}$

They reflect the effect of integrating out the massive BPS spectrum near $u = \Lambda^2$.

3. Generalization to other Gauge Groups

The above construction for $SU(2)$ Yang-Mills theory can be generalized in many ways; for example, one may add extra matter fields [23,24], and/or consider other gauge groups [25–27]. For lack of space, we will confine ourselves in the lectures to the extension of pure Yang-Mills theory to simply laced gauge groups of type ADE (though interesting phenomena can arise when matter is added [23]).

3.1. Simple Singularities

We will first outline the group theoretical aspects for $G = SU(n)$, and present the discussion in a particular way that follows [22]: namely by starting with the classical theory. Indeed, interesting features appear in a simplified fashion already at the classical level, and some of them will play an important role in the generalization to string theory.

Just like as for $G = SU(2)$, the scalar superfield component ϕ labels a continuous family of inequivalent ground states that constitutes the classical moduli space, \mathcal{M}_c . One can always rotate ϕ into the Cartan sub-algebra, $\phi = \sum_{k=1}^{n-1} a_k H_k$, with $H_k = E_{k,k} - E_{k+1,k+1}$, $(E_{k,l})_{i,j} = \delta_{ik} \delta_{jl}$. For

generic eigenvalues of ϕ , the $SU(n)$ gauge symmetry is broken to the maximal torus $U(1)^{n-1}$. However, if some eigenvalues coincide, then some larger, non-abelian group $H \subseteq G$ remains unbroken. Precisely which gauge bosons are massless for a given background $a = \{a_k\}$, can easily be read off from the central charge formula. For an arbitrary charge vector q , this formula reads:

$$Z = e_q(a) = q \cdot a, \quad \text{with} \quad m^2 = |Z|^2, \quad (3.1)$$

and in the present context we take for the charge vectors q of the gauge bosons the roots $\alpha \in \Lambda_R(G)$ in Dynkin basis.

The Cartan sub-algebra variables a_k are not gauge invariant and in particular not invariant under discrete Weyl transformations. Therefore, one introduces other variables for parametrizing the classical moduli space, which are given by the Weyl invariant Casimirs $u_k(a)$, $k = 2, \dots, n$. These variables parametrize the (complexified) Cartan sub-algebra modulo the Weyl group, ie, $\{u_k\} \cong \mathbb{C}^{n-1}/S(n)$, and can formally be generated as follows:

$$\begin{aligned} P_{A_{n-1}}^n &\equiv \det_{n \times n} [x\mathbf{1} - \phi] = \prod_{i=1}^n (x - e_{\lambda_i}(a)) \\ &= x^n - \sum_{l=0}^{n-2} u_{l+2}(a) x^{n-2-l} \\ &\equiv W_{A_{n-1}}(x, u). \end{aligned} \quad (3.2)$$

Here, λ_i are the weights of the n -dimensional fundamental representation, and $W_{A_{n-1}}(x, u)$ is nothing but the ‘‘simple singularity’’⁸ [5,28] associated with $SU(n)$, with

$$u_k(a) = (-1)^{k+1} \sum_{j_1 \neq \dots \neq j_k} e_{\lambda_{j_1}} e_{\lambda_{j_2}} \dots e_{\lambda_{j_k}}(a).$$

These symmetric polynomials are manifestly invariant under the Weyl group $S(n)$, which acts by permutation of the weights λ_i .

⁸More precisely, $W(x,0) = 0$ has a singularity of type A_{n-1} at the origin, which is resolved by switching on the u_k .

From the above we know that whenever $e_{\lambda_i}(a) = e_{\lambda_j}(a)$ for some i and j , there are, classically, extra massless non-abelian gauge bosons, since the central charge vanishes: $e_\alpha = 0$ for some root α . For such backgrounds the effective action becomes singular. The classical moduli space is thus given by the space of Weyl invariant deformations, except for such singular regions: $\mathcal{M}_0 = \{u_k\} \setminus \Sigma_0$. Here, $\Sigma_0 \equiv \{u_k : \Delta_0(u_k) = 0\}$ is the zero locus of the discriminant

$$\Delta_0(u) = \prod_{i < j}^n (e_{\lambda_i}(u) - e_{\lambda_j}(u))^2 = \prod_{\substack{\text{positive} \\ \text{roots } \alpha}} (e_\alpha(u))^2 \quad (3.3)$$

of the simple singularity (3.2). We schematically depicted (the real slices of) the singular loci Σ_0 for $n = 2, 3, 4$ in Fig.9.

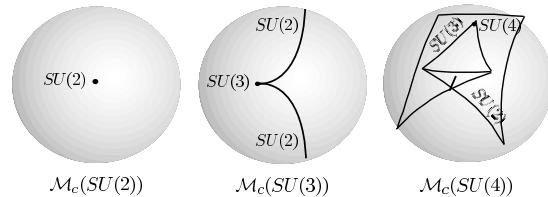


Figure 9. *Singular loci Σ_0 in the classical moduli spaces \mathcal{M}_c of pure $SU(n)$ $N = 2$ Yang-Mills theory. They are nothing but the bifurcation sets of the type A_{n-1} simple singularities, and reflect all possible symmetry breaking patterns in a gauge invariant way (for $SU(3)$ and $SU(4)$ we show only the real parts). The picture for $SU(4)$ is known in singularity theory as the ‘‘swallowtail’’.*

The discriminant loci Σ_0 are generally given by intersecting hypersurfaces of complex codimension one. On each such surface one has $e_{\alpha_i} = 0$ for some pair of roots $\pm\alpha_i$, so that there is an unbroken $SU(2)$. In total, there are $\frac{1}{2}n(n-1)$ of such branches $\Sigma_0^{\alpha_i}$. On the intersections of these branches one has, correspondingly, larger unbroken gauge groups. All surfaces intersect together in just one point, namely in the origin, where the gauge group $SU(n)$ is fully restored. Thus, what we learn is that all possible classical symmetry breaking patterns are encoded in the discriminant loci of the simple singularities, $W_{A_{n-1}}(x, u)$.

In previous sections we have seen that $SU(2)$ quantum Yang-Mills theory is characterized by an

auxiliary elliptic curve. In a more general context, one may view it as a “spectral”, or “level” manifold. The relationship between BPS states and cycles on an auxiliary manifold X seems in fact to be quite generic. As we will see, there is a whole variety of such manifolds, describing various different physical systems. In these notes, we will denote generic spectral manifolds of complex dimension d by X_d .

Indeed one may introduce here this concept to describe *classical* YM theory as well, and characterize BPS states (the non-abelian gauge bosons) by an auxiliary manifold $X = X_0$. This level manifold is zero dimensional and simply given by the following set of points:

$$X_0 = \{x : W_{A_{n-1}}(x, u) = 0\} = \{e_{\lambda_i}(u)\}. \quad (3.4)$$

It is singular if any two of the $e_{\lambda_i}(u)$ coincide, and the vanishing cycles are simply given by the formal differences: $\nu_\alpha = e_{\lambda_i} - e_{\lambda_j} = e_\alpha$, i.e., by the central charges (3.1) associated with the non-abelian gauge bosons. Obviously, massless gauge bosons are associated with vanishing 0-cycles of the spectral set X_0 . It is indeed well-known [5] that such 0-cycles ν_α generate the root lattice:

$$H_0(X_0, \mathbb{Z}) \cong \Gamma_R^{SU(n)}. \quad (3.5)$$

We depicted the level surface for $G = SU(3)$ in Fig.10. Such kind of pictures has a concrete group theoretical meaning – given the locations of the dots $\{x_i = \lambda_i \cdot a\}$, they just represent projections of weight diagrams.

We thus see a close connection between the vanishing homology of X_0 and $SU(n)$ weight space. Indeed, the intersection numbers of the vanishing cycles are just given by the inner products between root vectors, $\nu_{\alpha_i} \circ \nu_{\alpha_j} = \langle \alpha_i, \alpha_j \rangle$ (self-intersections counting +2), and the Picard-Lefschetz formula (2.37) coincides in this case with the well-known formula for Weyl reflections, with matrix representation: $M_{\alpha_i} = \mathbf{1} - \alpha_i \otimes w_i$ (where w_i are the fundamental weights).

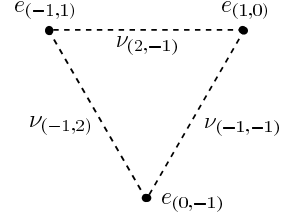


Figure 10. Level manifold X_0 for classical $SU(3)$ Yang-Mills theory, given by points in the x -plane that form a weight diagram. The dashed lines are the vanishing cycles associated with non-abelian gauge bosons (having corresponding quantum numbers, here in Dynkin basis). The masses are proportional to the lengths of the lines and thus vanish if the cycles collapse.

3.2. Classical Theory: Other Gauge Groups

The above considerations apply [5,26,27] more or less directly to the other simply laced Lie groups of types D and E .⁹ However, there are marked differences in that the corresponding simple singularities

$$\begin{aligned} W_{D_n} &= x_1^{n-1} + \frac{1}{2}x_1x_2^2 - \sum_{k=1}^{n-1} x_1^{n-1-k}u_{2k} - \tilde{u}_{n-1}x_2 \\ W_{E_6} &= x_1^3 + x_2^4 - u_2x_1x_2^2 - u_5x_1x_2 - u_6x_2^2 \\ &\quad - u_8x_1 - u_9x_2 - u_{12} \\ W_{E_7} &= x_1^3 + x_1x_2^3 - u_2x_1^2x_2 - u_6x_1^2 - u_8x_1x_2 \\ &\quad - u_{10}x_2^2 - u_{12}x_1 - u_{14}x_2 - u_{18} \\ W_{E_8} &= x_1^3 + x_2^5 - u_2x_1x_2^3 - u_8x_1x_2^2 - u_{12}x_2^3 \\ &\quad - u_{14}x_1x_2 - u_{18}x_2^2 - u_{24}x_2 - u_{30} \end{aligned} \quad (3.6)$$

involve an extra variable, x_2 (u_k denote the $r \equiv \text{rank}(G)$ independent Casimirs, which have the degrees k as indicated). And in contrast to A_{n-1} (cf., (3.2)), these have a priori no simple relationship to the characteristic polynomials

$$\begin{aligned} P_{ADE}^{\mathcal{R}} &\equiv \det[x\mathbf{1} - \phi]_{\mathcal{R}} = \prod_{\lambda_i \in \mathcal{R}} (x - \lambda_i \cdot a) \\ &= x^{\dim \mathcal{R}} + [\text{lower order terms in } x, u_k], \end{aligned} \quad (3.7)$$

where \mathcal{R} is some, say fundamental, representation of the gauge group G . For example, for $G = E_6$,⁹And essentially to non-simply laced groups as well, for which “boundary singularities” are relevant. See [5] for details.

$W_{E_6}(x_1, x_2)$ is of degree 12, while $P_{E_6}^{27}(x)$ is of order 27 – so these polynomials are really quite different from each other. The point is that the equations $W_{ADE} = 0$ and $P_{ADE}^{\mathcal{R}} = 0$ have the same relevant information content (in fact, for arbitrary representations \mathcal{R}); the simple singularities (3.6) are in a sense more efficient in encoding this information, in that the overall scaling degree is minimized (given by the dual Coxeter number h), at the expense of introducing another variable, x_2 .

In effect, both equations $W = 0$ and $P = 0$ can be taken to define an auxiliary spectral surface X . However, for D, E gauge groups the surfaces $W(x_1, x_2) = 0$ happen to be no longer zero dimensional.¹⁰ For D_n one can “integrate out” the variable x_2 and thereby relate $W = 0$ to $P = 0$. That is, we can simply eliminate x_2 via the “equation of motion” $\partial_{x_2} W_{D_n}(x_1, x_2) = 0$. Multiplying W_{D_n} by x_1 and substituting $x_1 = x^2$, we then indeed get

$$x^{2n} - \sum_{k=1}^{n-1} x^{2n-2k} u_{2k} - \frac{1}{2} \tilde{u}_{n-1}^2 \equiv P_{D_n}^{2n}(x, u) = 0.$$

The relation between $W = 0$ and $P = 0$ is however much more complicated for the exceptional groups; see [29] for E_6 .

3.3. Quantum $SU(n)$ Gauge Theory

We now turn to the quantum version of the $N = 2$ Yang-Mills theories, where the issue is to construct curves X_1 whose moduli spaces give the supposed quantum moduli spaces, \mathcal{M}_q . We have seen that the classical theories are characterized by simple singularities, so we may expect that the quantum versions should also have something to do with them. Indeed, for $G = SU(n)$ the appropriate manifolds were found in [25] and can be represented by

$$X_1 : y^2 = (W_{A_{n-1}}(x, u_i))^2 - \Lambda^{2n}, \quad (3.8)$$

which corresponds to special genus $g = n - 1$ hyperelliptic curves. Above, Λ is the dynamically

¹⁰Actually, as we will see later, the best way to think about this is to add a third variable x_3 and promote $W = 0$ to an “ALE space”.

generated quantum scale.

Since y^2 factors into $W_{A_{n-1}} \pm \Lambda^n$, the situation is in some respect like two copies of the classical theory, with the top Casimir u_n shifted by $\pm \Lambda^n$. Specifically, the points e_{λ_i} of the classical level surface (3.4) split as follows,

$$e_{\lambda_i}(u) \rightarrow e_{\lambda_i}^{\pm}(u, \Lambda) \equiv e_{\lambda_i}(u_2, \dots, u_{n-1}, u_n \pm \Lambda^n),$$

and become the $2n$ branch points of the Riemann surface (3.8). The curve can accordingly be represented by the two-sheeted x -plane with cuts running between pairs $e_{\lambda_i}^+$ and $e_{\lambda_i}^-$. See Fig.11 for an example.

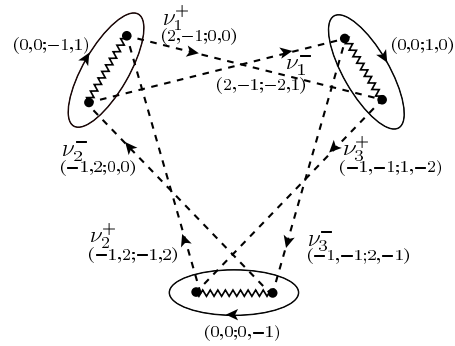


Figure 11. The spectral curve of quantum $SU(3)$ Yang-Mills theory is given by a genus two Riemann surface, which is represented here as a two-sheeted cover of the x -plane. It may be viewed as the quantum version of the classical, zero dimensional surface X_0 of Fig.10, whose points transmuted into branch cuts. The dashed lines represent the vanishing 1-cycles (on the upper sheet) that are associated with the six branches Σ_{\pm}^i of the singular locus $\Sigma_{\Lambda}(SU(3))$. The quantum numbers refer to $(g; q)$, where g, q are root vectors in Dynkin basis.

Moreover, the “quantum” discriminant, whose zero locus Σ_{Λ} gives the singularities in the quantum moduli space \mathcal{M}_q , is easily seen to factorize as follows:

$$\begin{aligned} \Delta_{\Lambda}(u_k, \Lambda) &= \prod_{i < j} (e_{\lambda_i}^+ - e_{\lambda_j}^+)^2 (e_{\lambda_i}^- - e_{\lambda_j}^-)^2 \\ &= \text{const.} \Lambda^{2n^2} \delta_+ \delta_- , \quad (3.9) \\ \delta_{\pm}(u_k, \Lambda) &\equiv \Delta_0(u_2, \dots, u_{n-1}, u_n \pm \Lambda^n) , \end{aligned}$$

is the shifted classical discriminant, (3.3). Thus, Σ_{Λ} consists of two copies of the classical singu-

lar locus Σ_0 , shifted by $\pm\Lambda^n$ in the u_n direction. Obviously, for $\Lambda \rightarrow 0$, the classical moduli space is recovered: $\Sigma_\Lambda \rightarrow \Sigma_0$. That is, when the quantum corrections are switched on, a single isolated branch $\Sigma_0^{\alpha_i}$ of Σ_0 (associated with massless gauge bosons of a particular $SU(2)$ subgroup) splits into two branches Σ_\pm^i of Σ_Λ (reflecting two massless dyons related to this $SU(2)$). For $G = SU(3)$, this is depicted in Fig.12.

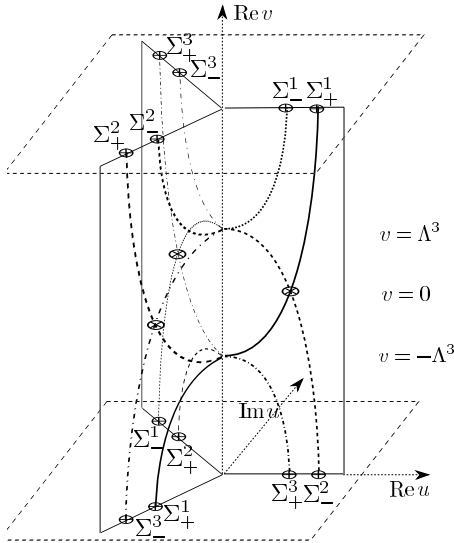


Figure 12. When switching to the exact quantum theory, the classical singular locus splits into two quantum loci that are associated with massless dyons; this is completely analogous to Fig.2. The distance is governed by the quantum scale Λ . Shown are here the six branches Σ_\pm^i for $G = SU(3)$.

According to the line of arguments given below equ. (2.35), all what it takes to determine the dyon spectrum associated with the $n(n-1)$ singular branches, is to determine the set of one-cycles ν_\pm^i that vanish on the Σ_\pm^i , with respect to some appropriate symplectic basis of α and β cycles. This can be done by tracing the exchange paths of the branch points $e_{\lambda_i}^\pm$ when we encircle the components of the discriminant Σ_\pm^i in the moduli space (starting from and ending at an arbitrary, but fixed base point).¹¹

¹¹Hypertext capable on-line readers may click [here](#) to obtain a Mathematica notebook that shows how this can be done in practice.

The result can be characterized in a very simple way: the punctured x -plane (cf., Fig.11) can be thought of as a “quantum deformation” of the classical level surface X_0 (cf., Fig.10), and thus inherits its group theoretical properties. We already mentioned above that the points e_{λ_i} of X_0 , associated with the projected weight vectors λ_i , turn into branch cuts, whose length is governed by the quantum scale, Λ (in fact, one obtains two, slightly rotated copies of the weight diagram).

Now, a basis of cycles can be chosen such that the coordinates of the “electric” α -type of cycles around the cuts are given precisely by the corresponding weight vectors λ_i . That is, we can associate charges $(g; q) = (0; \lambda_i)$ with the α -cycles. Moreover, the classical cycles of X_0 (related to the roots α_i), turn into pairs of “magnetic” β -type of cycles. By consistently assigning charge vectors to all vanishing cycles, we can then immediately read off the electric and magnetic quantum numbers of the massless dyons: they are given by specific combinations of root vectors. For $G = SU(3)$, this is indicated in Fig.11.

At this point a feature that is novel for $SU(n)$, $n > 2$, becomes evident: there are regions in \mathcal{M}_q where *mutually non-local* dyons become simultaneously massless [18]. Indeed, as can be inferred from Figs.10,11 for $G = SU(3)$, at $u \equiv u_2 = 0$, $v \equiv u_3 = \pm\Lambda^n$, dyons are massless whose vanishing cycles have non-zero intersection numbers, $\nu_i \circ \nu_j \neq 0$; this phenomenon persists for higher n as well. In other words, their Dirac-Zwanziger charge product (2.36) does not vanish, and this means, as mentioned before, that they are not local with respect to each other.

Whenever this happens, then by general arguments [30] the theory becomes conformally invariant. From (3.8) it is clear that near such an “Argyres-Douglas” point the curve looks locally like $y^2 = W_{A_{n-1}} = x^n + \dots$, and thus effectively behaves like a genus $g = (n-1)/2$ (for $n = \text{odd}$, $g = n/2 - 1$ for even n) curve that has a singularity of type A_{n-1} . This is the same singularity type that the classical level set X_0 (3.4) has at the conformally invariant point, $u_l = 0$. Indeed one may view the AD points as arising from

“splitting and shifting” the classical A_{n-1} singularities,¹² analogously to what saw in Fig.2 for $SU(2)$. However, whereas the classical theory has a gauge symmetry at the singularity, the SW theory appears not to have massless gauge bosons at the AD points [18]. Rather, the SW theory may have some sort of novel symmetry, but this is not yet completely settled.

To obtain the effective action (i.e., prepotential [31,24]), one must first determine the sections $a_i(u_k), a_{D,i}(u_k) \equiv \partial_{a_i} \mathcal{F}(a)$, appropriately defined as period integrals. For theories with more than one modulus, the existence of a prepotential poses an integrability condition, which can be solved by finding a suitable meromorphic one-form λ_{SW} .

More specifically, the genus of the hyperelliptic curve X_1 (3.8) is equal to $g = n - 1$, so that its $2n - 2$ periods can naturally be associated with

$$\vec{\pi} \equiv \begin{pmatrix} \vec{a}_D \\ \vec{a} \end{pmatrix}. \quad (3.10)$$

On such a curve there are $n - 1$ holomorphic differentials (abelian differentials of the first kind) $\omega_{n-i} = \frac{x^{i-1} dx}{y}$, $i = 1, \dots, g$, out of which one can construct $n - 1$ sets of periods $\int_{\gamma_j} \omega_i$. (Here γ_j , $j = 1, \dots, 2g$, is any basis of $H_1(X_1, \mathbb{Z})$.) All periods together can be combined in the $(g, 2g)$ -dimensional period matrix

$$\Pi_{ij} = \int_{\gamma_j} \omega_i. \quad (3.11)$$

If we chose a symplectic homology basis, i.e. $\alpha_i = \gamma_i$, $\beta_i = \gamma_{g+i}$, $i = 1, \dots, g$, with intersection pairing¹³ $(\alpha_i \circ \beta_i) = \delta_{ij}$, $(\alpha_i \circ \alpha_j) = (\beta_i \circ \beta_j) = 0$, and if we write $\Pi = (A, B)$, then $\tau \equiv A^{-1}B$ is the metric on the quantum moduli space. By Riemann’s second relation, $\text{Im}(\tau) \equiv 8\pi^2/g_{eff}^2$ is manifestly positive, which is important for unitarity of the effective $N = 2$ supersymmetric gauge theory.

¹²This generalized to a whole series of $d = 4$, $N = 2$ superconformal theories, classified by the ADE Lie algebras [30].

¹³We use the convention that a crossing between the cycles α, β counts positively to the intersection $(\alpha \circ \beta)$, if looking in the direction of the arrow of α the arrow of β points to the right.

The precise relation between the periods and the components of the section $\vec{\pi}$ is given by:

$$\begin{aligned} A_{ij} &= \int_{\alpha_j} \omega_i = \frac{\partial}{\partial u_{i+1}} a_j \\ B_{ij} &= \int_{\beta_j} \omega_i = \frac{\partial}{\partial u_{i+1}} a_{D_j} \end{aligned} \quad (3.12)$$

(where $i, j = 1, \dots, n - 1$). From the explicit expression (3.8) for the family of hyperelliptic curves, one immediately verifies that the integrability conditions $\partial_{i+1} A_{jk} = \partial_{j+1} A_{ik}$, $\partial_{i+1} B_{jk} = \partial_{j+1} B_{ik}$ are satisfied. It also follows that $\tau_{ij} \equiv \partial_{a_i} \partial_{a_j} \mathcal{F}(a)$. This reflects the special geometry of the quantum moduli space, and implies that the components of $\vec{\pi}$ can directly be expressed as integrals

$$a_{D_i} = \int_{\beta_i} \lambda_{SW}, \quad a_i = \int_{\alpha_i} \lambda_{SW}, \quad (3.13)$$

over a suitably chosen meromorphic differential. One may take, for example [32,33]:

$$\begin{aligned} \lambda_{SW} &= \frac{dx}{4\sqrt{2}\pi} \log \left[\frac{W_{A_{n-1}} + \sqrt{W_{A_{n-1}}^2 - \Lambda^{2n}}}{W_{A_{n-1}} - \sqrt{W_{A_{n-1}}^2 - \Lambda^{2n}}} \right] \\ &= \frac{1}{2\sqrt{2}\pi} \left(\frac{\partial}{\partial x} W_{A_{n-1}}(x, u_i) \right) \frac{x dx}{y} + \partial[*]. \end{aligned} \quad (3.14)$$

Explicit expressions for the prepotentials [21, 31,24] can be obtained by first solving Picard-Fuchs equations, and consequently matching the solutions with the asymptotic expansions of the period integrals (in analogy to what we discussed in section 2.5; recently, a more efficient method has been developed in ref. [34].) In fact, for a given group one can write down a whole variety of effective actions that are valid in appropriate coordinate patches in the moduli space; this is similar to what was shown in Fig.3 for $G = SU(2)$.

Specifically, in the semi-classical coordinate patch, where by definition the classical central charges are large, $e_{\alpha_i} \equiv \alpha_i \cdot a \gg \Lambda$, the prepoten-

tial has the form:¹⁴

$$\mathcal{F}(a_i) = \mathcal{F}_{\text{class}} + \mathcal{F}_{1\text{-loop}} + \mathcal{F}_{\text{non-pert}} , \quad (3.15)$$

where

$$\mathcal{F}_{\text{class}} = \frac{1}{2} \tau_0 (a^t \cdot C \cdot a)$$

$$\mathcal{F}_{1\text{-loop}} = \frac{i}{4\pi} \sum_{\substack{\text{positive} \\ \text{roots } \alpha}} e_\alpha^2 \log [e_\alpha^2 / \Lambda^2] \quad (3.16)$$

$$\mathcal{F}_{\text{non-pert}} = -\frac{i}{2\pi} \left(\sum_{\substack{\text{positive} \\ \text{roots } \alpha}} e_\alpha^2 \right) \sum_{\ell=1}^{\infty} \mathcal{F}_{2h\ell}(e_\alpha^{-1}) \Lambda^{2h\ell} .$$

Here $\mathcal{F}_{2h\ell}(e_\alpha^{-1})$ are Weyl invariant Laurent polynomials in the e_α of degree $-2h\ell$. For example, for $G = SU(3)$, $\mathcal{F}_6 = \frac{1}{4} \prod_\alpha e_\alpha^{-2}$; see refs. [22,34]. for some further explicit expressions for $\mathcal{F}_{2h\ell}$. The one-loop term, $\mathcal{F}_{1\text{-loop}}$, here obtained from solving a differential equation, indeed coincides exactly with what one obtains by a standard perturbative quantum field theory computation !

3.4. Fibrations of Weight Diagrams

There is an alternative representation of the SW curves X_1 , which is not manifestly hypergeometric and thus perhaps slightly less convenient to deal with, but which can easily be generalized to arbitrary gauge groups. As we will see, it is also precisely this geometrically more natural form of the curves that arises in string theory [6].

Inspired by the role of spectral curves in integrable systems [7,27,35], one is lead to consider SW curves (3.8) of the form [7,27]:

$$X_1 : z + \frac{\Lambda^n}{z} + 2 P_{A_{n-1}}^n(x, u_k) = 0 , \quad (3.17)$$

where the characteristic polynomial (3.8) for $SU(n)$ obeys ‘‘by accident’’ : $P_{A_{n-1}}^n \equiv W_{A_{n-1}}$. These curves are related to the hyperelliptic curves (3.8) by a simple reparametrization, $z \rightarrow y - P$, and thus are completely equivalent to them.

¹⁴This form is valid for all *ADE* groups; C denotes the Cartan matrix, τ_0 the bare coupling and h the dual Coxeter number ($h \equiv n$ for $SU(n)$).

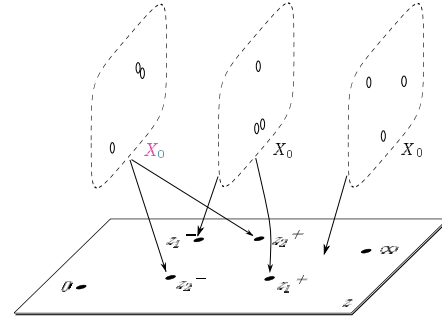


Figure 13. The Seiberg-Witten curve can be understood as fibration of a weight diagram (the classical surface X_0 of Fig.10 over \mathbb{P}^1 . Pairs of singular points in the base are associated with vanishing 0-cycles in the fiber, i.e., to root vectors $a_i - a_j$. In string theory, the local fibers will be replaced by appropriate ALE spaces with corresponding vanishing two-cycles.

Moreover, note also that the classical limit $\Lambda \rightarrow 0$ gives $X_1 : z + P(x) = 0$, which is an (equivalent) alternative to the classical level surfaces, $X_0 : P(x) = 0$.

A curve of the form (3.17) can be thought as fibration of the classical level set X_0 (3.4) over \mathbb{P}^1 , coordinatized by z and whose size is measured by $1/\Lambda$. There are $(n - 1)$ pairs of branch points in the z -plane, z_i^\pm , which are associated with the basic degenerations of X_0 , ie., with the simple roots α_i . There are two additional branch points z_0, z_∞ , and cuts run between, say z_i^- and z_0 , and between z_α^+ and z_∞ . See Fig.13 for $G = SU(3)$, where $X_1 : z + \Lambda^3/z + 2(x^3 - ux - v) = 0$ and

$$z_1^\pm = 2u^{\frac{3}{2}} + 3\sqrt{3}v \pm \sqrt{\left(2u^{\frac{3}{2}} + 3\sqrt{3}v\right)^2 - \Lambda^6}$$

$$z_2^\pm = -2u^{\frac{3}{2}} + 3\sqrt{3}v \pm \sqrt{\left(2u^{\frac{3}{2}} - 3\sqrt{3}v\right)^2 - \Lambda^6} .$$

The curve may also be viewed as a foliation, or n -sheeted covering of the z -plane, the sheets being associated to the points of X_0 , ie., to the weights λ_i , see Fig.14.¹⁵

The meromorphic differential takes the follow-

¹⁵One may draw the cuts also in different ways; we have chosen them here such that the massless monopoles are related to vanishing β -type cycles.

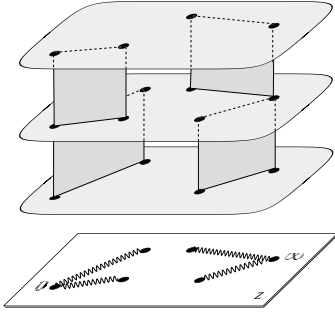


Figure 14. The genus two curve resulting from the fibration shown in Fig.13 can be viewed as a foliation with three leaves that are glued together over the cuts. The sheets are one-to-one to the weights of the fundamental representation of $SU(3)$.

ing particularly simple form,

$$\lambda_{SW} = \frac{1}{2\sqrt{2}\pi} x(z, u) \frac{dz}{z}. \quad (3.18)$$

which, via partial integration, is equivalent to (3.14). Note that there are actually n versions of λ_{SW} , since $x(z) \sim z^{1/n}$, each associated to one of the sheets of the foliation.

Having represented the $SU(n)$ curves in the form (3.17), the generalization to other simply laced groups is now easy to state [27]: one just takes the characteristic polynomial $P_{ADE}^{\mathcal{R}}$ (3.8) (for an arbitrary representation \mathcal{R}), and shifts the top Casimir u_h by $z + \frac{\Lambda^h}{z}$, i.e.,

$$X_1 : P_{ADE}^{\mathcal{R}}(x, u_k, u_h + z + \frac{\Lambda^h}{z}) = 0, \quad (3.19)$$

where h is the dual Coxeter number. As explained in [27], the choice of the representation is irrelevant. That is, there is for each gauge group an infinity of possible curves, with arbitrarily high genera; however, one can restrict attention to a particular subset of $r \equiv \text{rank}(G)$ α - and r dual β -periods, which carry all the relevant information.

One can in fact write down curves for the non-simply laced groups as well. These are obtained by fibering ADE level surfaces $P_{ADE}^{\mathcal{R}} = 0$ in a “non-split” fashion, i.e., in a way that encircling the singular points in the z -plane gives rise not

only to Weyl transformations acting on the fiber X_0 , but also to outer automorphisms. This is similar to the considerations of [36], and gives an orbifold prescription leading to a “folding” of the ADE Dynkin diagram into the corresponding non-simply laced one; it also appropriately modifies the curves (3.19)[27].

4. SW Geometry from String Duality

4.1. General Picture

So far, the auxiliary “spectral curves” (3.8),(3.17),(3.19) have been introduced in a somewhat *ad hoc* fashion, originally just in order to deal with the monodromy problem in a systematic way. One may in fact approach the SW theory without directly referring to a Riemann surface, for example along the lines of [37], but this gets pretty quickly out of hand for larger gauge groups. Thus the question comes up whether the SW curves have a more concrete physical significance – it would be rather absurd if all the geometrical richness of Riemann surfaces would be nothing more than a technical convenience, and would not have any deeper meaning.

This attitude is supported by growing recent experience that whenever we meet a geometrical object like a spectral manifold, *it represents a concrete physical object in some appropriate dual formulation of the theory*. For example, the classical level set X_0 (3.4), consisting of a discrete set of points, has been associated with the locations of D -branes in a dual formulation of gauge symmetry enhancement [38] – the dashed lines in Fig.10 then are nothing but open strings linking D -branes.

Indeed it turns out [6] that, in this sense, the natural home of SW geometry is string theory, but of a rather peculiar, “non-critical” kind [39, 40].

For ease of discussion it is most convenient to start with ordinary, “critical” superstrings that naturally live in ten dimensions. Remember that for these theories, complex manifolds play an ubiquitous role as compactification manifolds –

and this already hints at our aim to ultimately view the SW curves as some kind of compactification manifolds as well.

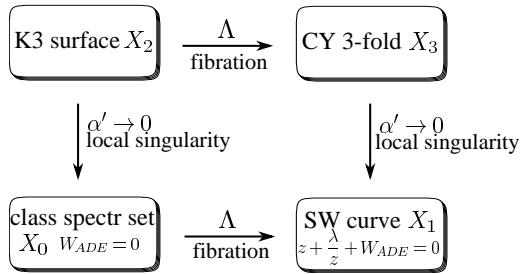


Figure 15. *Complex manifolds that are relevant in heterotic-type II duality. The vertical direction is a local scaling limit that includes switching off α' . The horizontal step corresponds to switching on space-time quantum corrections.*

More specifically, what we have in mind is a structure roughly as depicted in Fig.15. That is, starting from the spectral set X_0 , which describes classical Yang-Mills theory, one can go to the quantum version via fibering X_0 over \mathbb{P}^1 with size $1/\Lambda$ – this was described in section 3.4. However, one may also go vertically and view X_0 (and thus classical, $N = 4$ Yang-Mills theory itself) as arising from a particular multiple scaling limit of a $K3$ compactification (which includes the limit $\alpha' \rightarrow 0$). This $K3$ surface X_2 may then itself be taken as a fiber over \mathbb{P}^1 , to yield a Calabi-Yau threefold X_3 . One may thus expect to close the circle in Fig.15 by performing a suitable limit of X_3 , in order to get back to the SW curve. How this exactly works will be explained in the next couple of sections.

4.2. ALE Spaces and Heterotic-Type II String Duality in Six Dimensions

Fig.15 may look suggestive, but so far we did not specify the precise physical context to which it is supposed to apply. The correct framework is the duality [2,41,42] between the heterotic string and the type II string that appears when the strings are compactified on suitable manifolds. See [43] for a detailed review on this subject.

Let us first review the original Hull-Townsend

hypothesis, which states the non-perturbative equivalence of the heterotic string (compactified on T_4) with the type IIA string (compactified on $K3$). As is well-known, gauge symmetry arises in the six-dimensional heterotic string from the Narain lattice, $\Gamma_{20,4}$. More precisely, ADE type of gauge symmetries arise if the background moduli are such that the Narain lattice becomes ADE symmetric, i.e., if there are lattice vectors with $(\text{length})^2 = 2$. These are just the root vectors associated with the gauge group, and give rise to space-time gauge fields via the Frenkel-Kac mechanism (see eg., [44]).

On the type IIA string side, gauge symmetries arise from ADE type of singularities of $K3$ [2,45,46,38,47]. More specifically, if the $K3$ moduli are tuned appropriately, $K3$ can locally (near the singularity and in some suitable coordinate patch) be written as:

$$W_{K3} = \epsilon [\mathcal{W}_{ADE}^{ALE}] + \mathcal{O}(\epsilon^2) = 0, \quad (4.1)$$

where, in physical terms, $\epsilon \sim (\alpha')^\sigma \rightarrow 0$ for some power σ . Moreover, $S_{ADE} : \mathcal{W}_{ADE}^{ALE}(x_i) = 0$ is the **A**symptotically **L**ocal **E**uclidean space [48] with ADE singularity at the origin; it is a non-compact space obtained by excising a small neighborhood around the singularity on $K3$. This kind of spaces is essentially given by the ADE simple singularities (3.2), (3.6), up to “morsification” by extra quadratic pieces:

$$\begin{aligned} \mathcal{W}_{A_{n-1}}^{ALE} &= W_{A_{n-1}}(x_1, u_k) + x_2^2 + x_3^2 \\ &= x_1^n + x_2^2 + x_3^2 + \dots \\ \mathcal{W}_{D,E}^{ALE} &= W_{D,E}(x_1, x_2, u_k) + x_3^2. \end{aligned} \quad (4.2)$$

In fact, the ADE singularities can be characterized in a most uniform and natural way if they are written, exactly as above, as ALE space singularities in terms of three variables. By definition, $S_{ADE} = \mathbb{C}^3 / [\mathcal{W}_{ADE} = 0]$, but one may also write $S_{ADE} = \mathbb{C}^2 / \Gamma$, where $\Gamma = \mathcal{C}_n, \mathcal{D}_n, \mathcal{T}, \mathcal{O}, \mathcal{I} \subset SL(2, \mathbb{C})$ are the discrete isometry groups of the sphere that are canonically associated with the ADE groups [5,28].

The variables u_k provide a minimal resolution of these singularities, obtained by iterated blow

up's of points. That is, the ADE singularity at the origin in \mathbb{C}^3 is blown up into a connected union of 2-spheres, with self-intersections equal to -2 . Pairwise intersections are either null or transverse, and one may encode this information in a Dynkin diagram – surprisingly, these Dynkin diagrams agree precisely with those of the corresponding ADE groups; see Fig.16 for an example. In fact, the second homology group is isomorphic to the root lattice,

$$H_2(S_{ADE}, \mathbb{Z}) \cong \Gamma_R^{ADE}, \quad (4.3)$$

since it is equipped with a geometric intersection form given by (the negative of) the Cartan matrix. Indeed the vanishing 2-cycles of the ALE space behave exactly like the root vectors of the corresponding ADE group. Since the addition of quadratic pieces does not change the singularity type, the relevant features of the classical spectral surfaces X_0 thus apply here as well – this is what is meant by the left vertical arrow in Fig.15. Eq. (4.3) is precisely the two-dimensional version of (3.5) for A_{n-1} , and we may thus use Fig.10 to visualize the ALE space homology for A_{n-1} as well, if we simply view the dashed lines as 2-cycles and not as 0-cycles.

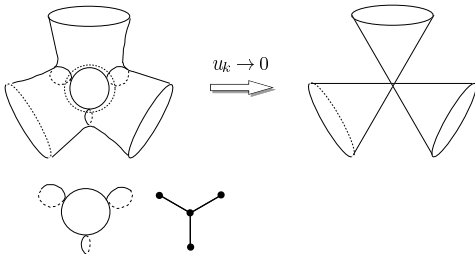


Figure 16. The vanishing 2-cycles of the D_4 -type of ALE space intersect in a Dynkin diagram pattern. At the top we have drawn only a real slice of the degeneration process.

That we meet here again the classical spectral set X_0 is no surprise in physics terms, since type IIA string compactification on $K3$ does give classical, unrenormalized $N = 2$ gauge theory in $d = 6$ (or $N = 4$ in $d = 4$ after additional toroidal compactification). The concrete physical mechanism that underlies the appearance of

gauge fields in space-time is the wrapping of type IIA 2-branes on the vanishing 2-cycles of the ALE space [2,45,46,38,47] – obviously, if the volumina of the 2-cycles shrink to zero, the corresponding BPS masses will vanish.

Summarizing, we see some sort of universality at work: for low-energy physics ($\alpha' \rightarrow 0$) and small symmetry breaking VEVs, only the local neighborhood of a singularity is relevant. That is, the local geometry of an ALE space, sitting somewhere on $K3$, singles out a subset of 2-cycles, namely those which tend to vanish near the singularity. The embedding of these vanishing cycles in the full 2-homology of $K3$ exactly mirrors the embedding of a singled-out ADE root lattice into the Narain lattice on the heterotic side, $H_2(S_{ADE}, \mathbb{Z}) \subset H_2(K3, \mathbb{Z}) \cong \Gamma_{20,4}$. The information that comes from more distant parts of $K3$, like effects of 2-branes wrapping around the other 2-cycles, is suppressed by powers of α' . This mirrors the effect of massive winding states on the heterotic side.

Note that in $d = 6$ the duality is between the heterotic string on T_4 and the type IIA string on $K3$, and not the type IIB string. Since the type IIB string does not have any 2-branes, clearly no gauge fields can arise from the vanishing 2-homology of $K3$. Rather, since type IIB strings have 3-branes, one needs to consider 3-branes wrapped around the vanishing 2-cycles. But what this gives is not massless particles, but *tensionless strings* in six dimensions [39,40].

Such kind of strings have not much to do with the perturbative ten-dimensional superstrings that we started with. Rather, these strings are *non-critical* and therefore do not involve gravity at all – indeed we have already taken the limit $\alpha' \rightarrow 0$. They couple to 2-form fields $B_{\mu\nu}^i$, in analogy to particles that couple to gauge fields A_μ^i . The antisymmetric tensor field $B_{\mu\nu}^i$ forms together with 5 scalars (plus some fermions) a tensor multiplet of $(0, 2)$ supersymmetry in six dimensions; there is one such multiplet for each 2-cycle in the ALE space (labelled by the index i).

The non-critical string represents a novel quan-

tum theory on its own, but is hard to study directly [39,40]. That is, since the field strength $H^{(3)} = dB$ to which it couples arises from a self-dual five-form in ten dimensions, that is (anti-)self-dual too, $*H^{(3)} = -H^{(3)}$. This implies that the coupling must be equal to one and thus that this string cannot be accessed in terms of usual (conformal field theoretic) perturbation theory.

As we will see later in section 5, it is these anti-self-dual strings that provide the natural dual, geometric representation of the SW theory that we are looking for.

4.3. $K3$ -Fibrations and SW Geometry

In the previous section we discussed $N = 2$ supersymmetric compactifications in six dimensions, which gives rise to “classical” ($N = 4$) Yang-Mills theory in $d = 4$. We now like to understand how $N = 2$ supersymmetric gauge theories emerge in four dimensions. For our purposes, the appropriate framework¹⁶ to describe such theories is the duality [42,41] between the heterotic string, compactified on $K3 \times T_2$, and the type IIA (IIB) string, compactified on a particular class of Calabi-Yau threefolds, X_3 (its mirror¹⁷ \tilde{X}_3). (See ref. [52,53] for some selection of further work related to this duality.) The particular choice of X_3 corresponds to the choice of gauge bundle data on the heterotic side.

Crucial is the insight that that the CY moduli space factors into two (generically) decoupled pieces:

$$\mathcal{M}_{X_3} = \mathcal{M}_V(s, t_i) \otimes \mathcal{M}_H(d, h_k), \quad (4.4)$$

where the vector multiplet moduli space \mathcal{M}_V has a complex, and the hypermultiplet moduli space \mathcal{M}_H a quaternionic structure – this follows directly from $N = 2$ supersymmetry [54]. It is well-known that

$$\begin{aligned} \dim_C \mathcal{M}_V(X_3) &= h_{11}(X_3) \\ \dim_C \mathcal{M}_H(X_3) &= h_{21}(X_3) + 1, \end{aligned} \quad (4.5)$$

¹⁶The are in fact various ways to obtain $N = 2$ $d = 4$ YM theories, see, for example, [49,50].

¹⁷For a review on mirror symmetry, see e.g., [51].

where, of course, the Hodge numbers h_{11} and h_{22} exchange under mirror symmetry, $X_3 \leftrightarrow \tilde{X}_3$. The shift “+1” accounts for the type IIA dilaton d .

The point is that the heterotic dilaton s does not enter in \mathcal{M}_H and the type IIA dilaton d does not enter in \mathcal{M}_V . Therefore, a tree-level computation in \mathcal{M}_H in the heterotic string will give the exact quantum result, while a tree-level computation in the type II string will give the exact quantum corrected vector multiplet moduli space, \mathcal{M}_V . More precisely, \mathcal{M}_V will not get any contributions from type IIA space-time quantum effects, but it will still get corrections from world-sheet instantons. But one can invoke mirror symmetry and obtain these corrections from a completely classical computation on the type IIB string side, see Fig.17. As is well-known [55], this boils down to evaluating period integrals or solving Picard-Fuchs equations, much like we did in previous sections for rigid Yang-Mills theories.

It is certainly \mathcal{M}_V that we are presently interested in, since we expect it to contain the SW moduli spaces \mathcal{M}_q for a variety of gauge groups. As explained before, these moduli spaces do get complicated corrections, which correspond, in heterotic language, to space-time instanton effects. But from the above we see that in the string framework, the functional complexity of the SW theory can equally well be attributed to type IIA world-sheet instanton corrections [32,42]. So what this means is that the $N = 2$ supersymmetric heterotic-type II duality implies a map between non-perturbative space-time instanton effects and world-sheet instanton effects !

To recover the SW physics from string theory, it is thus most natural to start with the type IIB formulation. As pointed out in [56,57], in IIB language the role of the vanishing 1-cycles of the SW curve is played by vanishing 3-cycles of the Calabi-Yau threefold (corresponding to “conifold” singularities). More specifically, the relevant periods are those of the canonical holomorphic 3-form Ω on \tilde{X}_3 [58]:

$$X^I = \int_{\Gamma_{\alpha_I}} \Omega, \quad F_J = \int_{\Gamma_{\beta_J}} \Omega, \quad (4.6)$$

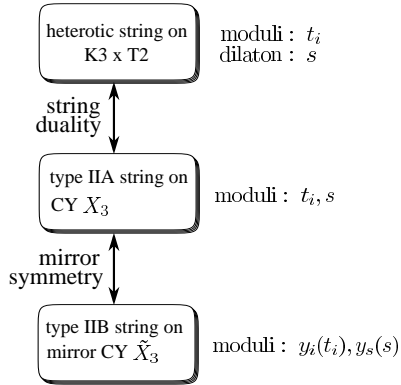


Figure 17. *Heterotic-type IIA duality plus mirror symmetry allows to compute non-perturbatively exact quantities in the heterotic string from tree-level computations in the type IIB string. In practice, in the vector multiplet sector indicated above, this amounts to solving Picard-Fuchs equations for the IIB moduli y_i, y_s in terms of the IIA (and heterotic) variables t_i, s . For the notation see below.*

where $\Gamma_{\alpha_I}, \Gamma_{\beta^J}, I, J = 1, \dots, h_{11}(\tilde{X}_3) + 1$ span an integral symplectic basis of H_3 , with the α -type of cycles being dual to the β -type of cycles: $\Gamma_{\alpha_I} \circ \Gamma_{\beta^J} = \delta^J_I$, etc. Concretely, assuming that the mirror threefold can be represented by the vanishing of a polynomial in some weighted projective space, $W_{\tilde{X}_3} = 0$, the holomorphic 3-form on \tilde{X}_3 can be written as:

$$\Omega = \int_{\gamma} \frac{1}{W_{\tilde{X}_3}} \omega, \quad \text{with} \quad (4.7)$$

$$\omega \equiv \sum_{A=1}^5 (-1)^A x^A dx^1 \wedge \dots \wedge \widehat{dx^A} \wedge \dots \wedge dx^5,$$

where γ is a small, one-dimensional curve winding around the hypersurface $W_{\tilde{X}_3} = 0$, and where the hat means that the indicated term is to be left out.

The periods (4.6) are defined up to symplectic basis transformations which belong to $Sp(2h_{11}(\tilde{X}_3) + 2, \mathbb{Z})$. The duality group, which represents the exact quantum symmetries of the theory and which is analogous the SW monodromy group $\Gamma_0(4)$ in section 2.4, must be a subgroup of this symplectic group. The precise form of this duality group depends, of course, on the choice of threefold. See refs. [59] for further

discussion of duality groups in this context.

In terms of the periods (4.6), the $N=2$ central charge, which enters in the BPS mass formula, then looks very similar to (2.15),

$$Z \sim M_I X^I + N^J F_J, \quad (4.8)$$

so that again vanishing cycles will lead to massless BPS states. More precisely, if some 3-cycle $\nu = M_I \Gamma_{\alpha_I} + N^J \Gamma_{\beta^J}$ vanishes in some region of the moduli space, then $Z \equiv \int_{\nu} \Omega = 0$ and we get, just like in the rigid SW theory, a massless hypermultiplet, with quantum numbers (N^J, M_I) . Physically, it arises from a type IIB 3-brane wrapped around ν [56].

Furthermore, by the general properties of vanishing cycles, governed by the Picard-Lefschetz formula (2.37), one finds a logarithmic behavior for the dual period $F_Z \equiv \partial \mathcal{F}(Z) / \partial Z = \frac{1}{2\pi i} Z \log[Z/\Lambda] + \dots$, so that the effective action $\mathcal{F}(Z)$ near the conifold singularity $Z = 0$ looks very similar to the SW effective action $\mathcal{F}_D(a_D)$ near the monopole singularity.

However, to really recover the SW geometry, we should not look just to a single conifold singularity (of local form $\sum_{i=1}^4 x_i^2 = u$), since it does not carry enough information. Indeed, by analogy just looking at the local singularity $x_1^2 + x_2^2 = u$ of the SW curve (or its $SU(n)$ extension, which is an Argyres-Douglas singularity), we cannot learn much about the global SW geometry. So what we need to look at is an appropriate “semi-local” neighborhood of the conifold singularity in the CY, in order to capture both SW type singularities (or both AD singularities, that is) at once. As we will see below, it is indeed not a local singularity, but rather a fibration of a local singularity, that is the right thing to consider.

Now, the CY manifolds that are relevant here have a very special structure: namely they must be $K3$ -fibrations [60–62]. Before we will briefly explain what this means mathematically, we first point out why such threefolds are physically important. That is, if and only if a threefold is a $K3$ -fibration, the effective prepotential (in the large

radius limit) has this particular asymptotic form:

$$\mathcal{F}(s, t) = \frac{1}{2}s Q_{ij} t^i t^j + \frac{1}{6} C_{ijk} t^i t^j t^k + \dots \quad (4.9)$$

Here, Q and C directly reflect the classical intersection properties of 2-cycles, and s, t^i are Kähler moduli of the CY, where s is singled out in that it couples only linearly. This means that s can naturally be identified (to leading order) with the semi-classical dilaton of the heterotic string, since the dilaton couples exactly in this way.

The $K3$ fibration structure turns out to be crucial for our purposes. To see this, let us assume, for simplicity, that the (mirror) threefold \tilde{X}_3 can be represented by some polynomial

$$\tilde{X}_3 : W_{\tilde{X}_3}(x_1, x_2, x_3, x_4, x_5; y_i, y_s) = 0 \quad (4.10)$$

in weighted projective space $WP_{1,1,2k_1,2k_2,2k_3}^{2d}$, with overall degree $2d \equiv 2(1 + k_1 + k_2 + k_3)$. Above, y_i denote the moduli, and, in particular, y_s denotes the special distinguished modulus that is related to the heterotic dilaton, $y_s \sim e^{-s}$. A list of this kind of $K3$ fibered threefolds was given in [60]. (For more general classes of fibrations, see [63]. Considerations similar to those below apply to these cases as well.)

The statement that \tilde{X}_3 is a $K3$ fibration of this particular type means that it can be written as

$$\begin{aligned} W_{\tilde{X}_3}(x_j; y_i, y_s) &= \frac{1}{2d} (x_1^{2d} + x_2^{2d} + \frac{2}{\sqrt{y_s}} (x_1 x_2)^d) \\ &+ \widehat{W}(\frac{x_1 x_2}{y_s^{1/d}}, x_k; y_i). \end{aligned} \quad (4.11)$$

Upon the variable substitution

$$\begin{aligned} x_1 &= \sqrt{x_0} \zeta^{1/2d} \\ x_2 &= \sqrt{x_0} \zeta^{-1/2d} y_s^{1/2d} \end{aligned} \quad (4.12)$$

this gives

$$W_{\tilde{X}_3}(x_j; \zeta, y_i) = \frac{1}{2d} \left(\zeta + \frac{y_s}{\zeta} + 2 \right) x_0^d + \widehat{W}(x_0, x_k; y_i).$$

That is, if we now alternatively view ζ as a modulus, and not as a coordinate, then

$W_{\tilde{X}_3}(\zeta, x_2, x_3, x_4, x_5; y_i, y_s) \equiv W_{K3}(x_2, x_3, x_4, x_5; \zeta + \frac{y_s}{\zeta}, y_i) = 0$ describes a $K3$ – this is precisely what is meant by fibration (of course, if we continue to view ζ as a coordinate, then this equation still describes the Calabi-Yau threefold). More precisely, ζ is the coordinate of the base \mathbb{P}^1 , and $y_s \rightarrow 0$ corresponds to the large base limit – obviously, the fibration looks in this limit locally trivial, and one expects then the theory to be dominated by the “classical” physics of the $K3$ (cf., the top horizontal step in Fig.15). This is the “adiabatic limit” [61], in which the $K3$ fibers vary only slowly over the base and where one can apply the original Hull-Townsend duality fiber-wise.

The left-over piece \widehat{W} is precisely such that $\frac{1}{d} x_0^d + \widehat{W}(x_0, x_k; y_i) = 0$ describes a $K3$ in canonical parametrization in WP_{1,k_1,k_2,k_3}^d . Now assuming that the $K3$ is singular of type ADE in some region of the moduli space, we can expand it around the critical point and thereby replace it locally by the ALE normal form (4.2) of the singularity, $\frac{1}{d} x_0^d + \widehat{W} \sim \epsilon \mathcal{W}_{ADE}^{ALE}(x_i, u_k)$. Going to the patch $x_0 = 1$ and rescaling¹⁸ $y_s = \epsilon^2 \Lambda^{2h}$ and $\zeta = \epsilon z$, we then obtain the following fibration of the ALE space:

$$\begin{aligned} W_{\tilde{X}_3}(x_j, z; u_k) &= \\ \epsilon \left(z + \frac{\Lambda^{2h}}{z} + 2 \mathcal{W}_{ADE}^{ALE}(x_j, u_k) \right) + \mathcal{O}(\epsilon^2) &= 0. \end{aligned} \quad (4.13)$$

This is not totally surprising: since the CY was a $K3$ fibration, considering a region in moduli space where the $K3$ can be approximated by an ALE space simply produces locally a corresponding fibration of the ALE space.

Now, focusing on $G = SU(n) \sim A_{n-1}$ and remembering the definition of the ALE space (4.2), we see that (4.13) is exactly the same as the fibered form of the SW curve (3.17), apart from the extra quadratic pieces in x_2, x_3 ! Since quadratic pieces do not change the local singularity type, this means that the local geometry of the threefold in the SW regime of the moduli space is indeed equivalent to the one of the Seiberg-Witten curve. However, just because of

¹⁸ This fixes $\epsilon = (\alpha')^{h/2}$.

these extra quadratic pieces, the SW curve itself is, strictly speaking, *not* geometrically embedded in the threefold, though this distinction is not very important.

One may in fact explicitly integrate out the quadratic pieces in (4.13), and verify [6] that the holomorphic three-form Ω of the threefold then collapses precisely to the meromorphic one-form λ_{SW} (3.18) that is associated with the SW curve:

$$\Omega = \frac{d\zeta}{\zeta} \wedge \left[\frac{dx_1 \wedge dx_2}{\frac{\partial W}{\partial x_3}} \right] \xrightarrow{\epsilon \rightarrow 0} x_1 \frac{dz}{z} \equiv \lambda_{SW}. \quad (4.14)$$

This implies that the periods $a_i, a_{D,i}$ are indeed among the periods of the threefold,

$$\begin{aligned} (X^I; F_J) &\equiv \int_{(\Gamma_{\alpha_I}; \Gamma_{\beta^J})} \Omega \\ \xrightarrow{\epsilon \rightarrow 0} \int_{(\alpha_i; \beta^j)} \lambda_{SW} &\equiv (a_i; a_{D,j}), \end{aligned} \quad (4.15)$$

and thus that the string effective action, $\mathcal{F} \equiv \frac{1}{2} X^I F_I$ [58], contains the SW effective action.

We should note there that we have tacitly assumed that the mirror \tilde{X}_3 is a $K3$ fibration. However, our starting point was really that the original threefold X_3 is a $K3$ fibration, since a priori it is type IIA strings on X_3 that are dual to the heterotic string. But in general, if X_3 is a $K3$ fibration, the mirror \tilde{X}_3 is not necessarily a $K3$ fibration as well. However, our above arguments are nevertheless correct, because all what counts is that *locally* near the relevant singularity, the mirror is a fibration of an ALE space. One can indeed show, using “local” mirror symmetry [64], that whenever we have an asymptotically free gauge theory on the type IIA side, the mirror \tilde{X}_3 has locally the required form.

As for the other simply laced groups, we face a complication similar to the one of section 3.2: namely, the CY geometry implies fibrations of ALE spaces, whereas the corresponding SW curves (3.19) involve the characteristic polynomials $R_{ADE}^{\mathcal{R}}$, which coincide with the simple singularities only for of $SU(n)$ (for the fundamental representation). For the other groups, the expressions (4.13) are quite different as compared to the

corresponding Riemann surfaces, and are not Riemann surfaces even if we drop the extra quadratic terms. But it can be shown that the independent periods of these spaces indeed do coincide with those of the curves (3.19) (see [29] for details on how this works for E_6). This represents a good test of the string duality, because that *predicts* that the fibered ALE spaces describe the rigid YM theories.

Moreover, note that these ideas carry over to gauge theories with extra matter, though we will very brief here. In the F -theoretical [53] formulation of gauge symmetry enhancement [65], there is a very systematic way to construct $N=2$ YM theory on the type IIA side, for almost any matter content. Via local mirror symmetry [64], this maps over to the type IIB side and directly produces the relevant SW curves, which generically exhibit extra matter fields. Similar to (4.13), one still has ADE singularities fibered over some base \mathbb{P}^1 , but in general the dependence of z will be more complicated.

Summarizing, not only pure $N=2$ gauge theory of ADE (and non-simply laced) type, but also gauge theories with extra matter fields, are geometrized in string theory and the corresponding Riemann surfaces can be constructed in a systematic fashion from appropriate threefolds.

There is a very simple generalization of the above,¹⁹ which however has no easy interpretation in terms of Yang-Mills theory. Remember that we focused above on $K3$ fibrations associated with weighted projected spaces of type $W\mathbb{P}_{1,1,2k_1,2k_2,2k_3}^{2d}$. There are many other types of $K3$ fibrations [63], for example related to $W\mathbb{P}_{1,\ell,(\ell+1)k_1,(\ell+1)k_2,(\ell+1)k_3}^{(\ell+1)d}$. These have the generic form

$$\begin{aligned} W_{\tilde{X}_3} &= \frac{1}{2d} \left(x_1^{(\ell+1)d} + \sum_k \tilde{y}_k x_1^k x_2^{(\ell+1)d-k} + \right. \\ &\quad \left. x_2^{(\ell+1)d/\ell} \right) + \widehat{W}(x_k; \tilde{y}_k, y_i) = 0, \end{aligned} \quad (4.16)$$

where the sum runs over appropriate values of k . Note that before, in eq. (4.11), we had just

¹⁹The rest of this section refers to unpublished work [66].

one variable y_s in the bracket. The difference is that we have now ℓ moduli of this sort, \tilde{y}_k . Still, one of the \tilde{y}_k is distinguished by the linear coupling property and thus is related to the heterotic dilaton; we will continue to denote it by $y_s \sim e^{-s}$. Also note that before, when we tuned $y_s \rightarrow 1$, the term in the bracket became a perfect square, and hence the threefold singular (of type A_1). This singularity leads in fact to an IR free $SU(2)$ gauge symmetry that is non-perturbative from the heterotic point of view (since $y_s = 1$ corresponds to strong coupling).

For the $K3$ fibrations above, we have now ℓ such parameters, and tuning them appropriately, there arises an analogous $SU(\ell + 1)$ “strong coupling” gauge symmetry [67]. But this is not the gauge symmetry that we are interested in here, we are rather still interested in the perturbative gauge symmetries that come from the $K3$ singularities. By taking now a singular limit similar to before, we arrive at the following modified fibration of the ALE spaces:

$$W_{\tilde{X}_3}(x_j, z; u_k) \sim \epsilon \left(z^\ell + \sum_{m=1}^{\ell-1} \Lambda^{(m)} z^m + \frac{\Lambda^{2h}}{z} + 2\mathcal{W}_{ADE}^{ALE}(x_j, u_k) \right) = 0. \quad (4.17)$$

This kind of fibrations has obviously a more complicated structure in the base \mathbb{P}^1 , and indeed we have now a whole series of extra moduli, $\Lambda^{(m)}$, that move the new singular points on the base around. If we send the extra Λ 's to infinity, we obviously recover the SW theories as explained before.

The interesting point is here that the physics associated to the extra parameters has nothing to do with the $K3$ fibers, and thus is intrinsically non-perturbative from the heterotic point of view. So what we have here is a marriage of the SW physics with physics inherited from the strong-coupling singularity. It results in SW type of curves that do *not* simply describe ordinary Yang-Mills physics. More concretely, in the classical limit, $\Lambda \rightarrow 0$, the discriminant of (4.17) looks

$$\Delta(u_k, \Lambda^{(m)}, \Lambda = 0) = (\Delta_0(u_k))^2 \tilde{\Delta}(u_k, \Lambda^{(m)})$$

$$\tilde{\Delta}(u_k, \Lambda^{(m)} = 0) \equiv (\Delta_0(u_k))^{\ell-1}, \quad (4.18)$$

where Δ_0 is the classical discriminant (3.3) associated with the corresponding ADE gauge group. The discriminant $\Delta(\Lambda = 0)$ describes a classical gauge symmetry $G = G_{ADE} \times U(1)^\ell$, with extra matter fields charged under the various group factors; in general this theory is non-asymptotically free.

However, for $\ell > 1$ the dependence of Δ on the $\Lambda^{(m)}$ is such that it does not describe any classical gauge theory (an exception is for $G = SU(2)$ with $\ell = 2$, where the theory is identical to the $SU(2)$ gauge theory with $N_f = 1$, with $\Lambda^{(1)}$ playing the role of the bare mass parameter). In other words, even though we can identify gauge symmetries and matter representations, the dependence of the theory on the extra parameters is in general not like the dependence on any mass parameters, or VEV's (because $\tilde{\Delta} \neq \prod_{\text{weights } \lambda} (a \cdot \lambda + m)$).

Rather, the moduli $\Lambda^{(m)}$ are novel, dilaton-like parameters that reflect non-perturbative physics of the heterotic string. Their effects persist even for weak coupling, $\Lambda \rightarrow 0$, and are thus like “small instanton” effects [68].

The lesson we draw from this is that “rigid” limits in string theory will in general not only reproduce known field theories like Yang-Mills theories and their curves, but also new kinds of SW like theories that do not have the interpretation in terms of conventional physics. Indeed, since the variety of possible CY singularities is quite large, one may expect to find a whole zoo of supersymmetric effective theories, and there is no reason why such theories should always be interpretable in terms of conventional field theories like gauge theories.

5. Anti-Self-Dual Strings on Riemann Surfaces

5.1. Geometry of Wrapped 3-Branes

In the previous section we have mentioned that the rôle of the SW monopole singularity is played in the CY threefold by a conifold singularity [56].

This corresponds in the type IIB formulation to a vanishing 3-cycle in the mirror, \tilde{X}_3 . The relevant BPS states are given by wrapping type IIB 3-branes around such 3-cycles.

What we are currently interested in is however not the full string theory (which includes gravity), but only the $\alpha' \rightarrow 0$ limit, where the local geometry is given by an ALE space fibered over \mathbb{P}^1 ; cf., (4.13). Thus we need to understand the properties of wrapped 3-branes in this local geometry in some more detail.

For this, reconsider the SW curve as fibration of a weight diagram over \mathbb{P}^1 ; cf., Fig.13. The difference to the present situation is that now the vanishing 0-cycles of the weight diagram are replaced by the vanishing 2-cycles of the ALE space. These 2-cycles, when fibered in this manner, produce 3-cycles in the CY in exactly the same way as the 0-cycles in Fig.13 produce 1-cycles on the SW curve; this is sketched in Fig.18.

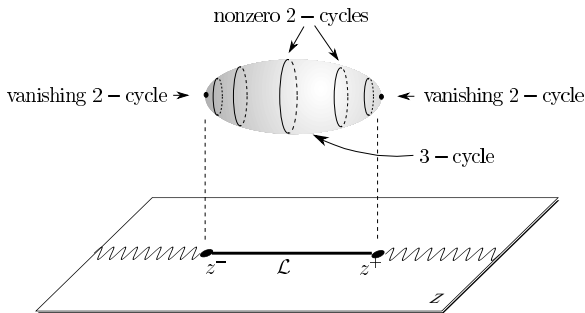


Figure 18. *Geometry of a 3-cycle in the CY, given by the fibration of an ALE 2-cycle. The 3-cycle arises by dragging the 2-cycle between branch points z^\pm on the base \mathbb{P}^1 . The 3-cycle thus projects in the base to an open line \mathcal{L} that joins the branch points – it represents a non-critical string with variable local tension.*

From this picture we can see that a 3-brane that is wrapped around the 3-cycle can be viewed as a fibration of a wrapped 2-brane over an open line segment \mathcal{L} in the base. This open line in the base is thus a “1-brane left-over”, obtained by wrapping two dimensions of the 3-brane around the ALE 2-cycle. But, according to what we said at the end of section 4.2, this corresponds precisely to a non-critical, anti-self-dual IIB string on the

base; its local tension varies along \mathcal{L} , depending on the size of the 2-cycle over it. Obviously, if the branch points z^\pm coincide in some region of the moduli space, the volume of the 3-brane and, in particular, the 1-brane \mathcal{L} vanishes, producing a massless “monopole” hypermultiplet in four dimensions.²⁰

Now, there is a partner of the open line \mathcal{L} on another sheet of the z -plane, which runs in the opposite direction. In effect, taking the sheets together and forgetting about the ALE fibers, these two lines corresponds to a *closed* anti-self dual string, wrapping around a β -cycle of a Riemann surface. This is indeed precisely how the SW curves were obtained in section 3.4 ! More generally, for $G = SU(n)$ there are n sheets, as well as $n - 1$ fundamental vanishing 2-cycles in the ALE space (which are associated with the simple roots of $SU(n)$). These gives rise to $n - 1$ different types of anti-self dual strings that run on the various sheets. It is now a simple mathematical fact of the representation of Riemann surfaces in terms of branched coverings, that the monodromy properties of these various strings are precisely such that they correspond to a *single* type of string that winds around the genus $g = n - 1$ Riemann surface given in (3.17).

Thus, what this discussion boils down to is that, effectively, 3-branes wrapped around the cycles of the threefold (4.13) are equivalent to 1-branes, or anti-self dual strings, wrapped around the Seiberg-Witten Riemann surface [6] ! This gives then finally a physical interpretation of the SW curves: just like the classical spectral surfaces X_0 (3.4) represent “ D -manifolds” [38] in a dual formulation of classical gauge symmetry breaking, the curves X_1 represent, in a dual formulation of the $N = 2$ gauge theory, compactification manifolds on which the six-dimensional, non-critical $(0, 2)$ supersymmetric string lives; see Fig.19. This is similar in spirit to ideas in refs. [69,39,70].

²⁰We consider here only β -type of 3-cycles, which give rise to (potentially massless) hypermultiplets. In fact, α -type of cycles, which describe electrically charged fields like the massive gauge multiplets W^\pm , project on closed lines in the base that wrap around the branch cuts – precisely as it is for rigid SW curves.

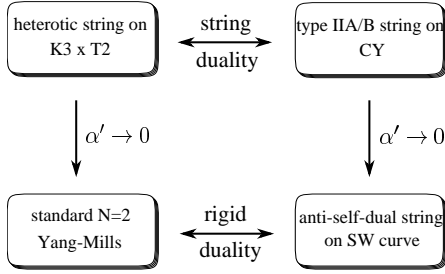


Figure 19. *The duality between conventional $N = 2$ Yang-Mills theory and anti-self-dual strings winding around the SW curve is just the rigid remnant of the non-perturbative duality between heterotic and type II strings.*

5.2. Geodesics and BPS States

We now focus on some of the properties of the anti-self dual string when wrapped around a SW curve; as we will see, this gives valuable insight into the $N = 2$ YM theory itself. There is a crucial point to make in this context, which can be easier appreciated if we ask first: if we compactify the six dimensional anti-self dual string on an ordinary flat torus T_2 , it is known [39] that this gives an $N = 4$ supersymmetric gauge theory in four dimensions (this theory has BPS states for all coprime (g, q) ; see Fig.20). Why are we then supposed to get only an $N = 2$ gauge theory, if we wrap the string on the SW curve, which is also a torus (for $SU(2)$) ?

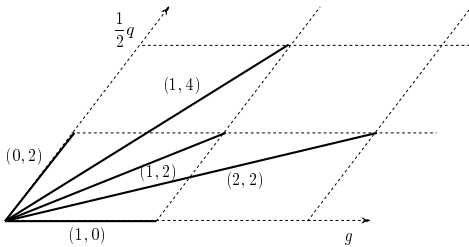


Figure 20. *On a torus with standard flat metric, there are string geodesics for all homology classes (g, q) where g and q are coprime. This reflects the BPS spectrum of $N = 4$ supersymmetric Yang-Mills theory.*

The answer is that the non-critical string cannot wrap the curve in an arbitrary fashion, rather it must wrap it in a particular way. Indeed, a BPS state always corresponds to a minimal volume, or “supersymmetric” cycle – otherwise, it does not

have the lowest energy in a given homology class. This means that the string trajectory (or space-part of the string world-sheet) must be a *geodesic* on the curve, with respect to a suitable metric. But, what is then here the right metric ? To see this, recall the condition for a “supersymmetric 3-cycle” [71] in the threefold:

$$\partial_\alpha X^\mu \partial_\beta X^\nu \partial_\gamma X^\rho \Omega_{\mu\nu\rho} e^{-\alpha' K} = \text{const.} \epsilon_{\alpha\beta\gamma} \quad (5.1)$$

Here, K is the Kähler potential (in which we have at present no further interest), Ω the holomorphic 3-form on the CY, and $\partial_\alpha X^\mu$ the pull-back from the compactified space-time to the 3-brane world volume. Now remember that in the limit $\alpha' \rightarrow 0$, the three-form Ω can be reduced by integration to give the meromorphic SW differential $\lambda_{SW} = x(z) \frac{dz}{z}$. Therefore, in the rigid limit the supersymmetric cycle condition (5.1) reduces to:

$$\frac{\partial}{\partial t} \lambda_z(z(t)) = \text{const.}, \quad \text{where } \lambda_{SW} \equiv \lambda_z(z) dz, \quad (5.2)$$

and where t parametrizes the space part of the string world-sheet. This is precisely the geodesic differential equation for $z(t)$, associated with the flat metric²¹ [6]

$$g_{z\bar{z}} = \lambda_z \bar{\lambda}_{\bar{z}}. \quad (5.3)$$

Now, in contrast to the usual flat metric on the torus used in the $N = 4$ supersymmetric compactification, this metric has *poles* since λ_{SW} is meromorphic. Thus, metrically the SW curves have “spikes” sticking out, which severely influence the form and kind of the possible geodesics.

The main effect from these singularities on the world-sheet metric is, for generic windings of type (g, q) , that the shortest trajectory in the homology class (g, q) may not be the “direct” one. Rather, the shortest trajectory may be one that is just a composition of fundamental trajectories;

²¹That this is the metric that governs the shape of the self-dual strings on the SW curve, can also be seen by some more direct geometrical reasoning, see [6]. Note that λ_{SW} is multi-valued on the z -plane, but is single valued on the curve X_1 . This means that g really is a metric on the SW curve, and not on the base space of the fibration.

see Fig.21 for an example. In other words, the lightest state in a given charge class (g, q) may not be a single-particle, but a multi-particle state ! When this happens, the state with charges (g, q) cannot be counted as a stable BPS state.

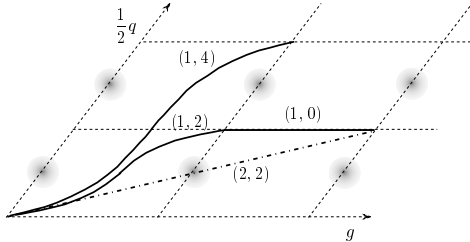


Figure 21. $N = 2$ Yang-Mills theory corresponds to a compactification torus with a metric given by $|\lambda_z|^2$. The poles in the differential λ deflect the string trajectories, with the effect that the shortest trajectories for $g > 1$ are composite of several fundamental ones. Specifically, the straight dashed line in charge class $(2, 2)$ is not the shortest trajectory in this class, but the composite one shown above is. On the other hand, we see from this picture that the states of type $(1, 2\ell)$ are good single-particle BPS states (in the semi-classical region).

We thus see that this dual formulation of $N = 2$ SYM theory gives a new method for determining quantum BPS spectra [6]. With conventional field theoretic methods, without the use of SW geometry, it is really hard to make statements about stable BPS spectra, see, for example [72]. The geodesic string method has been successfully applied [6] to re-derive the BPS spectrum [1,20] of the $SU(2)$ SYM theory that we had exhibited in section 2.5. More recently, it has been extended to discuss BPS spectra when extra matter is added [73]. It has also been studied [74] what happens if one starts with an $N = 2$ gauge theory with adjoint matter (which in total has $N = 4$ supersymmetry), and breaks $N = 4 \rightarrow N = 2$ supersymmetry by giving the adjoint matter field a mass. Then can explicitly see how most of the $N = 4$ BPS states decay into the allowed semi-classical states of the $N = 2$ Yang-Mills theory.

It is most interesting to study from this viewpoint what happens on the line of marginal stability \mathcal{C} discussed in section 2.5. Here, $a_D/a \in \mathbb{R}$,

and thus the period lattice spanned by (a_D, a) degenerates to a real line. That is, *all* possible string trajectories lie on top of each other. In particular, the string trajectory of the gauge field (and similarly for all the other semi-classical BPS states with $(g, q) = (1, 2\ell)$), is indistinguishable from the composite trajectory made out of the monopole $(1, 0)$ and the dyon $(1, 2)$; see Fig.22. This is very similar to the considerations of section 2.5, where we mentioned that for kinematical reasons the gauge field might decay into a monopole-dyon pair.

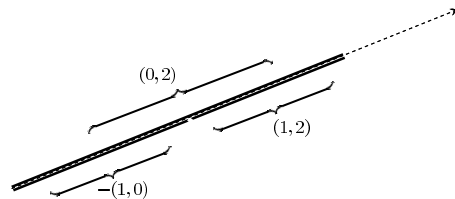


Figure 22. On the line \mathcal{C} of marginal stability the string representation degenerates, and the only indecomposable geodesics are the ones of the monopole and the dyon. Shown is here the trajectory $(0, 2)$ for the gauge boson, which cannot be distinguished from the one of the monopole-dyon pair.

However, the situation is quite different here, because we do not merely talk about kinematics, but about a dual representation of the BPS states. Thus, while Fig.22 may simultaneously also represent some simple kinematics, in the present context it shows that the *string representation itself* degenerates. Remember that even though we use classical physics (geometry) in the IIB formulation, by rigid string duality we supposedly capture the exact non-perturbative quantum behavior of the Yang-Mills theory. Therefore, what we see here is, roughly speaking, that on \mathcal{C} the (single particle) non-perturbative quantum state of the gauge field degenerates into a two-particle state made from the monopole and the dyon.

This clearly shows the conceptual power of the anti-self-dual string representation. Insights like the one above are extremely hard to obtain in ordinary quantum field theory, where the gauge field is elementary and the monopole/dyon solitonic (or vice versa). In contrast, in terms of non-critical strings, these fields are treated on equal

footing [39]. Other applications of this dual representation of $N = 2$ gauge theory include the description of non-abelian gauge flux tubes and confinement in terms of non-critical strings [75].

6. Conclusions

Note that the above construction of the SW theory is deductive – the SW theory in its full glory can be systematically derived from string theory, once one takes the heterotic-type II string duality [2] for granted and works out its consequences. This convergent evolution of *a priori* disparate physical ideas seems to indicate that we are really on the right track for understanding non-perturbative phenomena in supersymmetric field and string theory.

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