Notes on Relativity and Cosmology for PHY312

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Preface

Note: These lecture notes are available free of charge in color PDF from the course web site: (http://www.phy.syr.edu/courses/PHY312.03Spring/). The color version is particularly useful for getting the most out of complicated diagrams. To view PDF you will need to install Adobe Acrobat reader (if you don't have it already). It is free from Adobe at:

http://www.adobe.com/products/acrobat/readstep.html.

A letter of introduction

Dear Students,

This collection of lecture notes is intended as a guide to the material in PHY312: Relativity and cosmology. You are both lucky and unlucky to be taking this course. The point is that this course is essentially unique: I know of no other course anywhere that provides this thorough a treatment of both special and general relativity at a level accessibly with only elementary calculus. By your presence here, I take it that you are happy to have this opportunity!

However, because of the unique nature of this course there is simply no adequate textbook on the market. In the first edition of this course, we used two books (*Relativity* by A. Einstein and *Inside Relativity* by Mook and Vargish) as supplemental reading. However, most of the course content was conveyed directly through the lectures.

Then, in Spring 2000 I decided to type my lecture notes and distribute them to the class in order to provide more relevant reading materials. The aim was to allow students to spend more time listening and thinking during lectures and to reduce their need to devote mental energy to merely copying things said in class or written on the board. According to course evaluations this was a success and students found the notes quite valuable. By the end of the semester, a more or less complete (albeit sketchy) set of notes had been compiled.

Based on the recommendations of Spring 2000's students, I replaced one of the supplemental texts (Mook and Vargish) with this compilation of notes in Spring 2001. We continue to keep the Einstein book to allow you to get some of our story direct from the horse's mouth. However, these notes constitute the main 'text.'

While I have worked to edit these notes and make them more complete, they are still very much 'notes' as opposed to a textbook. Perhaps they will slowly evolve into a true textbook, but such a transformation will take several more years of teaching PHY312. So, please do not expect all of the usual bells and whistles (index, detailed outside references, self-contained treatments of all topics, etc.) Also, mistakes are likely. We caught many of them in 2001 and I am sure that we will catch more this time around!! The good news though (well, depending on your taste) is that I've felt free to write things things in a fairly informal style[1]. Hopefully, this will seem less dry than a real textbook.

While I ask you to be forgiving, I do want to continue to improve these notes for future students. So, if you have any comments, criticisms, or suggestions (ranging from typos to major structural changes), please do tell me about them (just phrase your comments kindly)! You can send them to me at marolf@phy.syr.edu or tell me about them in person. I would like to thank all past PHY312 students for their valuable feedback.

I hope you enjoy both these notes and the course.

Best Wishes,

Don Marolf

How to use these notes

I just want to make one thing very clear: Past students have told me that the notes were invaluable and reasonably well written. However, they also told me that the notes were next to useless without *also* attending the lectures. So, don't get caught in the trap of thinking 'well, I have the notes, maybe it's OK if I skip class today to take care of an important errand....' Yes, I will be lecturing (more or less) 'straight from the notes.' But you can help me to provide a dynamic and more useful class by voicing your questions and participating in class discussions. I'll look forward to talking with you.

Here is the right way to use these notes. By the way, I actually got this from the Spring 2001 students. While they did not actually have the notes until the day of the corresponding lecture (since the notes were not written until then!), this is what they told me they would have liked to do: First, read the notes before the lecture. Not everything will make sense at that point, so you shouldn't get obsessive about really 'getting it' then. However, it is important to get an overview with two goals in mind:

- Get an idea of what the notes contain so that you don't have to try to copy down every word in lecture.

- Identify the hard or subtle bits that will require real concentration in lecture. When you read the notes, think of questions that you would like to ask in lecture.

[1] Among other things, this means I will mark certain important paragraphs with random numbers of stars ($\star$) in order to draw your attention to them.

Maybe the following is a good way to look at it:

When reading the notes before class, the important thing is to identify the *questions*. Then we can answer those questions in class.

I would also recommend bringing these notes to class, so that you can mark them, underline things, and generally annotate the notes with what you learn in class discussions. In particular, you may want to add color to some of the black and white diagrams in the notes. Finally, you will want to use the notes as a reference after class, reading through them again to iron out fine points about which you are confused. Oh, and please do also make use of my office hours so that I can help you with this process. I am told that the Wednesday evening office hours are really, really useful.

Some parting comments

Our motto for the course: *THINK DEEPLY OF SIMPLE THINGS.*

I have shamelessly stolen this motto from one of my old professors (Arnold E. Ross, Dept. of Mathematics, The Ohio State University[2]). He used this for his number theory course, but it applies *at least* as well to relativity. After all, what will do is to spend the whole semester thinking about space and time. What could be simpler and more familiar? We will have to work hard to notice many important subtleties. We will need to think carefully about 'obvious' statements to see if they are in fact true. After all, (in the words of a past student) your view of reality is about to be 'stood on its head and turned inside out.'

Let me provide one final piece of advise. At times, the 'warped and twisted' view of reality that emerges from relativity may cause some despair. Some students feel that they 'just can't understand this stuff.' Often, the greatest obstacle a student faces in the quest for understanding is their idea of what it means to 'understand' something. What does it mean to you? To some it means to explain a new thing in terms of what they already know. Good luck with this in relativity. You will not explain relativity in terms of what you already 'know' about reality because, quite frankly, what you think you 'know' is *wrong.* Sorry, but that's the way it goes.

Then, what do I mean by understanding relativity? Well, first let me agree that obviously those things that you think you know about reality cannot be completely wrong. After all, those 'facts' have served you well all of your life! So, an important step is to grasp how all of the weird stuff of relativity is in fact consistent with your life experiences to date.

But there is another big step in understanding relativity. Your current 'knowledge' of reality is deeply embedded in your intuition. It is a part of the

[2] Arnold Ross died in the summer of 2002 at the age of 96. He actively taught mathematics until after his 94th birthday.

way you deal with almost every thing you encounter. You will have succeeded in understanding relativity when you have built up a *new* intuition based on what Einstein discovered. Yes, this can be done! Intuition is basically a reflex, albeit a mental as opposed to a physical one. We all know how to train our reflexes in sports: practice! Yep, this is a little speech to stress the importance of doing homework, and optional problems too.

Credits

PHY312 has it roots in a course originated by Austin Gleeson at the University of Texas at Austin. During our time there as graduate students and teaching assistants, D. Eric Smith and I had the opportunity to assist in the development of the course and in the design of course materials. I am very grateful to Austin Gleeson both for this experience and for his support during and since. Although PHY312 is a very different course, many of the goals were similar and many of the pedagogical ideas appearing in PHY312 and these notes stem from our work together. In particular, I am indebted to Eric for our many long discussions in which we struggled to find the most elementary way to phrase concepts and arguments while still capturing all of the subtleties inherent in working with curved spacetimes. It is fair to say that I obtained a deeper grasp of relativity though these conversations than from any course. I can only hope my own students have a similar experience, prodded by the homework problems below. Many of these problems, by the way, came originally from the course in Texas and are only slowly evolving over time.

Moving on to Syracuse, I would like to thank Eric Schiff, our current department chair, for encouraging me to develop this course, as well as Peter Saulson for his advice (on both pedagogical and experimental issues) and support. Finally, I would like to thank all of my past students, either at Syracuse or in Texas, for their valuable questions and comments.

Syllabus

For your convenience, I am including the course syllabus as a chapter of these notes. This may not flow especially smoothly into then rest of the book, but I think you'll like having this all together in a single package.

0.1 Introduction

There once was a lady named bright
who traveled much faster than light.
She left home one day
in a relative way
and returned the previous night.

This physics limerick captures an important fact about Einstein's theory of special relativity: according to Einstein, traveling faster than light does not just allow time travel, it is actually *equivalent* to time travel in a certain sense. You may not understand this now (since the course has not yet started) but you will by the end of the semester. This feature, time dilation, the famous $E = mc^2$, curved spacetime, black holes, and cosmology are some of the nifty things we'll be discussing in this course.

However, it is important to stress that this is not a course in science fiction. Relativity is an intrinsic part of such real-world phenomena as the Global Positioning System (GPS), nuclear energy, and the Laser Interferometric Gravitational wave Observatory (LIGO). All of the topics mentioned in the preceding paragraph are important physics concepts and we're going to discuss them *correctly*. Unfortunately, most sci-fi authors get these things wrong, so you'll have to try to forget anything that you may have 'learned' from sci-fi. (Trust me, you'll never watch or read science fiction the same way again after this course!!)

So, what does it take to understand these things properly?? There are two answers: 1) patience and 2) very careful thinking. The topics studied in this course are so far from our everyday experience that the intuition you have built up during your life so far will probably not be helpful to you – in fact, it will be your biggest hindrance. Many of the things we will study are counterintuitive, but this does not mean that they are wrong, unobservable, irrelevant to the 'real world,' or un-understandable. It is important to remind yourself of this early and often and to think very carefully about what you mean when you say (or are tempted to say) that something does or does not 'make sense.' The way to survive in a realm where your intuition does not apply is to learn to think clearly and accurately – that is, to base your reasoning on careful logic – and to build carefully from the few experimental facts you do have.

0.1.1 A Small Warning

Unfortunately, I feel the need to start with a small warning. My assumption is that students take this course because they are interested in the subject and would like to learn the material. I also assume that students are willing to apply themselves seriously to this task. Although the material is challenging, it can be mastered by anyone who can understand a bit of calculus. While it has sometimes been a struggle, there has never been a case of a student working hard in this course who was unable both to understand the subject and to perform well.

On the other hand, this is by no means a blow-off course. Occasionally, students wander into this course under the mistaken impression that this will be an easy way to fulfill a science elective for, say, an engineering degree. Perhaps it is the fact that this course is billed as being accessible to freshmen or the fact that it is billed as de-emphasizing mathematics (in comparison to most treatments of relativity) that draws such students. If you came with this mind, my advice is to either to get out now or to prepare yourself for a serious experience. While this course is not designed to be a bone-crusher, and while I do believe that it can be an excellent course even for non-technical people and non-scientists, Relativity and Cosmology should not be taken lightly.

I have put real work into this course and I will expect you to do the same.

0.1.2 Course Goals

The Goals for this course are:

1) To introduce the basic concepts of special and general relativity and to provide some skill in their use.

2) To promote skill in clear, precise, and analytical thinking.

3) To provide practice in altering one's opinions and intuitive picture of a structure in light of new evidence. In this case, the structure is none other than the 'well-known' framework of space and time.

0.1.3 Course Objectives

The following is a list of some specific skills that you will gain during the coming semester. At the moment, you may not even know what any of these mean but, by the end of the semester you should be able to:

1) Read and interpret a spacetime diagram.

2) Draw your own spacetime diagrams and be able to manipulate them to answer questions about and understand a number of effects in relativity.

3) Understand and be able to use the (local) equivalence principle; understand the basic properties of curved spacetime.

4) Describe the basics of time dilation, length contraction, cosmological models, and black holes.

0.2 Administrative Info

Lectures: Tuesdays and Thursdays 11:30-12:50, Physics Building Rm106. Trust me: you will want to bring a set of colored pens or pencils to class in order to work effectively with complicated diagrams.

Instructor: Don Marolf (Office: 265-3 Physics, office phone: x3882, home phone: 422-3902 [Note: This phone number is given incorrectly in the photocopied Notes from the bookstore. The number given here is correct!!], e-mail: marolf@physics.syr.edu)

Office Hours: Tuesdays 1:00pm - 2:30pm in Physics 263-5 and Wednesdays 7:00pm - 8:00pm+ in Physics 204. On Wednesday, I will generally stay until 10pm or so. However, if no students arrive before 8pm then I will leave and find some other way to spend my evening. If you can only arrive after 8pm and are concerned that I will depart before you can get there, feel free to contact me in advance and let me know that you are coming.

Texts: Unfortunately, there is no true textbook for a course like this.

- The closest things available is the set of *Notes on Relativity and Cosmology* that I have prepared. You can pick up a black and white copy from the bookstore – it's called the PHY312 course reader. You can also download the (color) PDF file from the course website at: http://physics.syr.edu/courses/PHY312.03Spring. It's a big file though (300+ pages).
- We will also use *Relativity* by A. Einstein as a supplementary text. Each chapter of my notes indicates which chapters of Einstein you should be reading simultaneously. This will show you a second perspective on the subject, and I think you will enjoy seeing what Albert himself has to say.

In addition, a number of further supplemental texts are recommended below if you would like to investigate a particular topic more deeply. These may be especially useful for your course project.

0.3 Coursework and Grading

Let me begin by reminding you that this is an upper division (≥ 300) course. This does *not* mean that the course is inaccessible to Freshmen and Sophomores but it *does* mean that students in this course will be treated as and expected to behave as mature learners: demonstrating initiative, asking questions, beginning and handing in homework promptly, and generally taking responsibility for their education. Correspondingly, the framework for this course will not be fixed as rigidly as it would be for any lower division course that I would teach.

The plan for this course is fairly simple. There will be weekly homework, two exams, and a project. The project will be worth 20% of the course grade, the two exams 35% each (making 70% together), and the homework will be worth 20%. Each assignment will receive a letter grade (A, A-, B+, etc.) and these grades will be averaged on a linear scale.

Note that the grading policy in this class may be different from that of other science/math courses you may have taken. While it is always difficult to state what is a 'fair' grading policy, I think it is clear what sorts of grading policies would *not* be reasonable for this course: *un-fair* grading policies would include

i) grading on a strict curve, allowing only 25% of the class to get an "A" no matter how well they do,

ii) grading on the 'usual' A=90-100, B=80-90, etc. scale – the material in this class is 'harder' than that.

You can be sure that neither of these will occur. When your first assignments are returned, you'll be able to see more concretely what the grading scale in this course does look like.

Homework: The homework will be assigned weekly. In general, it will be assigned on a Thursday and will be due the following Thursday. I will warn you in advance if solving some homework problems will require material from the intervening Tuesday's class. While I encourage you to work together on the homework, *the homework you turn in must represent your own understanding.* Homework that is simply copied will not be accepted.

I will hand out solutions to the homework on the day you turn it in. As a result, homework will be turned in on time at the beginning of class. If this is not possible due to some exceptional circumstances, it is your responsibility to contact me before class (as far in advance as possible) to discuss the matter.

Doing the homework will be an important part of learning the material in this course and I strongly recommend that you begin to work on it early and not leave the homework for the last minute. Especially because the topics we study are likely to be beyond your current intuition, you will need to devote some time and effort to working with these new ideas and building up a *new* kind of intuition. The homework is the primary means through which you do this. I will, however, try to keep the workload reasonable for this course.

Problem sets will be graded 'holistically,' so that you will only receive one grade for each assignment (as opposed to grading each problem separately). However, I will write a number of comments on your papers and, in addition, I will provide detailed solutions to most homework problems.

By the way, you are more than welcome to work on the homework together, or to come and work on it at my office hours – *especially the one on Wednesday evening*. The reason that I hold that office hour in a different room is that I would like the room to be available for you to simply come and sit and work on your homework, either individually or in groups. You can ask me questions as they arise. You're also welcome to meet people there to discuss the homework even if you don't plan to ask me anything. Please fell free to consider this as a weekly study and/or chat session for Relativity, Cosmology, and related matters. We can turn on the teapot and have a nice place to work and discuss physics. This has been quite popular in the past, and many students participate. Note: Due to the unique nature of this course, the physics clinic is unlikely to be a useful resource for PHY312.

Exams: The first exam will be Thursday March 6, just before Spring Break. *Note: You have now been warned of this date. Do* **not** *come to me with a story about how your bought a ticket to go somewhere for Spring Break and need to take the exam early so that you can leave.* The date of the second exam will depend on how things go, but will probably be Tuesday, April 22. The second exam will not be (explicitly) cumulative, though of course material from the last part of the course will build on material from the first part.

You should be aware that, despite the fact that there will be no cumulative final exam for this course, we *will* use the time assigned to PHY312 during finals week for you to display and present your course projects. This period is 5:00am -7:00pm, Tuesday May 6. The class time available for this course is far too short not to make use of every possible meeting time.

The Project: A significant part of your work in this class will be a project of your choice. The general guidelines are that the project should

a) in some way demonstrate your understanding of topics covered in this course,

b) be written in your own words,

c) be about the same amount of work as a term paper (which I would consider to be about 10-20 pages in a reasonable type-style like this one) and

d) include references to outside sources as appropriate for the project.

Projects will be graded on a combination of creativity, initiative, and the understanding of course material (and outside reading) that they demonstrate. The rule of thumb should be that your project should be something that you are proud of and not something that is quickly thrown together.

Each project must be approved by me, so you need to discuss your project with me before you begin. You are welcome to continue to consult with me while you work on your project. Each of you need to 'contract' with me for a project by March 25 (shortly after Spring Break). At this point, you and I will have more or less agreed on what your project will be. As a result, you should come and talk to me even earlier in order to bounce a few ideas around before making a final decision. It is a good idea to start thinking about your project early!!! You will be asked to give a (very) short presentation about your project at the end of the semester. **The projects are due at the start of the scheduled final exam period, 5pm on Tuesday, May 6.**

Some suggestions for projects are:

i) A standard report/term paper on a topic related to relativity, gravity, or cosmology.

ii) For web people: building a set of relativity/cosmology web pages either as an electronic 'term paper' or for educational purposes.

iii) For any programmers out there: writing a small computer game (something like asteroids???) that treats relativity correctly. (Be careful here: *several* people have tried this in the past, but have bitten of more than they could really handle.) Another suggestion would be just writing a piece of code the demonstrates some important aspect of gravity, relativity, or cosmology.

iv) For educators: preparing a set of instructional materials in relativity, cosmology, or related subjects for students at a level of your choosing.

v) For the writing/journalism crowd: a short story, poem, or 'magazine article' dealing with some of the topics discussed in this course. Note: Poetry need not be especially good as literature – the limerick on the front page of this syllabus would be of sufficient quality. Poetry may be set to music (i.e., lyrics for a song, possibly 'revised' lyrics to an existing song).

vi) For any artists: A work of visual, audio, or performing art could in principle be an excellent project. One year someone wrote a nice play. However, it must in some way involve your understanding of the course material. If you are interested in this kind of project, be extra sure to discuss it with me carefully.

vii) Anything else you can think of that follows the basic guidelines given above. I'm quite flexible. If you think of an interesting project, I'll be happy to discuss it with you and see if we can make it work!!!

The project is an opportunity for you to get what *you* want out of this course and to extend the course in a direction that you would like it to go. I will therefore expect that your project will demonstrate either substantial creativity on your part or some further reading beyond what we will cover in class. This reading can be from books (such as those suggested in section 0.4) or magazines such as Scientific American, Physics World, or Physics Today. For example, you might wish to study one of the following topics in more detail:

1) black holes (their structure, collisions, or hawking radiation)
2) the big bang
3) higher dimensional theories of physics (Kaluza-Klein theories or 'braneworlds')
4) the history of relativity
5) experimental tests of relativity, or experiments in cosmology. These can be recent experiments or a historical treatment of the early experiments.
6) current and/or future NASA/ESA missions to study relativity and/or cosmology (HUBBLE, CHANDRA, other x-ray satellites, COBE, MAP, PLANCK, LISA, balloon experiments such as last year's Boomerang, and others)
7) the mathematical structure of general relativity
8) gravitational waves
9) gravitational lensing
10) Sagittarius A^* (the black hole at the center of our galaxy)
11) X-ray telescopes and their observations of black holes
12) LIGO
13) Current ideas about 'more fundamental' theories of gravity (loop gravity, string theory, noncommutative geometry).
14) Closed Timelike Curves (aka 'time machines')
15) Wormholes
16) The Global Positioning System

Feel free to talk with me about where and how to locate references on these and other subjects. Some examples of past course projects can be found on the PHY312 web page (see http://physics.syr.edu/courses/PHY312.03Spring). I encourage you to build from what other students have done in the past.

However, your challenge will be to go beyond their work and create something new.

Warning: not all of these projects received high grades. If you look through them, it will probably be clear to you which projects are the best. Feel free to discuss them with me if you would like.

Additional Note: Let me repeat that the project must be your own work. In particular, wholesale copying of entire sentences from your sources is not allowed.

0.3.1 Creating your Project

Following the steps below will help you to create a successful course project.

1. (late Feb. or early March) Begin to think about what you would like to do. Read through the suggestions above and leaf through some of the suggestions for further reading below. Come and talk to me in order to bounce around a few ideas.

2. (by March 25) Settle on a rough plan and have me approve it. This will be our ‘contract.’

3. Read outside sources and begin creative work, consulting with me as appropriate. Prepare a rough draft, perhaps showing it to me. Consult the checklist below during this process.

4. Prepare the final version and have it ready to turn in by 5pm on Tuesday May 6. Again, consult the checklist below to make sure that your project is complete.

5. Be ready to give a *brief* (5 minute or less) summary of your project at 5:00pm on Tuesday, May 6.

0.3.2 Project Checklist

The checklist below will help you to be sure that your project is complete. Remember though, the important thing is that you discuss your project with me and that we agree on what items will be included. The checklist below is not a replacement for my commentary, it is merely a tool to help you interpret my comments an expectations.

Your project should:

1. Include an introduction which a) provides a general overview and b) shows how your project is related to relativity and/or cosmology: Make sure that you describe the connection of your project to these themes. You should do this even if you feel that the connection is completely obvious, though the description can be brief. You and I should agree on the connection in our contract.

2. Be well-defined and focused: Your project should have clearly stated goals and a declared target audience. Is your project an attempt to explain something to the general public, or is it a deep study of some issue? We should agree on this in our contract.

3. Be thorough: Your project should take the time to explore material in some depth. It must reach beyond what we have done in class through the use of your own creativity and/or through the study of additional sources. Again, we should discuss this in our contract. See 8 below if your main project is a poster.

4. Illustrate your understanding of course material: The project must be tied to some of the material discussed in lecture. Your project should show that you understand this material well.

5. Be original: Clearly the project should represent your own work, but in fact I mean more than this. Many of the past course projects are available on the PHY312 web page for your exploration. Your goal should be to add to the set of such projects by doing something new. This may mean doing a different sort of project than has been done before or it may mean improving or extending a previous project.

6. Be well referenced: As a rule of thumb, a project should draw from *at least* 5 outside references. *Also:* Be sure to cite a reference for each separate fact you use from it.

7. Include proper citations: Individual facts, tables, arguments, etc. should be referenced with the source where you found them. Any graphics which are not of your own making should also be individually referenced. You may use either footnotes or endnotes. It is not sufficient to simply put a bibliography at the end.

8. Include something to display on Tuesday, May 6 and something to turn in: I would like you to have something to show the other students which is not just a typed paper. If your main project is a paper, this might be a small poster (the size of a large piece of paper) listing the highlights and including a graphic or two. You should also have something to turn in which shows the depth of your research. For example, if your main project is a poster, you will be able to fit very little on the poster itself. In this case, please also submit a list of "things I wanted to include that would not fit on my poster." Each item on this list should have a short description, at least a couple of sentences long. *I will also want your original project*, including relevant computer files for web pages, computer programs, etc.

9. Be complete: In particular, if you design a web site or write a computer program, be sure that it is actually working before the due date!

0.4 Some Suggestions for Further Reading

Einstein's book *Relativity* covers only the very basics of relativity and cosmology. We will do quite a bit more in this course, and the lecture will be your main source of information for this extra material. This means that, together with the notes that I have prepared, your notes from class will be a primary source of study material. We will work with a number of complicated 'spacetime diagrams' and, while this will require no artistic ability whatsoever, I do recommend that you purchase a set of colored pens or pencils and bring them to class.

Unfortunately, it is difficult to find a single book which includes the right balance of material and is written on the right level for this course. I have chosen Einstein's *Relativity* and the notes that I have prepared as the main texts. A list of suggestions for further reading can be found below. Each of those books does *more* than we will cover in this course for some topics but *less* for others. These books will mainly be helpful if you would like to learn more about some particular aspect of relativity or cosmology, to satisfy your personal curiosity and/or for your course project. The following books are on reserve in the Physics Library (on the second floor of the Physics building). If you like them, you may want to order your own copies through the bookstore, Barnes and Nobles, or amazon.com. In my experience, all of these places charge the same price except that mail order places will charge you a few dollars for shipping. However, mail order services may be a few days faster in getting the book and tend to be more efficient than the SU bookstore.

1. *Cosmology: The Science of the Universe* by Edward Harrison (New York, Cambridge University Press, 1981). This book is written from a less mathematical and more observational perspective than the lectures in PHY312. It gives an excellent treatment of many aspects of cosmology from galaxies and quasars to the cosmic microwave background.

2. *The cosmic frontiers of general relativity* (Boston, Little, Brown, 1971) by W. Kauffman. Kauffman gives a very nice discussion of many topics in or affected by GR such as Black Holes (this part is very thorough but still quite readable!), gravitational lenses, gravitational waves, active galactic nuclei, stars, and neutron-stars.

3. *Flat and Curved Space-times* by George Ellis and Ruth Williams (New York, Oxford University Press, 1988). This is an excellent general book on special and general relativity for those who would like to see more mathematics than we will do in this course. In particular, it has a very nice treatment of black holes.

4. *Principles of Cosmology and Gravitation* by Michael V. Berry (New York, Cambridge U. Press, 1976). Berry's book provides a somewhat more technical treatment of General Relativity than we will experience in this course.

While it is less mathematical than that of Ellis and Williams, Berry does assume a certain familiarity with many concepts of physics. In his own words, the aim of the book is "to present a theoretical framework powerful enough to enable important cosmological formulae to be derived and numerical calculations to be performed." A down side of this book is that it is a bit old.

5. *Discovering Relativity for yourself, with some help from Sam Lilley* by Sam Lilley (New York, Cambridge University Press, 1981). This is an excellent book on the basics of special and general relativity, but it does not cover any advanced topics. It may be a good resource for working homework problems in this course.

6. *A traveler's Guide to Spacetime: An Introduction to the Special Theory of Relativity* by Thomas A. Moore (New York, McGraw-Hill Inc., 1995). This lovely book gives a good, modern, introduction to special relativity and may be helpful for someone who would like to read another discussion of this material. Unfortunately, it does not address accelerated reference frames or General Relativity.

7. *Special Relativity* by Anthony French (New York, Norton, 1968). This is an excellent traditional book on special relativity that will be of interest for those with more physics background. It has some very nice discussions of the experiments related to special relativity but readers should be prepared for a fair amount of algebra.

8. *Time, Space, and Things* by B.K. Ridley (New York, Cambridge University Press, 1984). This is a nice book about special relativity and particle physics which may be useful if you want a very non-technical presentation.

9. *The Ethereal Aether* by L.S. Swenson (Austin, University of Texas Press, 1972). This is a beautiful book about the history of the study of light, and especially of the experiments on the nature, velocity, etc. of light. It is not a relativity book per se, but it discusses a lot of the history of relativity as it discusses the relevant experiments. This includes not only the pre-Einstein history, but also the story of what happened during the time between when Einstein published his work and when it became fully accepted.

10. *Geometry, Relativity, and the Fourth Dimensions* by R. v.B. Rucker, (Dover, New York, 1977). This is a nice book that concentrates on the geometrical aspects of curved space and of four-dimensional spacetime, both flat and curved. It is easier to digest than many of the books above.

11. *Theory and experiment in gravitational physics* by Clifford M. Will, (Cambridge University Press, New York, 1993) QC178.W47 1993. This is the definitive work on experimental tests of General Relativity. A less technical version of the same work is: *Was Einstein Right? putting General Relativity to the test* by Clifford M. Will (Basic Books, New York, 1993).

Course Calendar

The following is a tentative calendar for PHY 312, Spring 2001.

Week 1: (1/14) Background material on pre-relativistic physics (coordinate systems, reference frames, the Newtonian assumptions about time and space, inertial frames, begin Newton's laws).

Week 2: (1/21) Newtonian mechanics: inertial frames and Newton's laws. Electricity, magnetism and waves. The constancy of the speed of light. The ether and the Michelson-Morely experiment.

Week 3: (1/28) The postulates of relativity. What do the postulates of relativity imply? Spacetime diagrams, simultaneity, light cones, and time dilation.

Week 4: (2/4) More work with spacetime diagrams: length contraction and more subtle problems – the train 'paradox,' the interval, proper time and proper distance, a bit on Minkwoskian geometry, the twin 'paradox.'

Week 5: (2/11) More on Minkowskian geometry. Begin acceleration. The headlight effect.

Week 6: (2/18) Acceleration in special relativity.

Week 7: (2/25) Dynamics (forces, energy, momentum, and $E = mc^2$).

Week 8: (3/4) Gravity, light, time, and the local equivalence principle. Exam 1.

Week 9: (3/11) *SPRING BREAK*

Week 10: (3/18) Nonlocal calculations in GR. Time dilation and GPS.

Week 11: (3/25) Curved spaces and curved spacetime.

Week 12: (4/1) The Metric: the mathematical description of a curved surface.

Week 13: (4/8) The Einstein equations and the Schwarzschild solution. The classic tests of GR: Mercury's orbit, the bending of light, and radar time delays. Begin black holes.

Week 14: (4/15) More on Black Holes: inside, outside, etc.

Week 15: (4/22) Second exam. A little cosmology.

Week 16: (4/29) More cosmology. If time permits, we may discuss compact universes, closed timelike curves, the periodic Milne Universe. Kaluza-Klein, higher dimensions, other extensions of Einstein's theories.

Brief Project Presentations: 5pm Tuesday, May 6.

Chapter 1

Space, Time, and Newtonian Physics

Read Einstein, Ch. 1 -6

The fundamental principle of relativity is the constancy of a quantity called c, which is the speed of light in a vacuum[1]:

$$c = 2.998 \times 10^8\, m/s, \quad \text{or roughly} \quad 3 \times 10^8\, m/s.$$

This is fast enough to go around the earth along the equator 7 times each second.

This speed is the same as measured by "everybody." We'll talk much more about just who "everybody" is. But, yes, this principle does mean that, if your friend is flying by at 99% of the speed of light, then when you turn on a flashlight *both* of the following are true:

- The beam advances away from you at $3 \times 10^8 m/s$.
- Your friend finds that the light beam catches up to her, at $3 \times 10^8 m/s$.

Now, this certainly sounds a bit strange. However, saying that something "sounds a bit strange" will not be enough for us in PHY312. We'll want to investigate this more deeply and find out *exactly* where this runs into conflict with our established beliefs.

To do this, we'll have to spend a little bit of time (just a week or so) talking about *'Newtonian'* physics; that is, the way people understood physics before Einstein came along. I know that Newtonian physics is old hat to some of you, but some people here have never studied any physics. In addition, we will emphasize different features than you focussed on if you saw this before in

[1]Light traveling through air, water, etc. does *not* travel at speed c, nor is the speed of light through air, water, etc. constant in the same way that c is.

PHY101 or PHY211. I suggest that you take the opportunity to reflect on this at a deeper level than you may have done before.

'Newtonian' physics is the stuff embodied in the work of Isaac Newton[2]. Now, there were a lot of developments in the 200 years between Newton and Einstein, but an important conceptual framework remained unchanged. It is this framework that we will refer to as Newtonian Physics and, in this sense, the term can be applied to all physics up until the development of Relativity by Einstein. Reviewing this framework will also give us an opportunity to discuss how people came to believe in such a strange thing as the constancy of the speed of light and why you should believe it too. (Note: so far I have given you no reason to believe such an obviously ridiculous statement.)

⋆ Many people feel that Newtonian Physics is just a precise formulation of their intuitive understanding of physics based on their life experiences. Granted, as those of you who have taken PHY101 or 211 know, there are many subtleties. But still, the basic rules of Newtonian physics ought to 'make sense' in the sense of meshing with your intuition.

Oh, you should actively ponder the question"What does it mean for something to 'make sense'?" throughout this course....

1.1 Coordinate Systems

We're going to be concerned with things like speed (e.g., speed of light), distance, and time. As a result, coordinate systems will be very important.

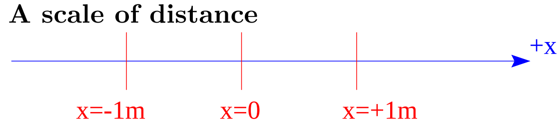
How many of you have worked with coordinate systems?

Let me remind you that a coordinate system is a way of labeling points; say, on a line. You need:

A zero

A positive direction

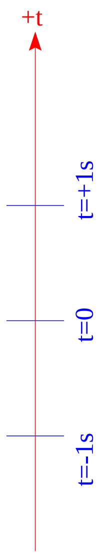
A scale of distance



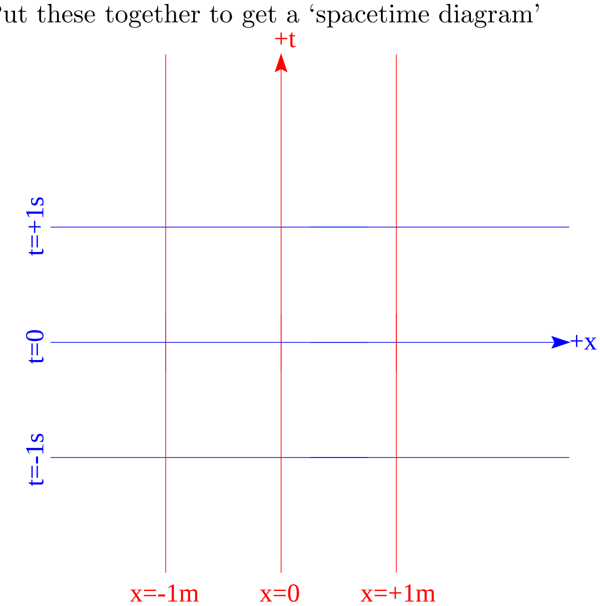
We're going to stick with one-dimensional motion most of the time. Of course, space is 3-dimensional, but 1 dimension is easier to draw and captures some of the most important properties.

In this course, we're interested in space **and time:**

[2]For our purposes, the most important part of this work was actually done by Galileo. However, we will also make use of the refinements added by Newton and the phrase 'Galilean Physics as refined by Newton' is just to long to use.



Put these together to get a 'spacetime diagram'



⋆ Note: For this class (as opposed to PHY211), t increases *upward* and x increases to the *right*. This is the standard convention in relativity and we adopt it so that this course is compatible with all books (and so that I can keep things straight, since it is the convention I am used to!!).

Also note:

- The *x-axis* is the line $t = 0$.
- The *t-axis* is the line $x = 0$.

$\star$ Think of them this way!!! It will make your life easier later.

1.2 Reference Frames

A particular case of interest is when we choose the line $x = 0$ to be the position of some object: e.g. let $x = 0$ be the position of a piece of chalk.

In this case, the coord system is called a *'Reference Frame'*; i.e., the reference frame of the chalk is the (collection of) coordinate systems where the chalk lies at $x = 0$ (All measurements are 'relative to' the chalk.)

$\star\star\star$ Note that I have said nothing about the motion of the chalk. We can talk about the chalk's reference frame whether it is "at rest," moving at constant velocity, or wiggling back and forth in a chaotic way. In both cases we draw the $x = 0$ line as a straight line *in the object's own frame of reference.*

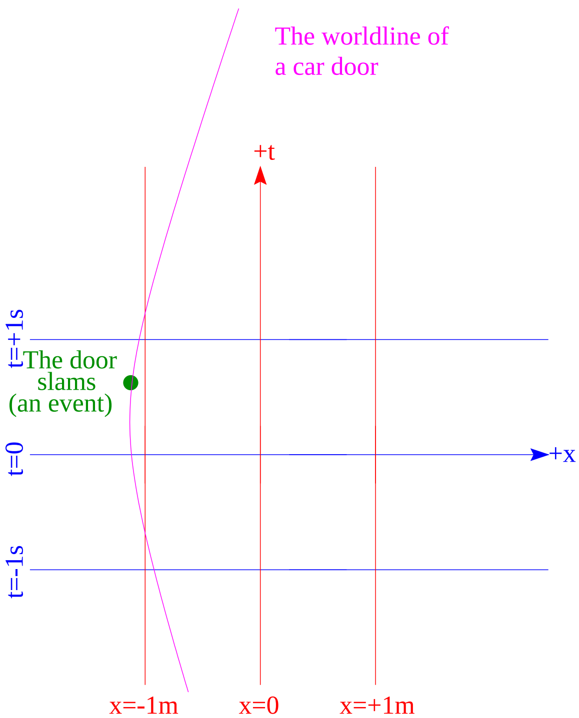
Also: The reference frame of a clock has $t = 0$ whenever the clock reads zero. (If we talk about the reference frame of an object like a piece of chalk, which is not a clock, we will be sloppy about when $t = 0$.)

$\star$ Note: A physicist's clock is really a sort of stopwatch. It reads $t = 0$ at some time and afterwards the reading increases all the time so that it moves toward $+\infty$. Before $t = 0$ it reads some negative time, and the distant past is $-\infty$. A physicist's clock does not cycle from 1 to 12.

Unfortunately, we're going to need a bit more terminology. Here are a couple of key definitions:

- **Your Worldline:** The line representing you on the spacetime diagram. In your reference frame, this is the line $x = 0$.

- **Event:** A point of spacetime; i.e., something with a definite position **and** time. Something drawn as a dot on a spacetime diagram. Examples: a firecracker going off, a door slamming, you leaving a house; see below.



That definition was a simple thing, now let's think deeply about it. Given an event (say, the opening of a door), how do we know where to draw it on a spacetime diagram (say, in your reference frame)? Suppose it happens in our 1-D world.

- *How can we find out what time it happens?*: One way is to give someone a clock and somehow arrange for them be present at the event. They can tell you at what time it happened.

- *How can we find out where (at what position) it happens?*: We could hold out a meter stick (or imagine holding one out). Our friend at the event in question can then read off how far away she is.

Note that what we have done here is to really *define* what we mean by the position and time of an event. This type of definition, where we define something by telling how to measure it (or by stating what a thing does) is called an *operational definition*. They are very common in physics. (*Food for thought:* Are there other kinds of precise definitions? How do they compare?)

Now, the speed of light thing is really weird. So, we want to be very careful in our thinking. You see, *something* is going to go terribly wrong, and we want to be able to see *exactly* where it is.

Let's take a moment to think deeply about this and to act like mathematicians. When mathematicians define a quantity they always stop and ask two questions:

1. Does this quantity actually exist? (Can we perform the above operations and find the position and time of an event?)
2. Is this quantity unique or, equivalently, is the quantity "well-defined?" (Might there be some ambiguity in our definition? Is there a possibility that two people applying the above definitions could come up with two different positions or two different times?)

Well, it seems pretty clear that we can in fact perform these measurements, so the quantities exist. This is one reason why physicists like operational definitions so much.

Now, how well-defined are our definitions are for position and time? [**Stop reading for a moment and think about this.**]

⋆ One thing you might worry about is that clocks and measuring rods are not completely accurate. Maybe there was some error that caused it to give the wrong reading. We will not concern ourselves with this problem. We will assume that there is some real notion of the time experienced by a clock and the length of a rod. Furthermore, we will assume that we have at hand 'ideal' clocks and measuring rods which measure these accurately without mistakes. Our real clocks and rods are to be viewed as approximations to ideal clocks and rods.

OK, what else might we worry about? Well, let's take the question of measuring the time. Can we give our friend just any old ideal clock? No..... it is very important that her clock be *synchronized* with our clock so that the two clocks agree[3].

And what about the measurement of position? Well, let's take an example. Suppose that our friend waits five minutes after the event and then reads the position off of the meter stick. Is that OK? What if, for example, she is moving relative to us so that the distance between us is changing? Ah, we see that it is very important for her to read the meter stick at the time of the event. It is also important that the meter stick be properly 'zeroed' at that same time.

So, perhaps a better definition would be:

time: If our friend has a clock synchronized with ours and is present at an event, then the time of that event in our reference frame is the reading of her clock at that event.

position: Suppose that we have a measuring rod and that, at the time that some event occurs, we are located at zero. Then if our friend is present at that event, the value she reads from the measuring rod at the time the event occurs is the location[4] of the event in our reference frame.

[3] Alternatively, if we knew that her clock was, say, exactly five minutes ahead of ours then we could work with that and correct for it. But the point is that we have to know the relationship of her clock to ours.

[4] Together with a + or - sign which tells us if the event is to the right or to the left. Note that if we considered more than one dimension of space we would need more complicated directional information (vectors, for the experts!).

Now, are these well-defined? After some thought, you will probably say 'I think so.' But, how can we be *sure* that they are well-defined? There are no certain statements without rigorous mathematical proof. So, since we have agreed to think deeply about simple things (and to check all of the subtleties!!!), let us try to prove these statements.

1.3 Newtonian Assumptions about Space and Time

Of course, there is also no such thing as a proof from nothing. This is the usual vicious cycle. Certainty requires a rigorous proof, but proofs proceed only from axioms (a.k.a. postulates or assumptions). So, where do we begin?

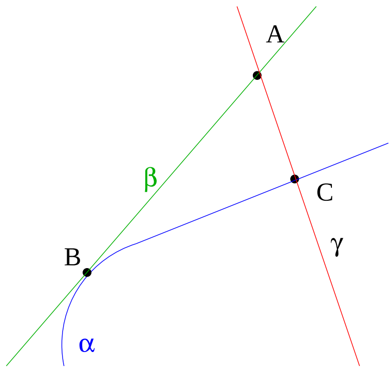
We could simply assume that the above definitions are well-defined, taking these as our axioms. However, it is useful to take even more basic statements as the fundamental assumptions and then prove that position and time in the above sense are well-defined. We take the fundamental Newtonian Assumptions about space and time to be:

T) All (*ideal*) clocks measure the same time interval between any two events through which they pass.

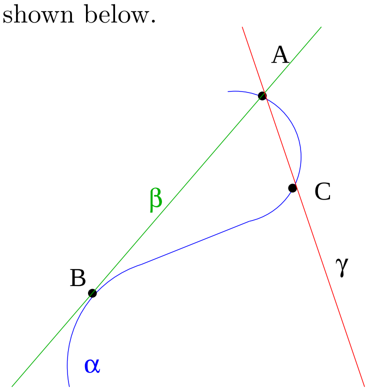
S) Given any two events at the same time, all (ideal) measuring rods measure the same distance between those events.

What do we mean by the phrase 'at the same time' used in (S)? This, after all requires another definition, and we must also check that *this* concept is well-defined. The point is that the same clock will not be present at two different events which occur at the same time. So, we must allow ourselves to define two events as occurring at the same time if any two synchronized clocks pass through these events and, when they do so, the two clocks read the same value. To show that this is well-defined, we must prove that the definition of whether event A occurs 'at the same time' as event B does not depend on exactly which clocks (or which of our friends) pass through events.

Corollary to T: *The time of an event (in some reference frame) is well-defined.* **Proof:** A reference frame is defined by some one clock α. The time of event A in that reference frame is defined as the reading at A on any clock β which passes through A and which has been synchronized with α. Let us assume that these clocks were synchronized by bringing β together with α at event B and setting β to agree with α there. We now want to suppose that we have some other clock (γ) which was synchronized with α at some other event C. We also want to suppose that γ is present at A. The question is, do β and γ read the same time at event A?



Yes, they will. The point is that clock α might actually pass through α as well as shown below.



Now, by assumption **T** we know that α and β will agree at event A. Similarly, α and γ will agree at event A. Thus, β and γ must also agree at event A.

Finally, a proof!!! We are beginning to make progress! Since the time of any event is well defined, the difference between the times of any two events is well defined. Thus, the statement that two events are 'at the same time in a given reference frame' is well-defined!!

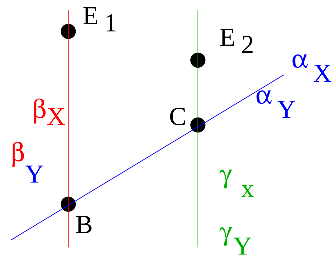
Er,..... But, might two events be at the same time in one reference frame but not in other frames?? Well, here we go again...

Second Corollary to T: Any two sets of synchronized ideal clocks measure the same time interval between a given pair of events.

Proof: From the first corollary, the time of an event defined with respect to a synchronized set of clocks is well-defined no matter how many clocks are in that synchronized set. Thus, we are free to add more clocks to a synchronized set as we like. This will not change the times measured by that synchronized set in any way, but will help us to construct our proof.

So, consider any two events E_1 and E_2. Let us pick two clocks β_X and γ_X from set X that pass through these two events. Let us now pick two clocks β_Y and γ_Y from set Y that follow the same worldlines as β_X and γ_X. If such clocks are not already in set Y then we can add them in. Now, β_X and β_Y were synchronized with some original clock α_X from set X at some events B and C.

Let us also consider some clock α_Y from set Y having the same worldline as α_X. We have the following spacetime diagram:



Note that, by assumption **T**, clocks α_X and α_Y measure the same time interval between B and C. Thus, sets X and Y measure the same time interval between B and C. Similarly, sets X and Y measure the same time intervals between B and E_1 and between C and E_2. Let $T_X(A,B)$ be the time difference between any two events A and B as determined by set X, and similarly for $T_Y(A,B)$. Now since we have both $T_X(E_1,E_2) = T_X(E_1,B) + T_X(B,C) + T_X(C,E_2)$ and $T_Y(E_1,E_2) = T_Y(E_1,B) + T_Y(B,C) + T_Y(C,E_2)$, and since we have just said that all of the entries on the right hand side are the same for both X and Y, it follows that $T_X(E_1,E_2) = T_Y(E_1,E_2)$. QED

In contrast, note that (**S**) basically states directly that position is well-defined.

1.4 Newtonian addition of velocities?

Let's go back and look at this speed of light business. Remember the *99%c* example? Why was it confusing?

Let V_{BA} be the velocity of B as measured by A. Similary V_{CB} is the velocity of C as measured by B and V_{CA} is the velocity of C as measured by A. What relationship would you guess between V_{BA}, V_{CB} and V_{CA}?

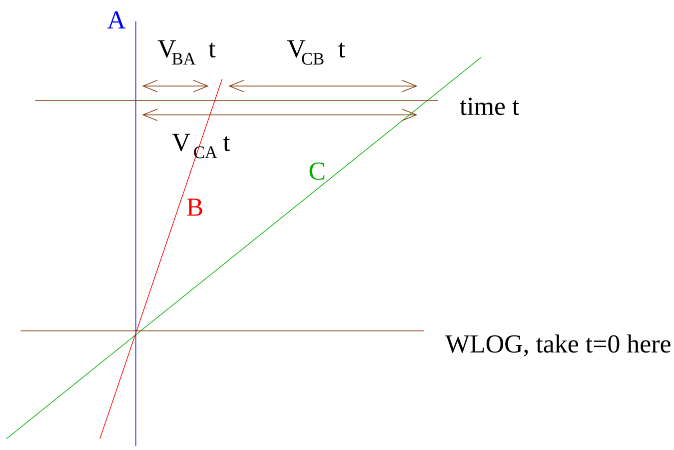
Most likely, your guess was:

$$V_{CA} = V_{CB} + V_{BA}, \tag{1.1}$$

and this was the reason that the 99%c example didn't make sense to you. But do you know that this is the correct relationship? Why should you believe in equation (1.1)?

The answer (still leaving the speed of light example clear as coal tar) is because (1.1) follows from assumptions S and T!!! Proof: Let A, B, C be clocks. For simplicity, suppose that all velocities are constant and that all three clocks pass through some one event and that they are synchronized there. The more general case where this does not occur will be one of your homework problems, so watch carefully!! Without Loss of Generality (WLOG) we can take this event to occur at $t = 0$.

The diagram below is drawn in the reference frame of A:



At time t, the separation between A and C is $V_{CA}t$, but we see from the diagram that it is is also $V_{CB}t + V_{BA}t$. Canceling the t's, we have

$$V_{CA} = V_{CB} + V_{BA}. \tag{1.2}$$

QED

$\star\star\star$ Now, our instructions about how to draw the diagram (from the facts that our ideas about time and position are well-defined) came from assumptions T and S, so the Newtonian formula for the addition of velocities is a *logical consequence* of T and S. If this formula does not hold, then at least one of T and S must be false! Of course, I have still not given you any real evidence to doubt (1.1) – I have only heavily foreshadowed that this will come. It is a good idea to start thinking now, based on the observations we have just made, about how completely any such evidence will make us restructure our notions of reality.

- *Q*: Where have we used T?
- *A:* In considering events *at the same time* (i.e., at time t on the diagram above).
- *Q*: Where have we used S?
- *A:* In implicitly assuming that d_{BC} is same as measured by anyone (A,B, or C).

1.5 Newton's Laws: Are all reference frames equal?

The above analysis was true for *all* reference frames. It made no difference how the clock that defines the reference frame was moving.

However, one of the discoveries of Newtonian Physics was that not all reference frames are in fact equivalent. *There is a special set of reference frames that are called* **Inertial Frames**. This concept will be extremely important for us throughout the course.

Here's the idea:

Before Einstein, physicists believed that the behavior of almost everything (baseballs, ice skaters, rockets, planets, gyroscopes, bridges, arms, legs, cells, ...) was governed by three rules called 'Newton's Laws of Motion.' The basic point was to relate the motion of objects to the 'forces' that act on that object. These laws picked out certain reference frames as special.

The first law has to do with what happens when there are no forces. Consider someone in the middle of a perfectly smooth, slippery ice rink. An isolated object in the middle of a slippery ice rink experiences zero force in the horizontal direction. Now, what will happen to such a person? What if they are moving?

> *Newton's first law of motion:*
> There exists a class of reference frames (called *inertial frames*) in which an object moves in a straight line at constant speed (at time t) if and only if zero (net) force acts on that object at time t.

$\star\star$ Note: When physicists speak about *velocity* this includes both the speed and the direction of motion. So, we can restate this as: There exists a class of reference frames (called *inertial frames*) in which the velocity of any object is constant (at time t) if and only if zero net force acts on that object at time t.

$\star\star\star$ *This is really an operational definition for an inertial frame.* Any frame in which the above is true is called inertial.

$\star$ The qualifier 'net' (in 'net force' above) means that there might be two or more forces acting on the object, but that they all counteract each other and cancel out. An object experiencing zero net force behaves identically to one experiencing no forces at all.

We can restate Newton's first law as:

> *Object A moves at constant velocity in an inertial frame* $\Leftrightarrow$ *Object A experiences zero net force.*

Here the symbol ($\Leftrightarrow$) means 'is equivalent to the statement that.' Trust me, it is good to encapsulate this awkward statement in a single symbol.

1.6 How can you tell if an object is in an inertial frame?

Recall Newton's first Law: There exists a class of reference frames (called *inertial frames*). If object A's frame is inertial, then object A will measure object B to have constant velocity (at time t) if and only if zero force acts on object B at time t.

To tell if you are in an inertial frame, think about watching a distant (very distant) rock floating in empty space. It seems like a safe bet that such a rock has zero force acting on it.

Examples: Which of these reference frames are inertial? An accelerating car? The earth? The moon? The sun? Note that some of these are 'more inertial' than others. $\star\star$ Probably the most inertial object we can think of is a rock drifting somewhere far away in empty space.

It will be useful to have a few more results about inertial frames. To begin, note that an object never moves in its own frame of reference. Therefore, it moves in a straight line at constant (zero) speed in an inertial frame of reference (its own). *Thus it follows from Newton's first law that, if an object's own frame of reference is inertial, zero net force acts on that object.*

Is the converse true? To find out, consider some inertial reference frame. Any object A experiencing zero net force has constant velocity v_A in that frame. Let us ask if the reference frame of A is also inertial.

To answer this question, consider another object C experiencing zero net force (say, our favorite pet rock). In our inertial frame, the velocity v_C of C is constant. Note that (by (1.1)) the velocity of C in the reference frame of A is just $v_C - v_A$, which is constant. Thus, C moves with constant velocity in the reference frame of A!!!! Since this is true for any object C experiencing zero force, A's reference frame is in fact inertial.

We now have:

Object A is in an inertial frame $\Leftrightarrow$ Object A experiences zero force $\Leftrightarrow$ Object A moves at constant velocity in any other inertial frame.

$\star\star\star$ Note that therefore any two inertial frames differ by a constant velocity.

1.7 Newton's other Laws

We will now complete our review of Newtonian physics by briefly discussing Newton's other laws, all of which will be useful later in the course. We'll start with the second and third laws. The second law deals with what happens to an object that does experience a new force. For this law, we will need the following.

Definition of acceleration, a, (of some object in some reference frame): $a = dv/dt$, the rate of change of velocity with respect to time. *Note that this includes* **any** *change in velocity, such as a change in direction.* $\star$ In particular, an object that moves in a circle at a constant speed is in fact *accelerating* in the language of physics.

Newton's Second Law: In *any* inertial frame,
(net force on an object) = (mass of object)(acceleration of object)

$$F = ma.$$

The phrase "in any inertial frame" above means that the acceleration must be measured relative to an inertial frame of reference. By the way, part of your homework will be to show that calculating the acceleration of one object in any two inertial frames always yields identical results. Thus, we may speak about acceleration 'relative to the *class* of inertial frames.'

$\star\star\star$ Note: We assume that force and mass are independent of the reference frame.

On the other hand, Newton's third law addresses the relationship between two forces.

Newton's Third Law: Given two objects (A and B), we have
(force from A on B at some time t) = - (force from B on A at some time t)

$\star\star$ This means that the forces have the same size but act in opposite directions.

Now, this is not yet the end of the story. There are also laws that tell us what the forces actually are. For example, Newton's Law of Universal Gravitation says:

Given any two objects A and B, there is a gravitational force between them (pulling each toward the other) of magnitude

$$F_{AB} = G\frac{m_A m_B}{d_{AB}^2}$$

with $G = 6.673 \times 10^{-11} Nm^2/kg^2$**.**

Important Observation: These laws hold in *any* inertial frame. As a result, there is no special inertial frame that is any different from the others. As a result, it makes no sense to talk about one inertial frame being more 'at rest' than any other. You could never find such a frame, so you could never construct an operational definition of 'most at rest.' Why then, would anyone bother to assume that a special 'most at rest frame' exists?

Note: As you will see in the reading, Newton discussed something called 'Absolute space.' However, he didn't need to and no one really believed in it. We will therefore skip this concept completely and deal with all inertial frames on an equal footing.

The above observation leads to the following idea, which turns out to be much more fundamental than Newton's laws.

Principle of Relativity: The Laws of Physics are the same in all inertial frames.

This understanding was an important development. It ended questions like 'why don't we fall off the earth as it moves around the sun at 67,000 mph?'

Since the acceleration of the earth around the sun is only $.006m/s^2$, the motion is close to inertial. This fact was realized by Galileo, quite awhile before Newton did his work (actually, Newton consciously built on Galileo's observations. As a result, applications of this idea to Newtonian physics are called 'Galilean Relativity.'

Now, the Newtonian Physics that we have briefly reviewed worked like a charm! It lead to the industrial revolution, airplanes, cars, trains, etc. It also let to the prediction and discovery of Uranus and Pluto, and other astronomical bodies. This last bit is a particularly interesting story to which we will return, and I would recommend that anyone who is interested look up a more detailed treatment. However, the success of Newtonian physics is a story for other courses, and we have different fish to fry.

1.8 Homework Problems

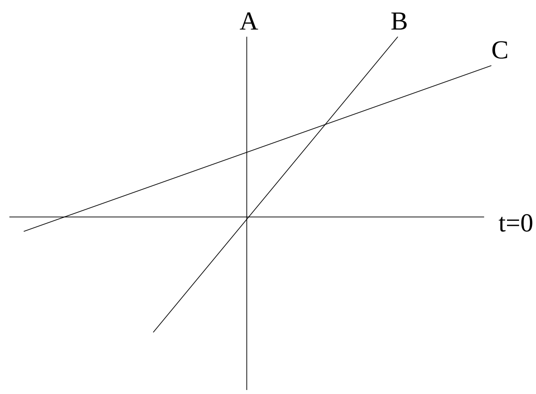
1-1. Suppose that your car is parked in front of a house. You get in, start the engine, and drive away. You step on the gas until the speedometer increases to 30mph, then you hold that reading constant.

Draw two spacetime diagrams, each showing both the house and the car. Draw one in your own frame of reference, and draw the other in the house's frame of reference. Be sure to label both diagrams with an appropriate scale.

1-2. Derive the Newtonian addition of velocities formula

$$v_{CA} = v_{CB} + v_{BA}$$

for the case shown below where the 3 objects (A, B, and C) do *not* pass through the same event. Note that, without loss of generality, we may take the worldlines of A and B to intersect at $t = 0$.



(If you like, you may assume that the velocities are all constant.) Carefully state when and how you use the Newtonian postulates about time and space.

1-3. To what extent do the following objects have inertial frames of reference? Explain how you determined the answer. *Note:* Do *not* tell me that an object is inertial "with respect to" or "relative to" some other object. Such phrases have no meaning as 'being inertial' is not a relative property. If this is not clear to you, please ask me about it!

a) a rock somewhere in deep space.

b) a rocket (with its engine on) somewhere in deep space.

c) the moon.

1-4. Which of the following reference frames are 'as inertial' as that of the SU campus? How can you tell?

a) A person standing on the ground.

b) A person riding up and down in an elevator.

c) A person in a car going around a curve.

d) A person driving a car at constant speed on a long, straight road.

1-5. Show that Newton's second law is consistent with what we know about inertial frames.

That is, suppose that there are two inertial frames (A and B) and that you are interested in the motion of some object (C). Let v_{CA} be the velocity of C in frame A, and v_{CB} be the velocity of C in frame B, and similarly for the accelerations a_{CA} and a_{CB}.

Now, suppose that Newton's second law holds in frame A (so that $F = ma_{CA}$, where F is the force on C and m is the mass of C). Use your knowledge of inertial frames to show that Newton's second law also holds in frame B; meaning that $F = ma_{CB}$. Recall that we assume force and mass to be independent of the reference frame.

1-6. Describe at least one way in which some law of Newtonian physics uses each of the Newtonian assumptions T and S about the nature of time and space.

1-7. In section 1.6 we used Newton's first law and the Newtonian rules for space and time to conclude that the following three statements are equivalent:

i) An object is in an inertial frame of reference.

ii) The net force on that object is zero.

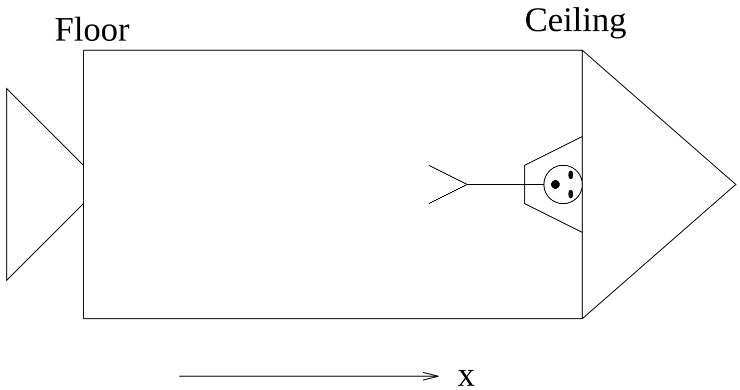
iii) The object moves in a straight line at constant speed in any inertial frame. (Note that this means that the worldline of our object is drawn as a straight line on any spacetime diagram that corresponds to an inertial frame.)

Give the argument for this, filling in any holes that you may have missed in lecture or in reading the text. *Please do this in your own words.* (Hint: The easiest way to prove that all three are equivalent is to prove that 1 implies 2, that 2 implies 3, and that 3 implies 1. That will make a complete cycle and show that any one of the above statements implies the other two).

1-8. It is always good to get more practice working with different reference frames and spacetime diagrams. This problem will provide you with some of that practice before we get to (Einstein) relativity itself.

Suppose that you are in a room in a rocket in deep space. When they are on, the rocket engines cause the rocket to be pushed 'upwards,' in the direction of the 'ceiling.' Draw spacetime diagrams showing the worldlines of the ceiling of the room, the floor of the room, and your head, and your feet in the following situations, using the reference frames specified.

Be sure to start your diagrams a short time before $t = 0$. You need only worry about where things are located in the direction marked x below (in other words, you only need to worry about the 'vertical' positions of the ceiling, floor, etc.). By the way, following our conventions you should draw this position coordinate along the *horizontal* axis, since time should run up the vertical axis.



Note: Your answer should consist of a *single* diagram for each part below.

a) The rocket engines are *off*. You are holding on to the ceiling (as shown below) and then let go at $t = 0$. Draw the diagram using an inertial frame that is at rest (not moving) relative to you before you let go.

b) As in (a), but this time you push yourself away from the ceiling as you let go.

c) This time, the rocket engines are *on* (the whole time, including the time before $t = 0$). You are holding on to the ceiling and let go at $t = 0$. Draw the diagram using an inertial frame that is at rest relative to you at the instant[5] just before you let go.

d) As in (c), but use the reference frame of the rocket.

e) As in (c), again using an inertial frame, but now suppose that you push yourself away from the ceiling as you let go.

f) In which of the above is your frame of reference inertial during the entire time shown on your diagram?

[5] If something is at rest relative to you at some instant, it has the same velocity as you do at that instant, but its acceleration may differ.

Chapter 2

Maxwell, Electromagnetism, and the Ether

Read Einstein, Ch. 7

As we said at the end of the last chapter, Newtonian physics worked just fine. And so, all was well and good until a scientist named Maxwell came along...

The hot topics in physics in the 1800's were electricity and magnetism. Everyone wanted to understand batteries, magnets, lightning, circuits, sparks, motors, and so forth (eventually to make power plants).

2.1 The Basics of E & M

Let me boil all of this down to some simple basics. People had discovered that there were two particular kinds of forces (Electric and Magnetic) that acted only on special objects. (This was as opposed to say, gravity, which acted on all objects.) The special objects were said to be *charged* and each kind of charge (Electric or Magnetic) came in two 'flavors':

electric: + and -

magnetic: N and S (north and south)

Like charges repel and opposite charges attract.

There were many interesting discoveries during this period, such as the fact that 'magnetic charge' is really just electric charge in motion[1].

[1] This turns out to explain why you never find just a N magnetic charge or just a S magnetic charge. They always occur together; any magnet will have both a N and a S magnetic pole. Again, this is another story (really one from PHY212), but ask me if you're interested.

As they grew to understand more and more, physicists found it useful to describe these phenomena not in terms of the **forces** themselves, but in terms of things called *"fields"*. Here's the basic idea:

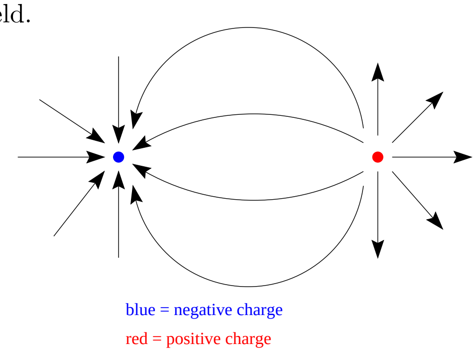
Instead of just saying that X and Y 'repel' or that there is a force between them, we break this down into steps:

- We say that X 'fills the space around it with an electric field E'
- Then, it is this electric field E that produces a force on Y.

(Electric force on Y) = (charge of Y)(Electric field at location of Y)

$$F_{\text{on } Y} = q_Y E$$

Note that changing the sign ($\pm$) of the charge changes the sign of the force. The result is that a positive charge experiences a force in the direction of the field, while the force on a negative charge is opposite to the direction of the field.



The arrows indicate the field. Red (positive charge) moves left with the field. Blue (negative charge) moves right against the field.

Similarly, a magnetic charge fills the space around it with a magnetic field B that then exerts a force on other magnetic charges.

$\star\star$ **Now, you may think that fields have only made things more complicated, but in fact they are a very important concept as they allowed people to describe phenomena which are not directly related to charges and forces.**

For example, the major discovery behind the creation of electric generators was *Faraday's Law.* This Law says that a magnetic field that changes in time produces an electric field. In a generator, rotating a magnet causes the magnetic field to be continually changing, generating an electric field. The electric field then pulls electrons and makes an electric current.

$\star$ Conversely, Maxwell discovered that an *electric* field which changes in time produces a magnetic field. Maxwell codified both this observation and Faraday's law in a set of equations known as, well, Maxwell's equations. Note the closed loop this makes. If we make the electric field change with time in the right way, it produces a magnetic field which changes with time. This magnetic field then produces an electric field which changes with time, which produces a magnetic field which changes with time..... and so on. This phenomenon is called an electromagnetic wave. Electromagnetic waves are described in more detail below in section 2.1.1, and all of their properties follow from Maxwell's equations. For the moment, we merely state an important property of electromagnetic waves: they travel with a precise (finite) speed. See section 2.1.1 for the derivation.

By the way: Consider a magnet in your (inertial) frame of reference. You, of course, find zero electric field. But, if a friend (also in an inertial frame) moves by at a constant speed, they see a magnetic field which 'moves' and therefore changes with time. Thus, Maxwell says that they must see an electric field as well! We see that a field which is purely magnetic in one inertial frame can have an electric part in another. But recall: all inertial frames are supposed to yield equally valid descriptions of the physics.

As a result, it is best not to think of electricity and magnetism as separate phenomena. Instead, we should think of them as forming a *single* "electromagnetic" field which is independent of the reference frame. It is the process of breaking this field into electric and magnetic parts which depends on the reference frame. *There is a strong analogy*[2] *with the following example:* The spatial relationship between the physics building and the Hall of Languages is fixed and independent of any coordinate system. However, if I am standing at the Physics building and want to tell someone how to walk to HOL, depending on which direction I am facing I may tell that person to "walk straight ahead across the quad," or I might tell them to "walk mostly in the direction I am facing but bear a little to the right." The relationship is fixed, but the description differs. For the moment this is just a taste of an idea, but we will be talking much more about this in the weeks to come. In the case of electromagnetism, note that this is consistent with the discovery that magnetic charge is really moving electric charge.

2.1.1 Maxwell's Equations and Electromagnetic Waves

The purpose of this section is to show you how Maxwell's equations lead to electromagnetic waves (and just what this means). This section is more mathematical than anything else we have done so far, and I will probably not go through the details in class. If you're not a math person, you should not be scared off by the calculations below. The important point here is just to get the general picture of how Maxwell's equations determine that electromagnetic waves travel at a constant speed.

Below, I will not use the 'complete' set of Maxwell's equations – instead, I'll use a slightly simplified form which is not completely general, but which is appropriate to the simplest electro-magnetic wave. Basically, I have removed all of the complications having to do with vectors. You can learn about such features in PHY212 or, even better, in the upper division electromagnetism course.

Recall that one of Maxwell's equations (Faraday's Law) says that a magnetic field (B) that changes is time produces an electric field (E). I'd like to discuss

[2]This analogy may be clear as mud at the moment, but will be clearer later in the course as we think about a number of similar effects.

some of the mathematical form of this equation. To do so, we have to turn the ideas of the electric and magnetic fields into some kind of mathematical objects. Let's suppose that we are interested in a wave that travels in, say, the x direction. Then we will be interested in the values of the electric and magnetic fields at different locations (different values of x) and a different times t. As a result, we will want to describe the electric field as a function of two variables $E(x, t)$ and similarly for the magnetic field $B(x, t)$. (As mentioned above, we are ignoring the fact that electric and magnetic fields are vector quantities; that is, that they are like arrows that point in some direction. For the vector experts, I have just picked out the relevant components for discussion here.)

Now, Faraday's law refers to magnetic fields that change with time. How fast a magnetic field changes with time is described by the derivative of the magnetic field with respect to time. For those of you who have not worked with 'multivariable calculus,' taking a derivative of a function of two variables like $B(x, t)$ is no harder than taking a derivative of a function of one variable like $y(t)$. To take a derivative of $B(x, t)$ with respect to t, all you have to do is to momentarily forget that x is a variable and treat it like a constant. For example, suppose $B(x, t) = x^2 t + xt^2$. Then the derivative with respect to t would be just $x^2 + 2xt$. When B is a function of two variables, the derivative of B with respect to t is written $\frac{\partial B}{\partial t}$.

It turns out that Faraday's law does not relate $\frac{\partial B}{\partial t}$ directly to the electric field. Instead, it relates this quantity to the derivative of the electric field *with respect to x*. That is, it relates the time rate of change of the magnetic field to the way in which the electric field varies from one position to another. In symbols,

$$\frac{\partial B}{\partial t} = \frac{\partial E}{\partial x}.. \tag{2.1}$$

It turns out that another of Maxwell's equations has a similar form, which relates the time rate of change of the electric field to the way that the magnetic field changes across space. Figuring this out was Maxwell's main contribution to science. This other equation has pretty much the same form as the one above, but it contains two 'constants of nature' – numbers that had been measured in various experiments. They are called ϵ_0 and μ_0 ('epsilon zero and mu zero'). The first one, ϵ_0 is related to the amount of electric field produced by a charge of a given strength when that charge is in a vacuum. Similarly, μ_0 is related to the amount of *magnetic* field produced by a certain amount of electric current (moving charges) when that current is in a vacuum[3]. The key point here is that

[3]Charges and currents placed in water, iron, plastic, and other materials are associated with somewhat different values of electric and magnetic fields, described by parameters ϵ and μ that depend on the materials. This is due to what are called 'polarization effects' within the material, where the presence of the charge (say, in water) distorts the equilibrium between the positive and negative charges that are already present in the water molecules. This is a fascinating topic (leading to levitating frogs and such) but is too much of a digression to discuss in detail here. See PHY212 or the advanced E & M course. The subscript 0 on ϵ_0 and μ_0 indicates that they are the vacuum values or, as physicists of the time put it, the values for 'free space.'

both of the numbers are things that had been measured in the laboratory long before Maxwell or anybody else had ever thought of 'electromagnetic waves.' Their values were $\epsilon_0 = 8.854 \times 10^{-12} \frac{C^2}{Nm^2}$ and $\mu_0 = 4\pi \times 10^{-7} \frac{Ns^2}{C^2}$.

Anyway, this other Equation of Maxwell's looks like:

$$\frac{\partial E}{\partial t} = \epsilon_0 \mu_0 \frac{\partial B}{\partial x}. \tag{2.2}$$

Now, to understand how the waves come out of all this, it is useful to take the derivative (on both sides) of equation (2.1) with respect to time. This yields some second derivatives:

$$\frac{\partial^2 B}{\partial t^2} = \frac{\partial^2 E}{\partial x \partial t} \tag{2.3}$$

Note that on the right hand side we have taken one derivative with respect to t and one derivative with respect to x.

Similarly, we can take a derivative of equation (2.2) on both sides with respect to x and get:

$$\frac{\partial^2 B}{\partial x^2} = (\epsilon_0 \mu_0) \frac{\partial^2 E}{\partial x \partial t}. \tag{2.4}$$

Here, I have used the interesting fact that it does not matter whether we first differentiate with respect to x or with respect to t: $\frac{\partial}{\partial t}\frac{\partial}{\partial x}E = \frac{\partial}{\partial x}\frac{\partial}{\partial t}E$.

Note that the right hand sides of equations (2.3) and (2.4) differ only by a factor of $\epsilon_0\mu_0$. So, I could divide equation (2.4) by this factor and then subtract it from (2.3) to get

$$\frac{\partial^2 B}{\partial t^2} - \frac{1}{\epsilon_0 \mu_0} \frac{\partial^2 B}{\partial x^2} = 0 \tag{2.5}$$

This is the standard form for a so-called 'wave equation.' To understand why, let's see what happens if we assume that the magnetic field takes the form

$$B = B_0 \sin(x - vt) \tag{2.6}$$

for some speed v. Note that equation (2.6) has the shape of a sine wave at any time t. However, this sine wave moves as time passes. For example, at $t = 0$ the wave vanishes at $x = 0$. On the other hand, at time $t = \pi/2v$, at $x = 0$ we have $B = -B_0$. A 'trough' that used to be at $x = -\pi/2$ has moved to $x = 0$. We can see that this wave travels to the right at constant speed v.

Taking a few derivatives shows that for B of this form we have

$$\frac{\partial^2 B}{\partial t^2} - \frac{1}{\epsilon_0 \mu_0} \frac{\partial^2 B}{\partial x^2} = (v^2 - \frac{1}{\epsilon_0 \mu_0}) B_0 \sin(x - vt). \tag{2.7}$$

This will vanish (and therefore solve equation (2.6)) if (and only if) $v = \pm 1/\sqrt{\epsilon_0 \mu_0}$. Thus, we see that Maxwell's equations do lead to waves, and that those waves travel at a certain speed[4] given by $1/\sqrt{\epsilon_0\mu_0}$. Maxwell realized this, and was curious how fast this speed actually is. Plugging in the numbers that had been found by measuring electric and magnetic fields in the laboratory, he found (as you can check yourself using the numbers above!) $1/\sqrt{\epsilon_0\mu_0} = 2.99... \times 10^8 m/s$. Now, the kicker is that, not too long before Maxwell, people had measured the speed at which light travels, and found that (in a vacuum) this speed was also $2.99... \times 10^8 m/s$!! Coincidence???

Maxwell didn't think so[5]. Instead, he jumped to the quite reasonable conclusion that light actually was a a kind of electromagnetic wave, and that it consists of a magnetic field of the kind we have just been describing (together with the accompanying electric field). We can therefore replace the speed v above with the famous symbol c that we reserve for the speed of light in a vacuum.

Oh, for completeness, I should mention that equations (2.1) and (2.2) show that our magnetic field must be accompanied by an electric field of the form $E = E_0 \sin(x - vt)$ where $E_0 = -B_0 c$.

2.2 The elusive ether

Recall the Principle of Relativity: The Laws of Physics are the same in all inertial frames.

So, the laws of electromagnetism (Maxwell's equations) ought to hold in any inertial reference frame, right? But then light would move at speed c in all reference frames, violating the law of addition of velocities... And this would imply that T and S are wrong!

How did physicists react to this observation? They said "Obviously, Maxwell's equations can only hold in a certain frame of reference." Consider, for example, Maxwell's equations in water. There, they also predict a certain speed for the waves as determined by ϵ and μ in water (which are different from the ϵ_0 and μ_0 of the vacuum). However, here there is an obvious candidate for a particular reference frame with respect to which this speed should be measured: the reference frame of the water itself. Moreover, experiments with moving water did in fact show that $1/\sqrt{\epsilon\mu}$ gave the speed of light through water only when the water was at rest[6]. The same thing, by the way, happens with regular surface waves on water (e.g., ocean waves, ripples on a pond, etc.). There is a wave equation not unlike (2.5) which controls the speed of the waves with respect to the water.

[4] The ($\pm$) sign means that the wave would travel with equal speed to the right or to the left.

[5] Especially since he also found that when ϵ and μ were measured say, in water, $1/\sqrt{\epsilon\mu}$ also gave the speed of light in water. The same holds for any material

[6] However, physicists like Fizeau did find some odd things when they performed these experiments. We will talk about them shortly.

So, *clearly*, c should be just the speed of light 'as measured in the reference frame of the vacuum.' Note that there is some tension here with the idea we discussed before that all inertial frames are fundamentally equivalent. If this is so, one would not expect empty space itself to pick out one as special. To reconcile this in their minds, physicists decided that 'empty space' should not really be completely empty. After all, if it were completely empty, how could it support electromagnetic waves? So, they imagined that all of space was filled with a fluid-like substance called the "*Luminiferous Ether*." Furthermore, they supposed that electromagnetic waves were nothing other than wiggles of this fluid itself.

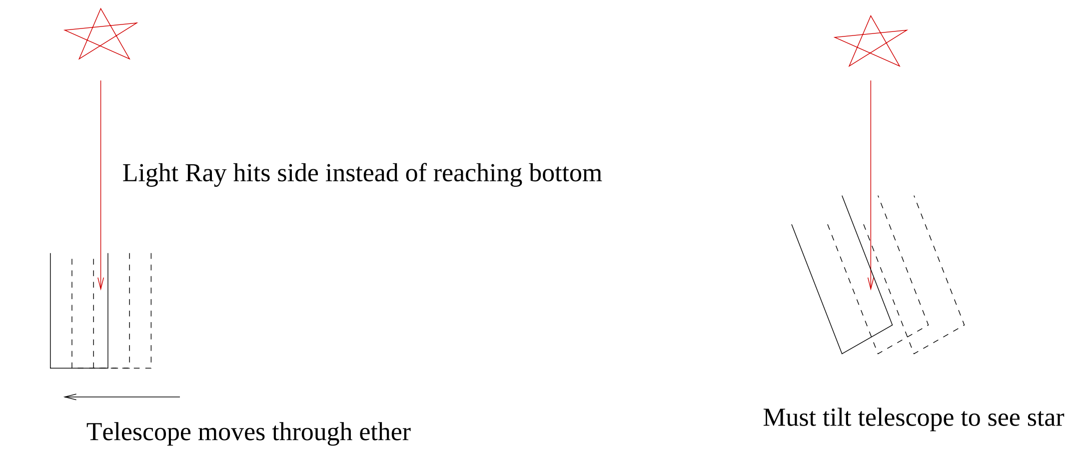
So, the thing to do was to next was to go out and look for the ether. In particular, they wanted to determine what was the ether's frame of reference. Was the earth moving through the ether? Was there an 'ether wind' blowing by the earth or by the sun? Did the earth or sun drag some of the ether with it as it moved through space?

The experiment that really got people's attention was done by Albert Michelson and Edward Morley in 1887. They were motivated by issues about the nature of light and the velocity of light, but especially by a particular phenomenon called the "aberration" of light. This was an important discovery in itself, so let us take a moment to understand it.

2.2.1 The Aberration of Light

Here is the idea: Consider a star very far from the earth. Suppose we look at this star through a telescope. Suppose that the star is "straight ahead" but the earth is moving sideways. Then, we will not in fact see the star as straight ahead. Note that, because of the finite speed of light, if we point a long thin telescope straight at the star, the light will not make it all the way down the telescope but will instead hit the side because of the motion of the earth. A bit of light entering the telescope and moving straight down, will be smacked into by the rapidly approaching right wall of the telescope, even if it entered on the far left side of the opening (see diagram below). The effect is the same as if the telescope was at rest and the light had been coming in at a slight angle so that the light moved a bit to the right. The only light that actually makes it to the bottom is light that is moving at an angle so that it runs away from the oncoming right wall as it moves down the telescope tube.

If we want light from the straight star in front of us to make it all the way down, we have to tilt the telescope. In other words, what we do see though the telescope is not the region of space straight in front of the telescope opening, but a bit of space slightly to the right.

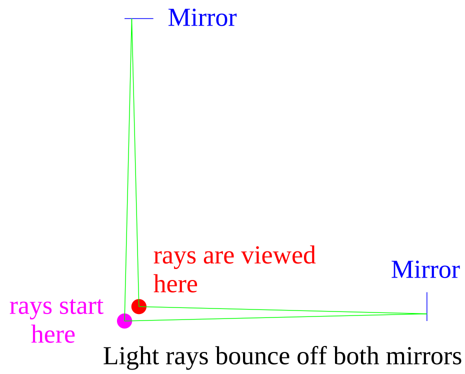


This phenomenon had been measured, using the fact that the earth first moves in one direction around the sun and then, six months later, it moves in the opposite direction. In fact, someone else (Fizeau) had measured the effect again using telescopes filled with water. The light moves more slowly through water than through the air, so this should change the angle of aberration in a predictable way. While the details of the results were quite confusing, *the fact that the effect occurred at all seemed to verify that the earth did move through the ether and, moreover, that the earth did not drag very much of the ether along with it.*

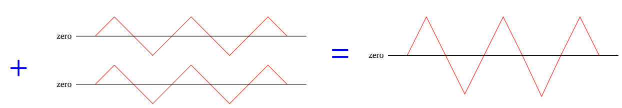
2.2.2 Michelson, Morely, and their experiment

Because of the confusion surrounding the details of Fizeau's results, it seemed that the matter deserved further investigation. Michelson and Morely thought that they might get a handle on things by measuring the velocity of the ether with respect to the earth in a different way. Have a look at their original paper (which I will provide as a handout) to see what they did in their own words.

Michelson and Morely used a device called an interferometer, which looks like the picture below. The idea is that they would shine light (an electromagnetic wave) down each arm of the interferometer where it would bounce off a mirror at the end and return. The two beams are then recombined and viewed by the experimenters. Both arms are the same length, say L.



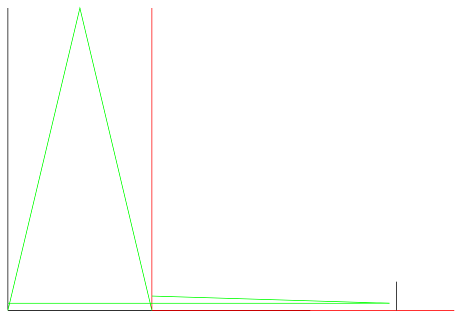
What do the experimenters see? Well, if the earth was at rest in the ether, the light would take the same amount of time to travel down each arm and return. Now, when the two beams left they were synchronized ("in phase"), meaning that wave crests and wave troughs start down each arm at the same time[7]. Since each beam takes the same time to travel, this means that wave crests emerge at the same time from each arm and similarly with wave troughs. Waves add together as shown below[8], with two crests combining to make a big crest, and two troughs combining to make a big trough. The result is therefore a a bright beam of light emerging from the device. This is what the experimenters should see.



On the other hand, if the earth is moving through the ether (say, to the right), then the right mirror runs away from the light beam and it takes the light longer to go down the right arm than down the top arm. On the way back though, it takes less time to travel the right arm because of the opposite effect. A detailed calculation is required to see which effect is greater (and to properly take into account that the top beam actually moves at an angle as shown below).

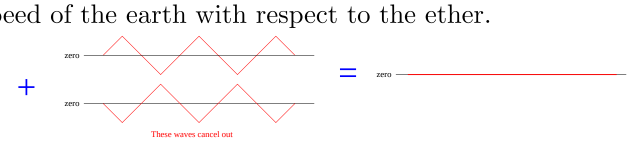
[7] Michelson and Morely achieved this synchronization by just taking one light beam and splitting it into two pieces.

[8] My apologies for the sharp corners, but triangular waves are a lot easier to draw on a computer than sine waves!



This one takes less time.

After doing this calculation one finds[9] that the light beam in the right arm comes back faster than light beam in the top arm. The two signals would no longer be in phase, and the light would not be so bright. In fact, if the difference were great enough that a crest came back in one arm when a trough came back in the other, then the waves would cancel out completely and they would see nothing at all! Michelson and Morely planned to use this effect to measure the speed of the earth with respect to the ether.



However, *they saw no effect whatsoever!* No matter which direction they pointed their device, the light seemed to take the same time to travel down each arm. Clearly, they thought, the earth just happens to be moving with the ether right now (i.e., bad timing). So, they waited six months until the earth was moving around the sun in the opposite direction, expecting a relative velocity between earth and ether equal to twice the speed of the earth around the sun. However, they still found that the light took the same amount of time to travel down both arms of the interferometer!

So, what did they conclude? They thought that maybe the ether is dragged along by the earth... But then, how would we explain the aberration effects?

Deeply confused, Michelson and Morely decided to gather more data. Despite the aberration effects, they thought the earth must drag some ether along with it. After all, as we mentioned, the details of the aberration experiments were a little weird, so maybe the conclusion that the earth did not drag the ether was not really justified..... If the earth did drag the ether along, they thought there might be less of this effect up high, like on a mountain top. So, they repeated their experiment at the top of a mountain. Still, they found no effect. There then followed a long search trying to find the ether, but no luck. Some people were still trying to find an ether 'dragged along very efficiently by everything' in the 1920's and 1930's. They never had any luck.

[9]See homework problem 2-2.

2.3 Homework Problems

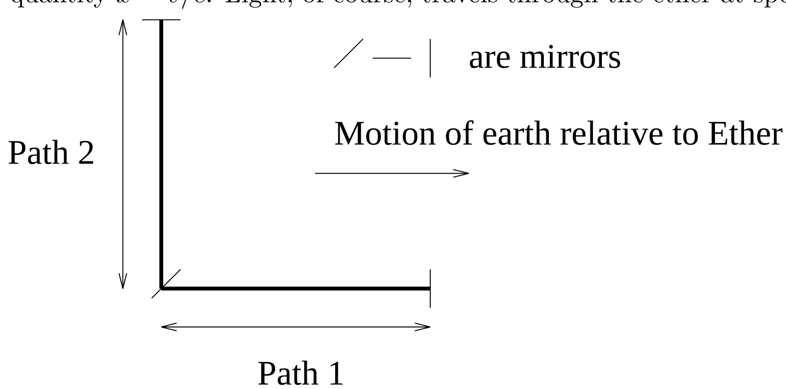
2-1. Calculate $1/\sqrt{\epsilon_0\mu_0}$ from the values given in the text.

2-2. Following the framework of Newtonian physics, suppose that that earth is moving through the luminiferous ether at a speed v. Suppose that you are Michelson (of the famous experiment) and that you have an interferometer with two arms, each of length L as shown below. If one arm is directed along the direction of the earth's motion and one arm is directed in a perpendicular direction,

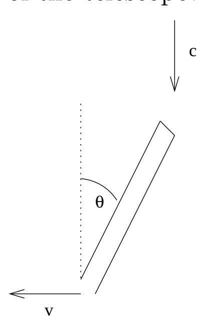
i How long will it take a bit of light to make a complete circuit (out and back) along each arm??

ii How big is the difference between these two times?

You will need the following 'expansions' (formula's that are approximately true when x is small): $(1+x)^{-1} = 1 - x + x^2$, $(1-x)^{-1} = 1 + x + x^2$, and $(1-x^2)^{-1/2} = 1 + \frac{1}{2}x^2$. You will need these formulas for the small quantity $x = v/c$. Light, of course, travels through the ether at speed c.



2-3. Calculate the size of the stellar aberration effect in Newtonian physics for light coming straight down at speed c and a telescope moving straight sideways at speed v. At what angle θ must we point the telescope to see the star? Hint: In what direction is the light moving in the reference frame of the telescope?



Chapter 3

Einstein and Inertial Frames

Read Einstein, ch. 8-14, Appendix 1

In the last chapter we learned that *something is very wrong.* The stellar aberration results indicate that the earth is moving through the ether. But, Michelson and Morely found results indicating that the ether is dragged along **perfectly** by the earth. All of the experiments seem to have been done properly, so how can we understand what is going on? Which interpretation is correct?

3.1 The Postulates of Relativity

In 1905, Albert Einstein tried a different approach. He asked *"What if there is no ether?"* What if the speed of light in a vacuum really is the same in every inertial reference frame? He soon realized, as we have done, that this means that we must abandon **T** and **S**, our Newtonian assumptions about space and time.

Hopefully, you are sufficiently confused by the Michelson and Morely and aberration results that you will agree to play along with Einstein for awhile. This is what we want to do in the next few sections. We will explore the consequences of Einstein's idea. Surprisingly, one can use this idea to build a consistent picture of what is going on that explains both the Michelson-Morely and the aberration results. It turns out that this idea makes a number of other weird and ridiculous-sounding predictions as well. Perhaps even more surprisingly, these predictions have actually been confirmed by countless experiments over the last 100 years.

We are about to embark on a very strange path, one that runs counter to the intuition that we accumulate in our daily lives. As a result, we will have to tread carefully, taking the greatest care with our logical reasoning. As we saw in chapter 1, careful logical reasoning can only proceed from clearly stated assumptions (a.k.a. 'axioms' or 'postulates'). We're throwing out almost everything that we thought we understood about space and time. So then, what should we keep?

We'll keep the bare minimum consistent with Einstein's idea. We will take our postulates to be:

I) The laws of physics are the same in every inertial frame.

II) The speed of light in an inertial frame is always $c = 2.99... \times 10^8 m/s$.

We also keep Newton's first law, which is just the definition of an inertial frame:

There exists a class of reference frames (called *inertial frames*) in which an object moves in a straight line at constant speed if and only if zero net force acts on that object.

Finally, we will need a few properties of inertial frames. We therefore postulate the following familiar statement.

Object A is in an inertial frame $\Leftrightarrow$ *Object A experiences zero force* $\Leftrightarrow$ *Object A moves at constant velocity in any other inertial frame.*

Since we no longer have S and T, we can no longer derive this last statement. It turns out that this statement does in fact follow from even more elementary (albeit technical) assumptions that we could introduce and use to derive it. This is essentially what Einstein did. However, in practice it is easiest just to assume that the result is true and go from there.

Finally, it will be convenient to introduce a new term:

Definition An "observer" is a person or apparatus that makes measurements.

Using this term, assumption II becomes: The speed of light is always $c = 2.9979 \times 10^8 m/s$ as measured by any inertial observer.

By the way, it will be convenient to be a little sloppy in our language and to say that two observers with zero relative velocity are in the same reference frame, even if they are separated in space.

3.2 Time and Position, take II

Recall that in Chapter 1 we used the (mistaken!) old assumptions **T** and **S** to show that our previous notions of time and position were well-defined. Thus, we can no longer rely even on the definitions of 'time and position of some event in some reference frame' as given in chapter 1. We will need new definitions based on our new postulates.

For the moment, let us stick to inertial reference frames. What tools can we use? We don't have much to work with. The only assumption that deals with time or space at all is postulate II, which sets the speed of light. Thus, we're going to somehow base out definitions on the speed of light.

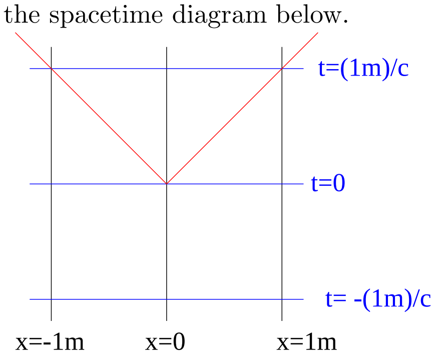
We will use the following:

To define position in a given inertial frame: Build a framework of measuring rods and make sure that the zero mark always stays with the object that defines the (inertial) reference frame. Note that, once we set it up, this framework will move with the inertial observer without us having to apply any forces. The measuring rods will move with the reference frame. An observer (say, a friend of ours who rides with the framework) at an event can read off the position (in this reference frame) of the event from the mark on the rod that passes through that event.

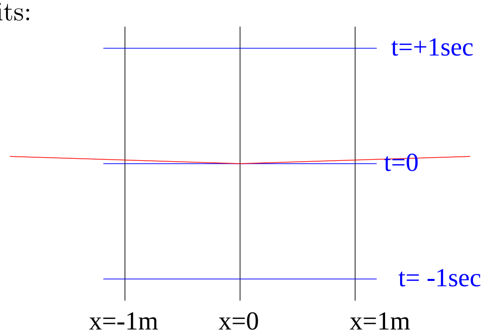
To define time in a given inertial frame: Put an ideal clock at each mark on the framework of measuring rods above. Keep the clocks there, moving with the reference frame. The clocks can be synchronized with a pulse of light emitted, for example, from $t = 0$. A clock at x knows that, when it receives the pulse, it should read $|x|/c$.

These notions are manifestly well-defined. We do not need to make the same kind of checks as before as to whether replacing one clock with another would lead to the same time measurements. This is because the rules just given do not in fact allow us to use any other clocks, but only the particular set of clocks which are bolted to our framework of measuring rods. Whether other clocks yield the same values is still an interesting question, but not one that affects whether the above notions of time and position of some event *in a given reference frame* are well defined.

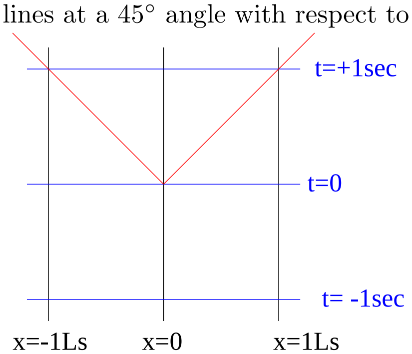
Significantly, we have used a different method here to synchronize clocks than we did in chapter 1. The new method based on a pulse of light is available now that we have assumption II, which guarantees that it is an accurate way to synchronize clocks in an inertial frame. This synchronization process is shown in the spacetime diagram below.



Note that the diagram is really hard to read if we use meters and seconds as units:



Therefore, it is convenient to use units of seconds and *light-seconds*: $1Ls = (1\ sec)c = 3x \times 10^8 m = 3 \times 10^5 km$. This is the distance that light can travel in one second, roughly 7 times around the equator. Working in such units is often called "choosing $c = 1$," since light travels at $1Ls/sec$. We will make this choice for the rest of the course, so that light rays will always appear on our diagrams as lines at a 45° angle with respect to the vertical; i.e. $slope = 1$.

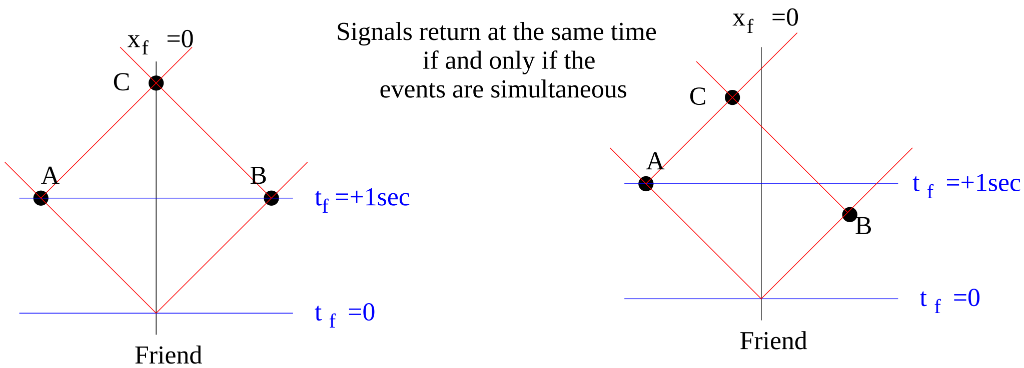


3.3 Simultaneity: Our first departure from Galileo and Newton

The above rules allow us to construct spacetime diagrams in various reference frames. An interesting question then becomes just how these diagrams are related. Let us start with an important example. Back in chapter 1, we went to some trouble to show that the notion that two events happen 'at the same time' does not depend on which reference frame (i.e., on which synchronized set of clocks) we used to measure these times. Now that we have thrown out **T** and **S**, will this statement still be true?

Let us try to find an operational definition of whether two events occur 'simultaneously' (i.e., at the same time) in some reference frame. We can of course read the clocks of our friends who are at those events and who are in our reference frame. However, it turns out to be useful to find a way of determining which events are simultaneous with each other directly from postulate II, the one about the speed of light. Note that there is no problem in determining whether or not two things happen (like a door closing and a firecracker going off) at the same *event*. The question is merely whether two things that occur at different events take place simultaneously.

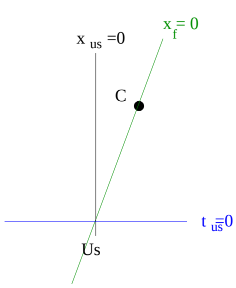
Suppose that we have a friend in an inertial frame and that she emit a flash of light from her worldline. The light will travel outward both to the left and the right, always moving at speed c. Suppose that some of this light is reflected back to her from event A on the left and from event B on the right. The diagram below makes it clear that the two reflected pulses of light reach her at the same time if and only if A and B are simultaneous. So, if event C (where the reflected pulses cross) lies on her worldline, she knows that A and B are in fact simultaneous in our frame of reference.



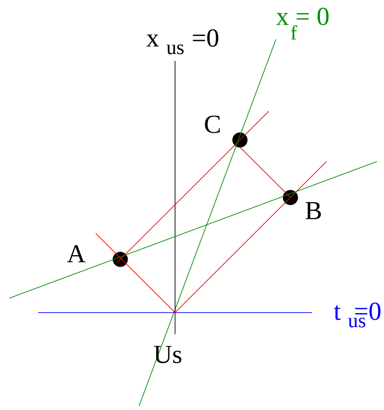
In fact, even if we are in a different reference frame, *we* can tell that A and B are simultaneous in our friend's frame if event C lies on her worldline.

Suppose that we are also inertial observers who meet our friend at the origin event and then move on. What does the above experiment look like in our frame? We'll take the case of the figure on the left above.

Let's start by drawing our friend's worldline and marking event C. We don't really know where event C should appear, but it doesn't make much difference since I have drawn no scale on the diagram below. All that matters is that event C is *on* our friend's worldline ($x_f = 0$).

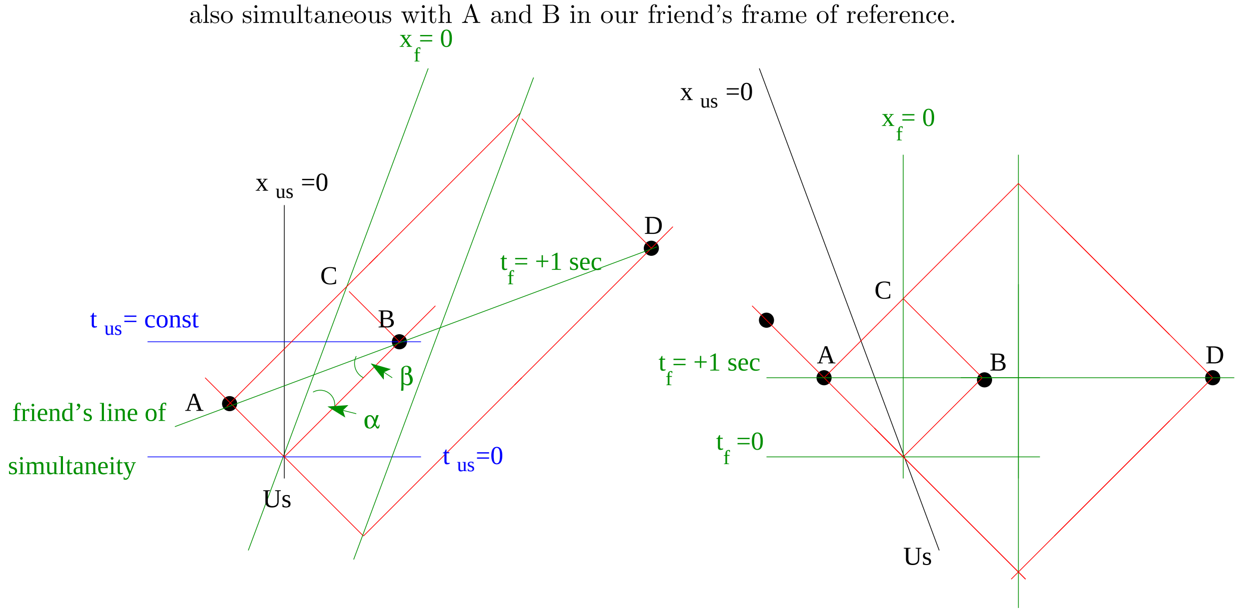


Now let's add the light rays (45° lines) from the origin and from event C. The events where these lines cross must be A and B, as shown below.



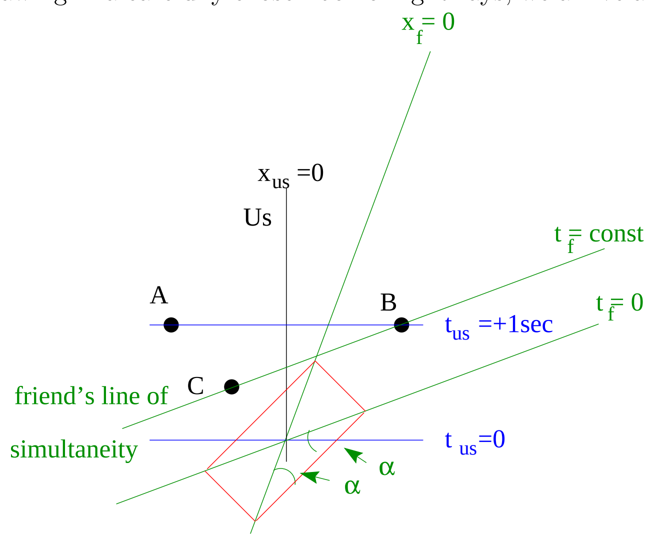
Note that, on *either diagram*, the worldline $x_f = 0$ makes the same angle with the light cone as the line of simultaneity $t_f = const$. That is, the angles α and β above are equal. You will in fact derive this in one of your homework problems.

$\star\star$ By the way, we also can find other pairs of events on our diagram that are simultaneous in our friend's reference frame. We do this by sending out light signals from another observer in the moving frame (say, located $1m$ to the right of our friend). For example, the diagram below shows another event (D) that is also simultaneous with A and B in our friend's frame of reference.



In this way we can map out our friend's entire line of simultaneity – the set of all events that are simultaneous with each other in her reference frame. The result is that the line of simultaneity for the moving frame does indeed appear as a straight line on our spacetime diagram. This property will be very important in what is to come.

Before moving on, let us get just a bit more practice and ask what set of events our friend (the moving observer) finds to be simultaneous with the origin (the event where the her worldline crosses ours)? We can use light signals to find this lines as well. Let's label that line $t_f = 0$ under the assumption that our friend chooses to set her watch to zero at the event where the worldlines cross. Drawing in a carefully chosen box of light rays, we arrive at the diagram below.



Note that we could also have used the rule noticed above: that the worldline and any line of simultaneity make equal angles with the light cone.

As a final comment, note that while we know that the line of simultaneity drawn above ($t_f = const$) represents *some* constant time in the moving frame, we do not yet know which time that is! In particular, we do not yet know whether it represents a time greater than one second or a time less than on second. We were able to label the $t_f = 0$ line with an actual value only because we explicitly *assumed* that our friend would measure time from the event (on that line) where our worldlines crossed. We will explore the question of how to assign actual time values to other lines of simultaneity shortly.

Summary: We have learned that events which are simultaneous in one inertial reference frame are not in fact simultaneous in a different inertial frame. We used light signals and postulate II to determine which events were simultaneous in which frame of reference.

3.4 Relations between events in Spacetime

It will take some time to absorb the implications of the last section, but let us begin with an interesting observation. Looking back at the diagrams above, note that a pair of events which is separated by "pure space" in one inertial frame (i.e., is simultaneous in that frame) is separated by *both* space and time in another. Similarly, a pair of events that is separated by "pure time" in one frame (occurring at the same location in that frame) is separated by both space and time in any other frame. This may remind you a bit of our discussion of electric and magnetic fields, where a field that was purely magnetic in one frame involved both electric and magnetic parts in another frame. In that case we decided that is was best to combine the two and to speak simply of a single "electromagnetic" field. Similarly here, it is best not to speak of space and time separately, but instead only of "spacetime" as a whole. The spacetime separation is fixed, but the decomposition into space and time depends on the frame of reference.

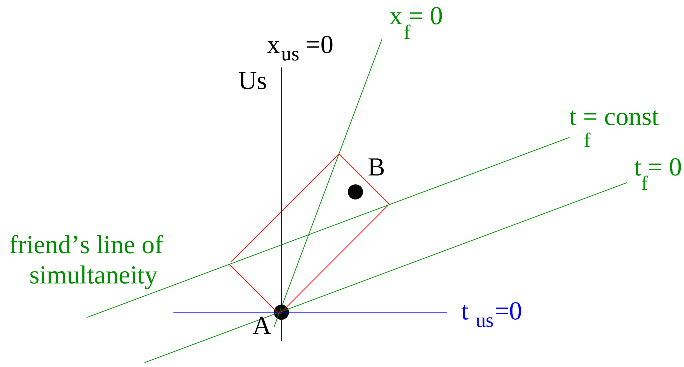
Note the analogy to what happens when you turn around in space. The notions of Forward/Backward vs. Right/Left get mixed up when you turn (rotate) your body. If you face one way, you may say that the Hall of Languages is "straight ahead." If you turn a bit, you might say that the Hall of Languages is "somewhat ahead and somewhat to the left." However, the separation between you and the Hall of Languages is the same no matter which way you are facing. As a result, Forward/Backward and Right/Left are not strictly speaking separate, but rather fit together to form two-dimensional space.

This is *exactly* what is meant by the phrase "space and time are not separate, but fit together to form four-dimensional spacetime." As a result, "time is the fourth dimension of spacetime."

So then, how do we understand the way that events are related in this spacetime? In particular, we have seen that simultaneity is not an absolute concept in spacetime itself. There is no meaning to whether two events occur at the same time unless we state which reference frame is being used. If there is no absolute meaning to the word 'simultaneous,' what about 'before' and 'after' or 'past' and 'future?'

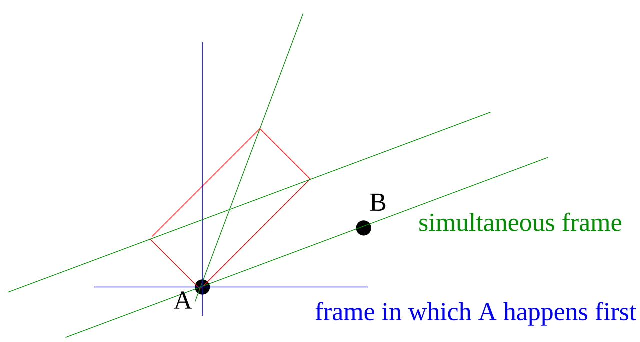
Let's start off slowly. We have seen that if A and B are simultaneous in your (inertial) frame of reference (but are not located at the same place), then there is another inertial frame in which A occurs before B. A similar argument (considering a new inertial observer moving in the other direction) shows that there is another inertial frame in which B occurs before A. Looking back at our diagrams, the same is true if A occurs just slightly before B in your frame of reference.....

However, this does not happen if B is on the light cone emitted from A, or if B is *inside* the light cone of A. To see this, remember that since the speed of light is $c = 1$ in *any* inertial frame, the light cone looks the same on everyone's spacetime diagram. A line more horizontal than the light cone therefore represents a 'speed' greater than c, while a line more vertical than the light cone represents a speed less than c. Because light rays look the same on everyone's spacetime diagram, the distinction between these three classes of lines must also be the same in all reference frames.



Thus, it is worthwhile to distinguish three classes of relationships that pairs of events can have. These classes and some of their properties are described below. Note that in describing these properties we limit ourselves to inertial reference frames that have a relative speed less than that of light[1].

case 1) A and B are **outside** each other's light cones.



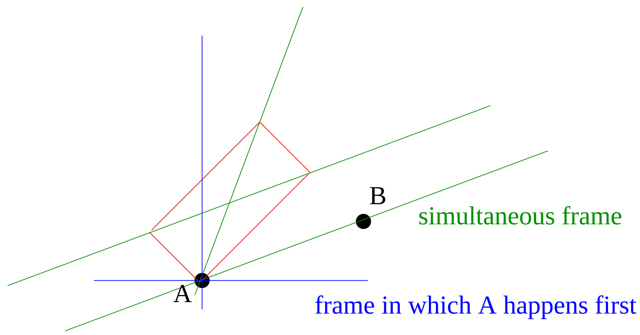
In this case, we say that they are *spacelike* related. Note that the following things are true in this case:

a) There is an inertial frame in which A and B are simultaneous.

b) There are also inertial frames in which event A happens first as well as frames in which event B happens first (even more tilted than the simultaneous frame shown above). However, A and B remain outside of each other's light cones in all inertial frames.

case 2) A and B are *inside* each other's light cones in all inertial frames.

[1]We will justify this in a moment.



In this case we say that they are *timelike* related. Note that the following things are true in this case:

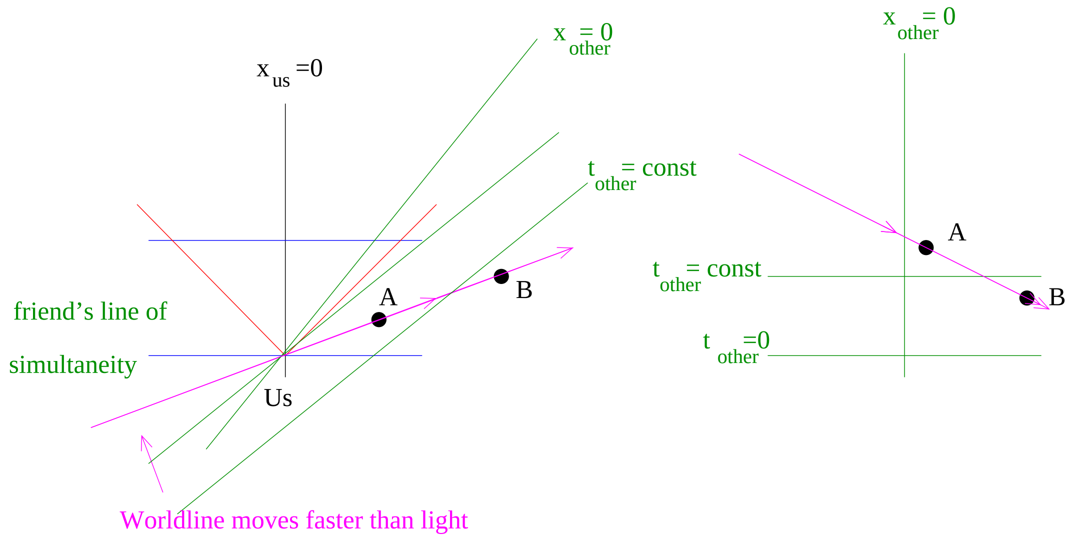
a) There is an inertial observer who moves through both events and whose speed in the original frame is less than that of light.

b) All inertial observers agree on which event (A or B) happened first.

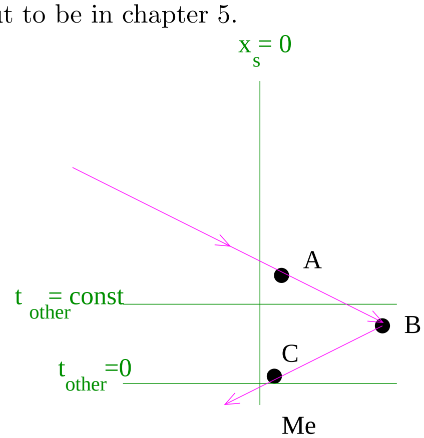
c) As a result, we can meaningfully speak of, say, event A being to the past of event B.

case 3) A and B are **on** each other's light cones. In this case we say that they are *lightlike* related. Again, all inertial observers agree on which event happened first and we can meaningfully speak of one of them being to the past of the other.

Now, why did we consider only inertial frames with relative speeds less than c? Suppose for the moment that our busy friend (the inertial observer) could in fact travel at $v > c$ (i.e., faster than light) as shown below at left. I have marked two events, A and B that occur on her worldline. In our frame event A occurs first. However, the two events are spacelike related. Thus, there is another inertial frame (t_{other}, x_{other}) in which B occurs before A as shown below at right. This means that there is some inertial observer (the one whose frame is drawn at right) who would see her traveling backwards in time.

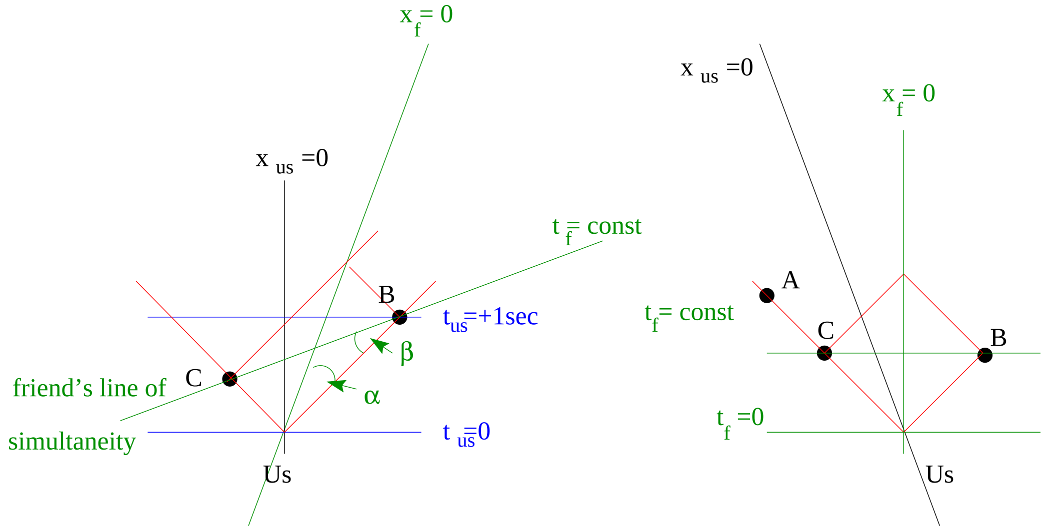


This was too weird even for Einstein. After all, if she could turn around, our faster-than-light friend could even carry a message from some observer's future into that observer's past. This raises all of the famous 'what if you killed your grandparents' scenarios from science fiction fame. The point is that, in relativity, travel faster than light *is* travel backwards in time. For this reason, let us simply ignore the possibility of such observers for awhile. In fact, we will assume that no information of any kind can be transmitted faster than c. I promise that we will come back to this issue this later. The proper place to deal with this turns out to be in chapter 5.



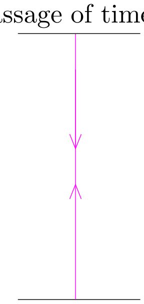
3.5 Time Dilation

We are beginning to come to terms with simultaneity but, as pointed out earlier, we are still missing important information about how different inertial frames match up. In particular, we still do not know just what value of constant t_f the line marked “friend’s line of simultaneity” below actually represents.



In other words, we do not yet understand the rate at which some observer’s clock ticks in another observer’s reference frame.

To work this out, it is useful to tie time measurements directly to our axioms, just as we found it useful to tie simultaneity directly to our axioms in section 3.4. That is, we should somehow make a clock out of light! For example, we can bounce a beam of light back and forth between two mirrors separated by a known distance. Perhaps we imagine the mirrors being attached to a rod of fixed length L. Since we know how far apart the mirrors are, we know how long it takes a pulse of light to travel up and down and we can use this to mark the passage of time. We have a clock.



3.5.1 Rods in the perpendicular direction

A useful trick is to think about what happens when this ‘light clock’ is held perpendicular to the direction of relative motion. This direction is simpler than the direction of relative motion itself. For example, two inertial observers actually do agree on which events are simultaneous in that direction. To see this, suppose that I am moving straight toward you (from the front) at some constant speed. Suppose that you have two firecrackers, one placed one light second to your left and one placed one light second to your right. Suppose that both explode at the same time in your frame of reference. Does one of them explode earlier in mine?

No, and the easiest way to see this is to argue by *symmetry*: the only difference between the two firecrackers is that one is on the right and the other is on the left. Since the motion is forward or backward, left and right act exactly the same in this problem. Thus, the answer to the question 'which is the earliest' must not distinguish between left and right. But, there are only three possible answers to this question: left, right, and neither. Thus, the answer must be 'neither', and both firecrackers explode at the same time in our reference frame as well[2].

Now, suppose we ask about the length of the meter sticks. Let's ask whose meter stick you measure to be longer. For simplicity, let us suppose that you conduct the experiment at the moment that the two meter sticks are in contact (when they "pass through each other"). Note that both you and I agree on which events constitute the meter sticks passing through each other, since this involves only simultaneity in the direction along the meter sticks and, in the present case, this direction is perpendicular to our relative velocity.

On the one hand, since we both agree that we are discussing the same set of events, we must also agree on which meter stick is longer. This is merely a question of whether the event at the end of your meter stick is inside or outside of the line of events representing my meter stick.

Said more physically, suppose that we put a piece of blue chalk on the end of my meter stick, and a piece of red chalk on the end of yours. Then, after the meter sticks touch, we must agree on whether there is now a blue mark on your stick (in which case yours in longer), there is a red mark on my stick (in which case that mine is longer), or whether each piece of chalk marked the very end of the other stick (in which case they are the same length).

On the other hand, the laws of physics are the same in all inertial frames. In particular, suppose that the laws of physics say that, if you (as an inertial observer) take a meter stick 1m long in its own rest frame and move it toward you, then that that meter stick appears to be *longer* than a meter stick that is at rest in your frame of reference. Here we assume that it does not matter in which direction (forward or backward) the meter stick is moving, as all direction in space are the same. In that case, the laws of physics must *also* say that, if I (as an inertial observer) take a meter stick 1m long in its own rest frame and move it toward me, that that meter stick again appears to be *longer* than a meter stick that is at rest in **my** of reference. Thus, if you find my stick to be longer, I must find your stick to be longer. If you find my stick to be shorter,

[2]It is useful to contrast this with what happens in the direction of relative motion. There *is* a physical difference between the end 'in front' (i.e., in the direction of the relative velocity) and the end 'in back' (in the direction opposite to the relative velocity). So, in that case it was consistent for the answer to be that the event *in front* of the moving observer occurs later. We say that the velocity *breaks* (i.e., destroys) the symmetry between front and back while the right/left symmetry remains.

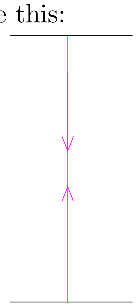
then I must find your stick to be shorter. As a result, consistency requires both of us find the two meter sticks to be of the same length.

We conclude that the length of a meter stick is the same in two inertial frames for the case where the stick points in the direction perpendicular to the relative motion.

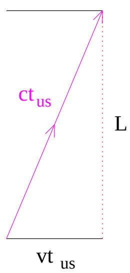
3.5.2 Light Clocks and Reference Frames

The property just derived makes it convenient to use such meter sticks to build clocks. Recall that we have given up most of our beliefs about physics for the moment, so that in particular we need to think about how to build a reliable clock. The one thing that we have chosen to build our new framework upon is the constancy of the speed of light. Therefore, it makes sense to use light to build our clocks. We will do this by sending light signals out to the end of our meter stick and back. For convenience, let us assume that the meter stick is one light-second long. This means that it will take the light one second to travel out to the end of the stick and then one second to come back. A simple model of such a light clock would be a device in which we put mirrors on each end of the meter stick and let a short pulse of light bounce back and forth. Each time the light returns to the first mirror, the clock goes 'tick' and two seconds have passed.

Now, suppose we look at our light clock from the side. Let's say that the rod in the clock is oriented in the vertical direction. The path taken by the light looks like this:



However, what if we look at a light clock carried by our inertial friend who is moving by at speed v? Suppose that the rod in her clock is also oriented vertically, with the relative motion in the horizontal direction. Since the light goes straight up and down in *her* reference frame, the light pulse moves up and forward (and then down and forward) in our reference frame. In other words, it follows the path shown below. This should be clear from thinking about the path you see a basketball follow if someone lifts the basketball above their head while they are walking past you.



The length of each side of the triangle is marked on the diagram above. Here, L is the length of her rod and t_{us} is the time (as measured by us) that it takes the light to move from one end of the stick to the other. To compute two of the lengths, we have used the fact that, in our reference frame, the light moves at speed c while our friend moves at speed v.

The interesting question, of course, is *just how long is this time* t_{us}. We know that the light takes 1 second to travel between the tips of the rod as measured in our friend's reference frame, but what about in ours? It turns out that we can calculate the answer by considering the length of the path traced out by the light pulse (the hypotenuse of the triangle above). Using the Pythagorean theorem, the distance that we measure the light to travel is $\sqrt{(vt_{us})^2 + L^2}$. However, we know that it covers this distance in a time t_{us} at speed c. Therefore, we have

$$c^2 t_{us}^2 = v^2 t_{us}^2 + L^2, \tag{3.1}$$

or,

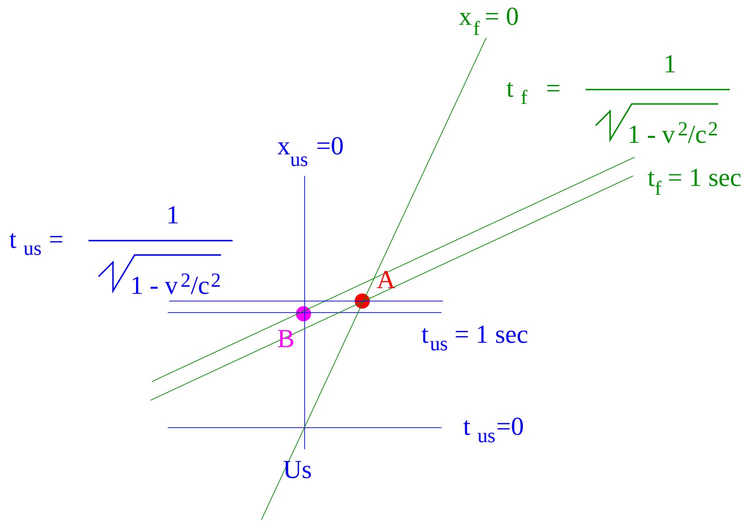
$$L^2/c^2 = t_{us}^2 - (v/c)^2 t_{us}^2 = (1 - [v/c]^2) t_{us}^2. \tag{3.2}$$

Thus, we measure a time $t_{us} = \frac{L}{c\sqrt{1-(v/c)^2}}$ between when the light leaves one mirror and when it hits the next! This is in contrast to the time $t_{friend} = L/c = 1 second$ measured by our friend between these same to events. Since this will be true for *each* tick of our friend's clock, we can conclude that:

Between any two events where our friend's clock ticks, the time t_{us} that we measure is related to the time t_{friend} measured by our friend by through

$$t_{us} = \frac{t_{friend}}{\sqrt{1-(v/c)^2}}. \tag{3.3}$$

Finally, we have learned how to label another line on our diagram above:



Note that I have marked two special events on the diagram. The dot labeled A is *the event* where the moving (friend's) clock ticks $t = 1$ second. It is an event *on the friend's worldline*. The dot labeled B is *the event* where our clock ticks $t = 1$ second. It is an event *on our worldline.*

3.5.3 Proper Time

We have seen that[3] different observers in different inertial frames measure different amounts of time to pass between two given events. We might ask if any one of these is a "better" answer than another? Well, in some sense the answer must be 'no,' since the principle of relativity tells us that all inertial frames are equally valid. However, there can be a *distinguished* answer. Note that, if one inertial observer actually experiences both events, then inertial observers in other frames have different worldlines and so cannot pass through both of these events. It is useful to use the term *proper time* between two events to refer to the time measured by an inertial observer *who actually moves between the two events.* Note that this concept exists only for timelike separated events.

Let's work through at a few cases to make sure that we understand what is going on. Consider two observers, *red* and **blue**. The worldlines of the two observers intersect at an event, where both set their clocks to read $t = 0$.

1) Suppose that *red* sets a firecracker to go off on *red's* worldline at $t_{red} = 1$. At what time does **blue** find it to go off? Our result (3.3) tells us that $t_{blue} = 1/\sqrt{1-(v/c)^2}$**.**

2) Suppose now that **blue** sets a firecracker to go off on **blue's** worldline at $t_{blue} = 1$. At what time does *red* find it to go off? From (3.3) we now have $t_{red} = 1/\sqrt{1-(v/c)^2}$.

[3]Still assuming that Einstein's idea is right.

3) Suppose that (when they meet) **blue** plants a time bomb in *red's* luggage and sets it to go off after $1sec$. What times does **blue** find it to go off? The time bomb will go off after it experiences $1sec$ of time. In other words, it will go off at the point along its worldline which is $1sec$ of *proper* time later. Since *red* is traveling along the same worldline, this is $1sec$ later according to *red* and on *red's* worldline. As a result, (3.3) tells us that this happens at $t_{blue} = 1/\sqrt{1-(v/c)^2}$.

4) Suppose that (when they meet) *red* plants a time bomb in **blue's** luggage and he wants it to go off at $t_{red=1}$. How much time delay should the bomb be given? This requires figuring out how much proper time will pass **on blue's worldline** between *red's* lines of simultaneity $t_{red=0}$ and $t_{red=1}$. Since the events are **on blue's worldline, blue** plays the role of the moving friend in (3.3). As a result, the time until the explosion as measured by **blue** should be $t_{blue} = \sqrt{1-(v/c)^2}$, and this is the delay to set.

3.5.4 Why should you believe all of this?

So far, we have just been working out consequences of Einstein's idea. We have said little about whether you should actually believe that this represents reality. In particular, the idea that clocks in different reference frames measure different amounts of time to pass blatantly contradicts your experience, doesn't it? Just because you go and fly around in an airplane does not mean that your watch becomes unsynchronized with the Cartoon Network's broadcast schedule, does it?

Well, let's start thinking about this by figuring out how big the time dilation effect would be in everyday life. Commercial airplanes move at about[4] $300m/s$. So, $v/c \approx 10^{-6}$ for an airplane. Now[5], $\sqrt{1-(v/c)^2} \approx 1 - \frac{1}{2}(v/c)^2 + ... \approx 1 - 5 \times 10^{-13}$ for the airplane. This is less than 1 part in a trillion.

Tiny, eh? You'd never notice this by checking your watch against the Cartoon Network. However, physics is a very precise science. It turns out that it *is* in fact possible to measure time to better than one part in a trillion. A nice form of this experiment was first done in the 1960's. Some physicists got two identical atomic clocks, brought them together, and checked that they agreed to much better than 1 part in a trillion. Then, they left one in the lab and put the other on an airplane (such clocks were big, they bought a seat for the clock on a commercial airplane flight) and flew around for awhile. When they brought the clocks back together at the end of the experiment, the moving clock had in fact 'ticked' less times, measuring less time to pass in precise accord with our calculations above and Einstein's prediction.

⋆⋆ You should not underestimate the significance of what I have just said. Just a moment ago, we were merrily exploring Einstein's crazy idea. While Einstein's

[4] Those of you with physics background may recognize this as roughly the speed of sound in air. Travel faster than the speed of sound is difficult and therefore expensive, so most commercial planes lurk as just below sound speed.

[5] Note that I am using the expansion from problem (2-2). It really is handy.

suggestion clearly fits with the Michelson-Morely experiment, we still have not figured out how it fits with the aberration experiments. So, we were just exploring the suggestion to see where it leads. It led to a (ridiculous??) prediction that clocks in different reference frames measure different amounts of time to pass. Then, I tell you that this prediction has in fact been experimentally tested, and that Einstein's idea passed with flying colors. *Now,* you should begin to believe that all of this crazy stuff really is true. Oh, and there will be plenty more weird predictions and experimental verifications to come.

Another lovely example of this kind of thing comes from small subatomic particles called muons (pronounced moo-ons). Muons are "unstable," meaning that they exist only for a short time and then turn into something else involving a burst of radiation. You can think of them like little time bombs. They live (on average) about 10^{-6} seconds. Now, muons are created in the upper atmosphere when a cosmic ray collides with the nucleus of some atom in the air (say, oxygen or nitrogen). In the 1930's, people noticed that these particles were traveling down through the atmosphere and appearing in their physics labs. Now, the atmosphere is about 30,000m tall, and these muons are created near the top. The muons then travel downward at something close to the speed of light. Note that, if they traveled at the speed of light $3 \times 10^8 m/s$, it would take them a time $t = 3 \times 10^4 m/(3 \times 10^8 m/s) = 10^{-4} sec.$ to reach the earth. *But, they are only supposed to live for* 10^{-6}*seconds!* So, they should only make it 1/100 of they way down before they explode. [By the way, the explosion times follow an exponential distribution, so that the probability of a muon "getting lucky" enough to last for 10^{-4} seconds is $e^{-100} \approx 10^{-30}$. This is just about often enough for you to expect it to happen a few times in the entire lifetime of the Universe.]

The point is that the birth and death of a muon are like the ticks of its clock and should be separated by 10^{-6} seconds as measured in the rest frame of the muon. In other words, the relevant concept here is 10^{-6} seconds of proper time. In our rest frame, we will measure a time $10^{-6} sec/\sqrt{1-(v/c)^2}$ to pass. For v close enough to c, this can be as large (or larger than) 10^{-4} seconds.

This concludes our first look at time dilation. In the section below, we turn our attention to measurements of position and distance. However, there remain several subtleties involving time dilation that we have not yet explored. We will be revisiting the subject soon.

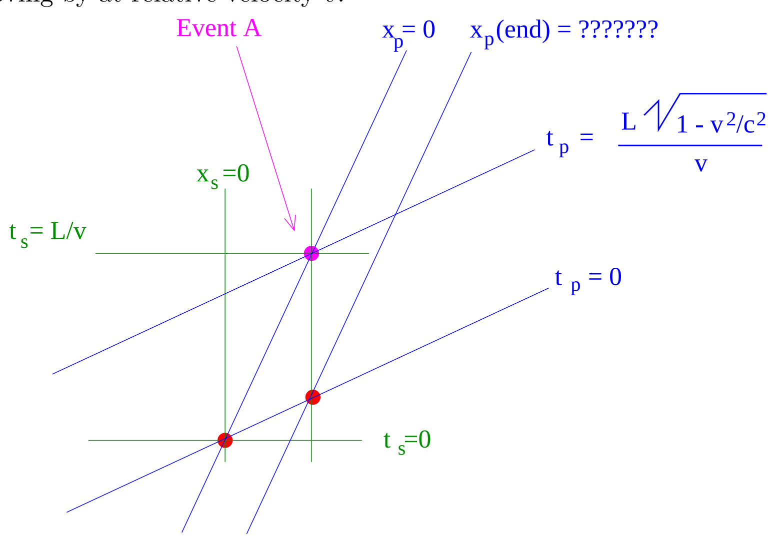
3.6 Length Contraction

In the last section, we learned how to relate times measured in different inertial frames. Clearly, the next thing to understand is distance. While we had to work fairly hard to compute the amount of time dilation that occurs, we will see that the effect on distances follow quickly from our results for time.

Let's suppose that two inertial observers both have measuring rods that are at rest in their respective inertial frames. Each rod has length L in the frame in which it is at rest (it's "rest frame").

In section 3.5.1, we saw that distances in the direction perpendicular to the relative motion are not affected. So, to finish things off, this time we must consider the case where the measuring rods are aligned *with* the direction of the relative velocity.

For definiteness, let us suppose that the two observers each hold their meter stick at the leftmost end. The relevant spacetime diagram is shown below. As usual, we assume that the two observers clocks both read $t = 0$ at the event where their worldlines cross. We will call our observers 'student' and 'professor.' We begin by drawing the diagram in the student's rest frame and with the professor moving by at relative velocity v.



Now, the student must find that the professor takes a time L/v to traverse the length of the student's measuring rod. Let us refer to the event (marked in magenta) where the moving professor arrives at the right end of the student's measuring rod as "event A." Since this event has $t_s = L/v$, we can use our knowledge of time dilation (3.3) to conclude that the professor assigns a time $t_P = (L/v)\sqrt{1-(v/c)^2}$ to this event.

Our goal is to determine the length of the student's measuring rod in the professor's frame of reference. That is, we wish to know what position $x_P(end)$ the professor assigns to the rightmost end of the students rod when this end crosses the professor's line of simultaneity $t_P = 0$.

To find this out, note that from the professor's perspective it is the student's rod that moves past him at speed v. Recall that we determined above that the professor finds that it takes the rod a time $t_P = (L/v)\sqrt{1-(v/c)^2}$ to pass by. Thus, the student's rod must have a length $L_P = L\sqrt{1-(v/c)^2}$ in the professor's frame of reference. The professor's rod, of course, will similarly be shortened in the student's frame of reference.

So, we see that distance measurements also depend on the observer's frame of reference. Note however, that given any inertial object, there is a special inertial frame in which the object is at rest. The length of an object in its own rest frame is known as its *proper length*. The length of the object in any other inertial frame will be shorter than the object's proper length. We can summarize what we have learned by stating:

> An object of proper length L moving through an inertial frame at speed v has length $L\sqrt{1 - v^2/c^2}$ as measured in that inertial frame.

There is an important subtlety that we should explore. Note that the above statement refers to an *object*. However, we can also talk about *proper distance* between two events. When two events are spacelike related, there is a special frame of reference in which the events are simultaneous and the separation is "pure space" (with no separation in time). The distance between them in this frame is called the *proper distance* between the events. It turns out that this distance is in fact *longer* in any other frame of reference.....

Why longer? To understand this, look back at the above diagram and compare the two events at either end of the *students'* rod that are simultaneous in the **professor's** frame of reference. Note that the proper distance is the distance measured in the professor's reference frame, which we just concluded is shorter than the distance measured by the student. The difference here is that we are now talking about **events** (points on the diagram) where as before we were talking about *objects* (whose ends appear as **worldlines** on the spacetime diagram). The point is that, when we talk about measuring the length of an object, different observers are actually measuring the distance between different pairs of events.

3.7 The Train Paradox

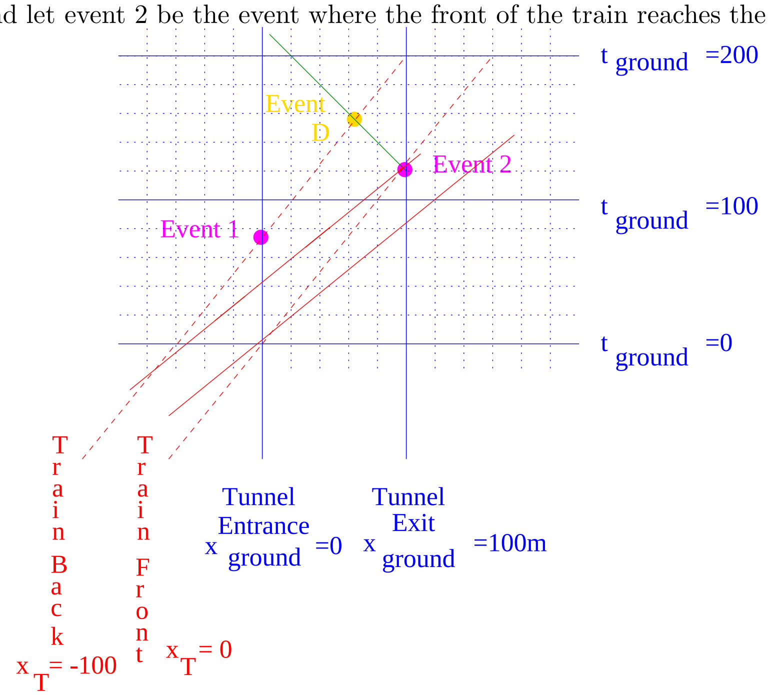
Let us now test our new skills and work through some subtleties by considering an age-old parable known as the train paradox. It goes like this:

Once upon a time there was a *really* fast Japanese bullet train that ran at 80% of the speed of light. The train was 100m long in its own rest frame. The train carried as cargo the profits of SONY corporation from Tokyo out to their headquarters in the countryside. The profits were, of course, carried in pure gold.

Now, some less than reputable characters found out about this and devised an elaborate scheme to rob the train. They knew that the train would pass through a 100m long tunnel on its route. Watching the train go by, they measured the train to be quite a bit less than 100m long and so figured that they could easily trap it in the tunnel.

Of course, the people on the train found that, when the train was in motion, it was the train that was 100m long while the tunnel was significantly shorter. As a result, they had no fear of being trapped in the tunnel by train robbers. Now, do you think the robbers managed to catch the train?

Let's draw a spacetime diagram using the tunnel's frame of reference. We can let E represent the tunnel entrance and X represent the tunnel exit. Similarly, we let B represent the back of the train and F represent the front of the train. Let event 1 be the event where the back of the train finally reaches the tunnel and let event 2 be the event where the front of the train reaches the exit.



Suppose that one robber sits at the entrance to the tunnel and that one sits at the exit. When the train nears, they can blow up the entrance just after event 1 and they can blow up the exit just before event 2. Note that, in between these two events, the robbers find the train to be completely inside the tunnel.

Now, what does the train think about all this? How are these events described in its frame of reference? Note that the train finds event 2 to occur long *before* event 1. So, can the train escape?

Let's think about what the train would need to do to escape. At event 2, the exit to the tunnel is blocked, and (from the train's perspective) the debris blocking the exit is rushing toward the train at half the speed of light. The only way the train could escape would be to turn around and back out of the tunnel. Recall that the train finds that the entrance is still open at the time of event 2.

Of course, both the front and back of the train must turn around. How does the back of the train know that it should do this? It could find out via a phone call from an engineer at the front to an engineer at the back of the train, or it could be via a shock wave that travels through the metal of the train as the front of the train throws on its brakes and reverses its engines. The point is though that some signal must pass from event 2 to the back of the train, possibly relayed along the way by something at the front of the train. Sticking to our assumption that signals can only be sent at speed c or slower, the earliest possible time that the back of the train could discover the exit explosion is at the event marked D on the diagram. Note that, at event D, the back of the train *does* find itself inside the tunnel and also finds that event 1 has already occurred. The entrance is closed and the train cannot escape.

There are two things that deserve more explanation. The first is the above comment about the shock wave. Normally we think of objects like trains as being perfectly stiff. However, this is not really so. Let's think about what happens when I push on one end of a meter stick. I press on the atoms on the end, which press on the atoms next to them, which press on the atoms next to them ..... Also, it takes a (small but finite) amount of time for each atom to respond to the push it has been given on one side and to move over and begin to push the atom on the other side. The result is known as a "shock wave" that travels at finite speed down the object. Note that an important part of the shock wave are the electric forces that two atoms use to push each other around. Thus, the shock wave can certainly not propagate faster than an electromagnetic disturbance can. As a result, it must move at less than the speed of light.

For the other point, let's suppose that the people at the front of the train step on the brakes and stop immediately. Stopping the atoms at the front of the train will make them push on the atoms behind them, stopping them, etc. The shock wave results from the fact that atoms just behind the front slam into atoms right at the front; the whole system compresses a bit and then may try to reexpand, pushing some of the atoms farther back.

⋆⋆ What we saw above is that the shock wave cannot reach the back of the train until event D. Suppose that it does indeed stop the back of the train there. The train has now come to rest in the tunnel's frame of reference. Thus, after event D, the *proper* length of the train is less than 100m!!!!

In fact, suppose that we use the lines of simultaneity in the train's original frame of reference (before it tries to stop) to measure the proper length of the train. Then, immediately after event 2 the front of the train changes its motion, but the back of the train keeps going. As a result, in this sense the proper length of the train starts to shrink immediately after event 2. This is how it manages to fit itself into a tunnel that, in this frame, is less than 100m long.

What has happened? The answer is in the compression that generates the shock wave. The train really has been physically compressed by the wall of debris at the exit slamming into it at half the speed of light[6]! This compression is of course accompanied by tearing of metal, shattering of glass, death screams of passengers, and the like, just as you would expect in a crash. The train is completely and utterly destroyed. The robbers will be lucky if the gold they wish to steal has not been completely vaporized in the carnage.

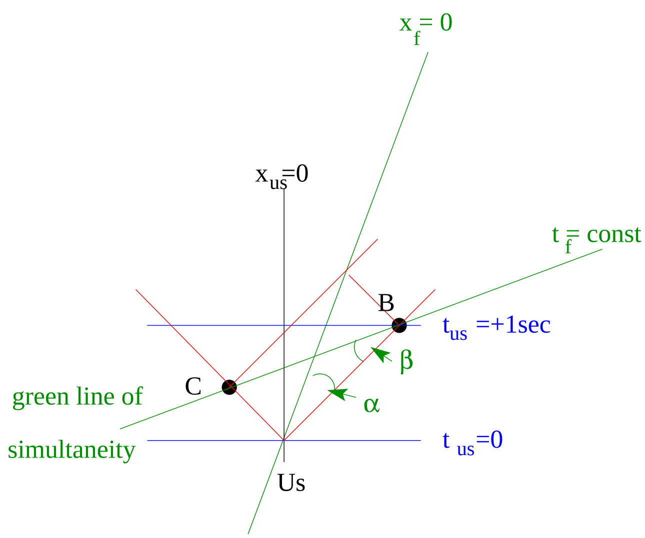
Now, you might want to get one more perspective on this (trying to show some hidden inconsistency?) by analyzing the problem again in a frame of reference

[6] Or the equivalent damage inflicted through the use of the train's brakes.

that moves with the train at all times, even slowing down and stopping as the train slows down and stops. However, we do not know enough to do this yet since such a frame is not inertial. We will get to accelerating reference frames in chapter 5.

3.8 Homework Problems

3-1. Use your knowledge of geometry (and/or trigonometry) to show that the angle α between the worldline of an inertial observer and the lightcone (drawn in an inertial frame using units in which light rays travel at 45 degrees) is the same as the angle β between that observer's line of simultaneity and the lightcone. (Hint: Many people find it easier to use trigonometry to solve this problem than to use geometry.)



3-2. Suppose that you and your friend are inertial observers (that is, that both reference frames are inertial). Suppose that two events, A and B, are simultaneous in your own reference frame. Draw two spacetime diagrams in your reference frame. Include your friend's worldline in both. For the first, arrange the relative velocity of you and your friend so that event A occurs *before* event B in your friend's reference frame. For the second, arrange it so that event B occurs first in your friend's frame. In both cases, include one of your friend's lines of simultaneity on the diagram.

3-3. Draw a spacetime diagram in an inertial reference frame.

a) Mark any two spacelike related events on your diagram and label them both 'S.'

b) Mark any two lightlike related events on your diagram and label them both 'L.'

c) Mark any two timelike related events on your diagram and label them both 'T.'

3-4. You and your friend are again inertial observers and this time your relative velocity is $\frac{1}{2}c$. Also, suppose that both of your watches read $t = 0$ at the instant that you pass each other. At the event where your worldlines cross, a firecracker explodes (or a light bulb is turned on and off very quickly, or something else happens to create a brief burst of light).

a) Draw a spacetime diagram in your reference frame showing your friend's worldline and the outgoing light from the explosion.

b) Let events A and B be the events on the left and right light rays respectively that are one second (as determined by your own reference frame) after the explosion. Label these two events on your diagram.

c) Let event C be the the event on the left light ray that is simultaneous with B as determined by your friend. Mark this event on the diagram.

3-5. As in problem (3-4), you and your friend are inertial observers with relative velocity $\frac{1}{2}c$ and your watches both "tick" $t = 0$ at the instant that you pass each other. For this problem, it is important that you use a large scale to draw the spacetime diagram so that $t = 1$ is far from the origin.

a) Draw the two worldlines on a spacetime diagram in your frame of reference, and draw and label (i) the event where your watch "ticks" $t = 1$ and (ii) the event where your friend's watch "ticks" $t = 1$. At what time is this second event in your reference frame? Did you draw it in the right place?

b) On the same diagram, sketch the line of events which are at $t = 1$ *second* as determined by your system of reference. This is your $t = 1$ line of simultaneity.

c) On the same diagram, sketch the line of events which are at $t = 1$ *second* as determined by your friend's system of reference. (Hint: At what time in your system of reference does his watch tick $t = 1$?) This is your friend's $t = 1$ line of simultaneity.

d) Consider the event where your friend's $t = 1$ line of simultaneity crosses *your* worldline. What time does your friend assign to this event? What time do you assign to this event?

e) Consider the event where your $t = 1$ line of simultaneity crosses *your friend's* worldline. What time do you assign to this event? What time does your friend assign to this event?

3-6. Suppose that you are in an airplane and that you are watching another airplane fly in the opposite direction. Your relative velocity is roughly $600 m/s$. Calculate the size of the time dilation effect you would observe if you measured the ticking of a clock in the other airplane. If that clock ticks once each second, how much time passes in your reference frame between two of it's ticks?

Hint: Your calculator may not be able to deal effectively with the tiny numbers involved. As a result, if you just try to type formula (3.3) into your calculator you may get the value 1. What I want to know is how much the actual value is different from 1. In cases like this, it is helpful to use Taylor series expansions. The expansions you need to complete this problem were given in problem (2-2).

3-7. A muon is a particle that has a lifetime of about 10^{-6} seconds. In other words, when you make a muon, it lives for that long (as measured in its own rest frame) and then decays (disintegrates) into other particles.

a) The atmosphere is about $30km$ tall. If a muon is created in the upper atmosphere moving (straight down) at $\frac{1}{2}c$, will it live long enough to reach the ground?

b) Suppose that a muon is created at the top of the atmosphere moving straight down at $.999999c$. Suppose that you want to catch this muon at the surface and shoot it back up at $.999999c$ so that it decays just when it reaches the top of the atmosphere. How long should you hold onto the muon?

3-8. For this problem, consider three inertial observers: You, Alice, and Bob. All three of you meet at one event where your watches all read zero. Alice recedes from you at $\frac{1}{2}c$ on your left and Bob recedes from you at $\frac{1}{2}c$ on your right. Draw this situation on a spacetime diagram in your frame of reference. Also draw in a light cone from the event where you all meet. Recall that you measure distances using your own lines of simultaneity, *and note that you find each of the others to be* 'halfway between you and the light cone' *along any of your lines of simultaneity.*

Now, use the above observation to draw this situation on a spacetime diagram in *Alice's* frame of reference. Use this second diagram to estimate the speed at which Alice finds *Bob* to be receding from her. What happens if you draw in another observer (again meeting all of you at $t = 0$) and traveling away from Bob on the right at $\frac{1}{2}c$ as measured in Bob's frame of reference??

3-9. Consider a specific version of the 'train paradox' that we discussed in class. Suppose that we a have tunnel of length $100m$ (measured in its rest frame) and a train whose length is $100m$ (measured in its rest frame). The train is moving along the track at $.8c$, and the robbers want to trap it using the following scheme: one robber will sit at the entrance to the tunnel and blow it up just after the back end of the train has entered, while the other robber will sit at the exit of the tunnel and blow it up just before the front end of the train gets there.

i) Draw a spacetime diagram of all of this in the (inertial) reference frame of the tunnel. Be sure to include the explosions and the response of the train. Use the graph paper to make your diagram accurate.

ii) Draw a spacetime diagram of all of this using the inertial frame that matches the train while it is moving smoothly down the tracks. Be sure to include the explosions and the response of the train. Use the graph paper to make your diagram accurate.

iii) Describe the events shown on your diagrams from the point of view of the robbers (either one). Be sure to state the order in which the events occur in their frame of reference, to discuss whether the train can escape, and to explain why or why not.

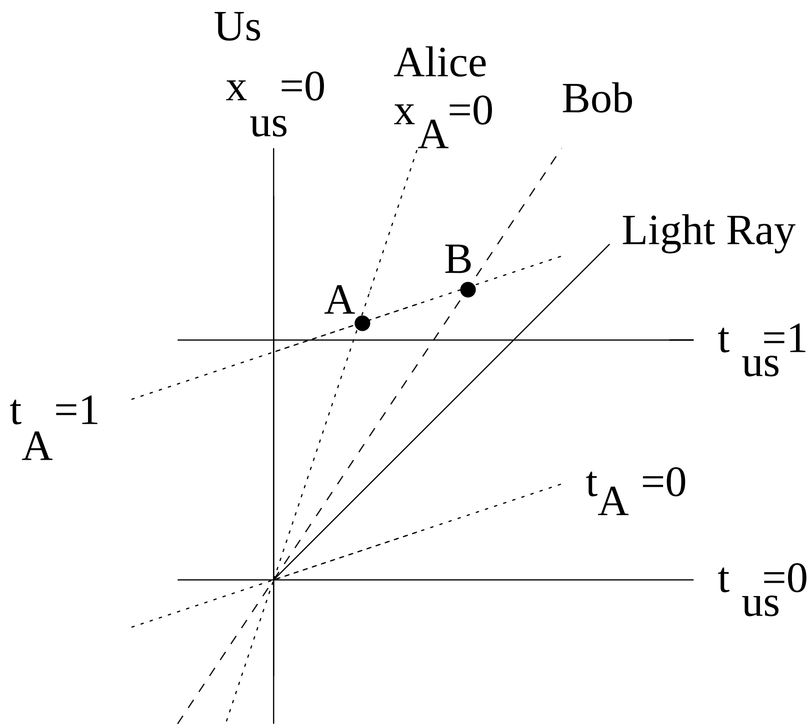
iv) Describe what happens from the point of view of someone at the front of the train. Be sure to state the order in which the events occur in their frame of reference, to discuss whether the train can escape, and to explain why or why not.

v) Finally, briefly describe what happens from the point of view of someone at the back of the train.

3-10. The starting point for our discussion of relativity was the observation that velocities do *not* combine in the naive way by just adding together. Clearly then, a good question is "just how do they combine?" In this problem, you will derive the formula for the 'relativistic composition of velocities.' Let me point out that you have all of the tools with which to do this, since you know how to translate both distances and times between reference frames. This is just a quantitative version of problem (3-8), but we now work in the more general case where v_{AB} is arbitrary. Feel free to choose units so that $c = 1$. Then you can ignore all the factors of c to make the algebra easier.

Let's proceed in the following way. Think about three inertial observers: you (and let's say that I'm traveling with you), Alice, and Bob. Suppose as that you all three meet at some event and that you find Alice to be moving away from you (to the right) at speed v_A, while Alice finds Bob to be moving away from her (to the right) at speed v_{BA}. The question is, what is the speed v_B with which Bob is moving away from you?? We want to find a formula for v_B in terms of v_A and v_{BA}.

As usual, let's start by drawing a diagram:



I have drawn a few lines of simultaneity to remind us of what is going on, and I have marked two events (A and B) on the diagram. Event A is where Alice's clock ticks 1, and event B is the event *on Bob's worldline* that is at $t_A = 1$ (that is, the event that Alice finds to be simultaneous with event A). Since Bob recedes from Alice at speed v_{BA}, in Alice's reference system event B has coordinates $(t_A = 1, x_A = v_{BA})$.

(a) What are the values of x_{us} and t_{us} at event A?

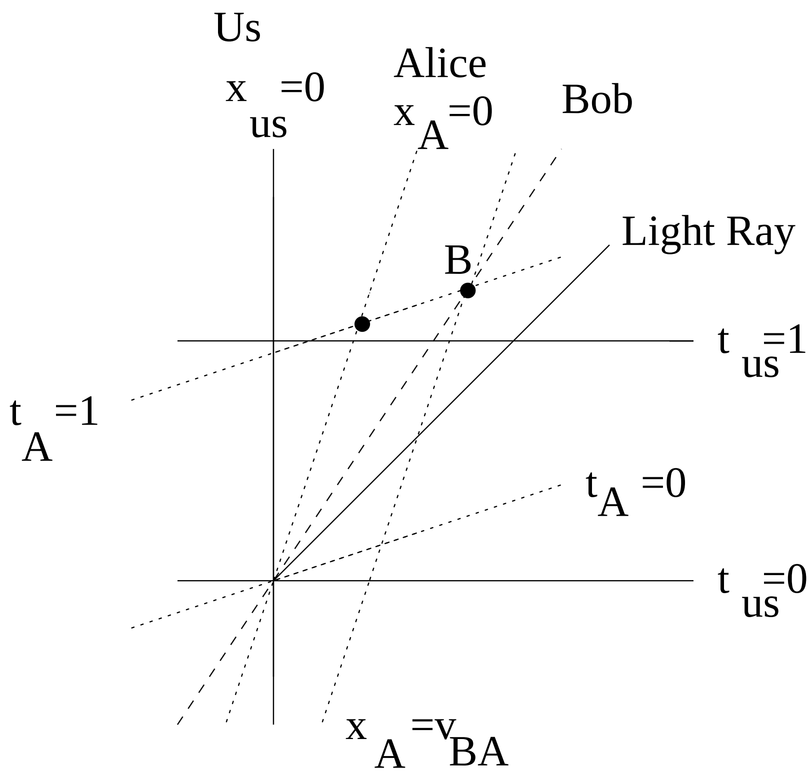
Now, recall that Bob passes through the event $(x_{us} = 0, t_{us} = 0)$. So, if we knew the coordinates (in our reference frame) of *any* other event on Bob's worldline, we could calculate his velocity (as measured by us) using $v = \Delta x/\Delta t$. Let us use event B for this purpose. We already know the coordinates of event B in *Alice's* system of reference, but we still need to figure out what the coordinates are in *our* system of reference.

(b) Since event B lies on Alice's $t_A = 1$ line, a good place to start is to write down the equation for this line in terms of *our* coordinates (x_{us} and t_{us}). That is, determine the values of m and b for which the equation

$$t_{us} = mx_{us} + b$$

describes the line $t_A = 1$. Note that b is just the time (t_{us}) at which the line intersects the line $x_{us} = 0$ and m is the slope of the line. If you are not sure what the slope is, you may find it helpful to use the results of part (a) and your value of the parameter b to compute the slope.

Now that you have done part (b), recall that two intersecting lines uniquely determine a point. So, if we could find another line whose equation we know (in terms of our coordinates), we could solve the two equations together to find the coordinates of event B. (Note that Bob's worldline won't work for this, because we don't yet know what his speed is in our reference frame.) What other line could we use? Well, we know that event B occurs a distance $v_{BA}(1sec)$ to Alice's right in *Alice's* frame of reference. This means that it lies along the line $x_A = v_{BA}$:



(c) Find the values of m and $x_{us,0}$ for which the equation

$$t_{us} = m(x_{us} - x_{us,0})$$

describes the line $x_A = v_{BA}$. Note that m is the slope of the line and $x_{us,0}$ is the place where the line of interest intersects the line $t_{us} = 0$. It may also be useful to note that, if Alice holds a rod of proper length v_{BA} and carries it with her, holding it at one end, then the line $x_A = v_{BA}$ is the worldline of the other end.

(d) Solve the equations from (b) and (c) to find the coordinates of event B.

(e) Use the results of part *d* to show that

$$v_B = \frac{v_A + v_{BA}}{1 + v_A v_{BA}/c^2}, \tag{3.4}$$

or, in units where $c = 1$,

$$v_B = \frac{v_A + v_{BA}}{1 + v_A v_{BA}}. \tag{3.5}$$

3-11. What happens in the relativistic addition of velocities formula (derived in problem (10) when the two velocities are both very small? Try it for $v_{BA} = .01c$ and $v_A = .01c$. How much does the relativistic result differ from the Newtonian result (1.2) in this case?

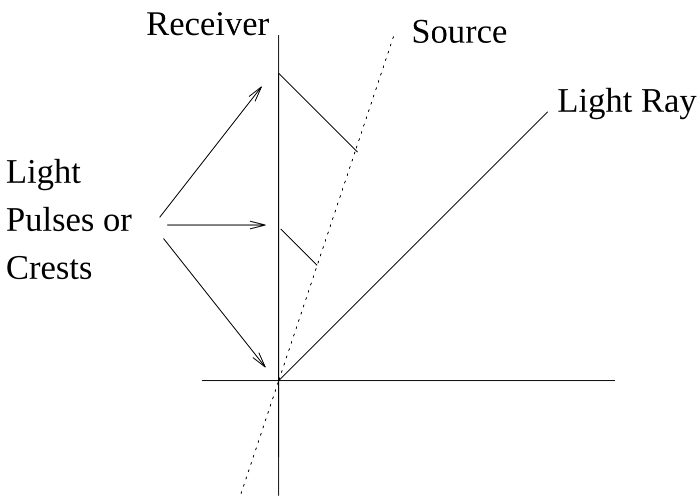
3-12. What happens in the relativistic addition of velocities formula (derived in problem (10) when one of the velocities (say, v_{BA}) is the speed of light??

3-13. The most common use of relativity in every day life involves what is called the 'Doppler effect.' This is an effect in which the *frequency* (= rate at which crests pass by you) of a wave depends on the reference frame. There is also a Doppler effect in Newtonian physics, but Newtonian physics would predict a different amount of the effect. The Doppler effect is used in various devices such as 'Doppler radar' and police radar guns to measure the speed of storm systems or cars. For this problem, derive the formula for the *relativistic* Doppler effect

$$\tau_R = \sqrt{\frac{c+v}{c-v}}\tau_S. \tag{3.6}$$

where v is the velocity of the source (S) away from the receiver (R). Here, I have expressed the formula using the period τ (the time between wave crests) as measured by each observer.

Hint: A good way to figure this out is to think about a strobe light carried by the source which emits light pulses at regular intervals τ_S. Then we can ask what the time interval τ_R is (as measured by the receiver) between the events where the light pulses are received. If we then think about each of the strobe pulses as tracking a wave crest, this will give us the Doppler effect formula.



For comparison, the Newtonian formula is

$$\tau_R = \left(\frac{c + v_S}{c + v_R} \right) \tau_S \tag{3.7}$$

where c is the speed of the wave relative to the medium that carries it (i.e., sound in air, waves on water), v_S is the velocity of the source relative to the medium, and v_R is the velocity of the receiver relative to the medium. [For fun, you might try deriving that one, too ...]

3-14. By expanding equations (3.6) and (3.7) in a Taylor series, show that the Newtonian and relativistic effects agree to first order in $\frac{v}{c} = \frac{v_S - v_R}{c}$. This means that engineers who design radar devices do not in fact need to understand relativity. You can do this using the expansions from problem (2-**??**). Note that "to first order" means that we ignore any terms proportional to v^2, v_R^2, v_S^2 or higher powers of v, v_R, v_S.

Chapter 4

Special Relativity is Minkowskian Geometry

Read Einstein, ch. 16,17, Appendix 2

Let's take a look at where we are. In chapter 2 we were faced with the baffling results of the Michelson-Morely experiment and the stellar aberration experiments. In the end, we decided to follow Einstein and to allow the possibility that space and time simply do not work in the way that our intuition predicts. In particular, we took our cue from the Michelson-Morely experiment which seems to say that the speed of light in a vacuum is the same in all inertial frames and, therefore, that velocities do not add together in the Newtonian way. We wondered "How can this be possible?"

We then spent the last chapter working out "how this can be possible." That is, we have worked out what the rules governing time and space must actually be in order for the speed of light in a vacuum to be the same in all inertial reference frames. In this way, we discovered that different observers have different notions of simultaneity, and we also discovered time dilation and length contraction. Finally, we learned that some of these strange predictions are actually correct and have been well verified experimentally.

It takes awhile to really absorb what is going on here. The process does take time, though at this stage of the course the students who regularly come to my office hours are typically moving along well. If you are having trouble making the transition, I encourage you to come and talk to me.

There are lots of levels at which one might try to "understand" the various effects[1]. Some examples are:

[1] This is as opposed to why you should believe in these effects. The fundamental reason to believe them must always be that they predicted new phenomena (like time dilation) which were then experimentally verified.

Logical Necessity: Do see that the chain of reasoning leading to these conclusions is correct? If so, and if you believe the results of Michelson and Morely that the speed of light is constant in all inertial frames, then you must believe the conclusions.

External Consistency: Since I have never seen any sign of these strange things in my life, how can they be true? How can all of this be consistent with my own experience? Understanding at this level involves determining how big the affects actually would be in your everyday life. You will quickly find that they are seldom more than one part in a billion or a trillion. At this level, it is no wonder that you never noticed.

Internal Consistency: How can these various effects possibly be self-consistent? How can a train 100m long get stuck inside a tunnel that, in it's initial frame of reference, is less than 100m long? To understand things at this level involves working through various 'paradoxes' such as the train paradox of the last chapter and the paradox that we will address below in section 4.2.

Step Outside the old Structure: People often say "OK, but *WHY* are space and time this way?" Typically, when people ask this question, what they mean is "Can you explain why these strange things occur in terms of things that are familiar to my experience, or which are reasonable to my intuition?" It is important to realize that, in relativity, this is most definitely *not possible* in a direct way. This is because all of your experience has built up an intuition that believes in the *Newtonian* assumptions about space and time and, as we have seen, these cannot possibly be true! Therefore, you must remove your old intuition, remodel it completely, and then put a new kind of intuition back in your head.

Finding the new logic: If we have thrown out all of our intuition and experience, what does it mean to "understand" relativity? We will see that relativity has a certain logic of its own. What we need to do is to uncover the lovely structure that space and time really do have, and not the one that we want them to have. In physics as in life, this is often necessary. Typically, when one understands a subject deeply enough, one finds that the subject really does have an intrinsic logic and an intrinsic sense that are all its own. This is the level at which finally see "what is actually going on." This is also the level at which people finally begin to "like" the new rules for space and time.

Starting with the next section we'll begin to see a little bit of the underlying structure, the "new logic" of relativity. There are some technicalities involved, so we won't do it all at once. We'll do just a bit, and then go on to talk about one of the classic 'paradoxes' – the so-called twin paradox in section 4.2. Then we'll return to the "new logic" in section 4.3. The new logic goes by the name of Minkowskian Geometry.

4.1 Minkowskian Geometry

Minkowski was a mathematician, and he is usually credited with emphasizing the fact that time and space are part of the same "spacetime" whole in relativity. He also who emphasized the fact that this spacetime has a special kind of geometry. It is this geometry which is the underlying structure and the new logic of relativity.

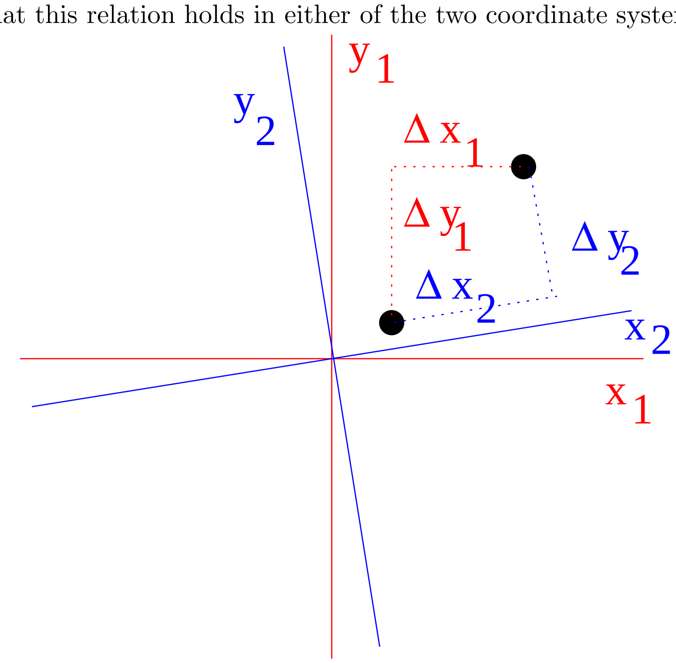
Understanding this geometry will provide both insight and useful technical tools. For this reason, we now pursue what at first sight will seem like a technical aside in which we first recall how the familiar *Euclidean* geometry relates quantities in different coordinate systems. We can then build an analogous technology in which Minkowskian geometry relates different inertial frames.

4.1.1 Invariants: Distance vs. the Interval

Recall that a fundamental part of familiar Euclidean geometry is the Pythagorean theorem. One way to express this result is to say that

$$(distance)^2 = \Delta x^2 + \Delta y^2, \tag{4.1}$$

where *distance* is the distance between two points and Δx, Δy are respectively the differences between the x coordinates and between the y coordinates of these points. Here the notation Δx^2 means $(\Delta x)^2$ and not the change in x^2. Note that this relation holds in either of the two coordinate systems drawn below.

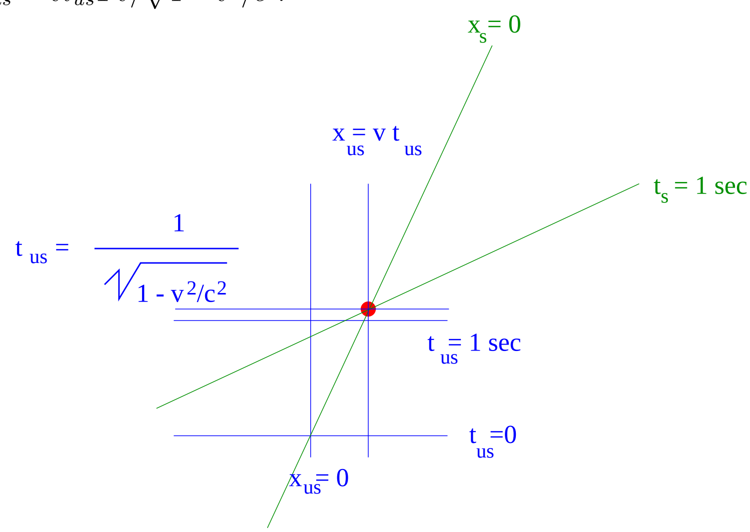


Now, recall that I have made analogies between changing reference frames and rotations. Note that when I perform a rotation, distances do not change, and if I compare coordinate systems (with one rotated relative to the other) I find

$$\Delta x_1^2 + \Delta y_1^2 = \Delta x_2^2 + \Delta y_2^2.$$

Let's think about an analogous issue involving changing inertial frames. Consider, for example, two inertial observers. Suppose that our friend flies by at speed v. For simplicity, let us both choose the event where our worldlines intersect to be $t = 0$. Let us now consider the *event* (on his worldline) where his clock 'ticks' $t_f = T$. Note that our friend assigns this event the position $x_f = 0$ since she passes through it.

What coordinates do we assign? Our knowledge of time dilation tells us that we assign a longer time: $t_{us} = T/\sqrt{1 - v^2/c^2}$. For position, recall that at $t_{us} = 0$ our friend was at the same place that we are ($x_{us} = 0$). Therefore, after moving at a speed v for a time $t_{us} = T/\sqrt{1 - v^2/c^2}$, our friend is at $x_{us} = vt_{us}Tv/\sqrt{1 - v^2/c^2}$.



Now, we'd like to examine a Pythagorean-like relation. Of course we can't just mix x and t in an algebraic expression since they have different units. But, we have seen that x and ct do mix well! Thinking of the marked event where our friend's clock ticks, is it true that $x^2 + (ct)^2$ is the same in both reference frames? Clearly no, since both of these terms are larger in our reference frame than in our friend's ($x_{us} > 0$ and $ct_{us} > ct_f$)!

Just for fun, let's calculate something similar, but slightly different. Let's compare $x_f^2 - (ct_f)^2$ and $x_{us}^2 - (ct_{us})^2$. I know you don't actually want to read through lines of algebra here, so please stop reading and do this calculation yourself.

$\star\star$ You did do the calculation, didn't you?? If so, you found $-c^2T^2$ in both cases!!!!

What we have just observed is that whenever an inertial observer passes through two events and measures a proper time T between them, *any* inertial observer finds $\Delta x^2 - c^2\Delta t^2 = -c^2T^2$. But, given any two timelike separated events, an inertial observer could in fact pass through them. So, we conclude that the quantity $\Delta x^2 - c^2\Delta t^2$ computed for a pair of timelike separated events is the same in all inertial frames of reference. Any quantity with this property is called an 'invariant' because it does not vary when we change reference frames.

A quick check (which I will let you work out) shows that the same is true for spacelike separated events. For lightlike separated events, the quantity $\Delta x^2 - c^2\Delta t^2$ is actually zero in all reference frames. As a result, we see that for *any* pair of events the quantity $\Delta x^2 - c^2\Delta t^2$ is completely independent of the inertial frame that you use to compute it. This quantity is known as the $(interval)^2$.

$$\boxed{(interval)^2 = \Delta x^2 - c^2\Delta t^2 \qquad (4.2)}$$

The language here is a bit difficult since this can be negative. The way that physicists solve this in modern times is that we always discuss the $(interval)^2$ and never (except in the abstract) just "the interval" (so that we don't have to deal with the square root). The interval functions like 'distance,' but in *spacetime*, not in space.

Let us now explore a few properties of the interval. As usual, there are three cases to discuss depending on the nature of the separation between the two events.

timelike separation: In this case the squared interval is negative. As noted before, for two timelike separated events there is (or could be) some inertial observer who actually passes through both events, experiencing them both. One might think that her notion of the amount of time between the two events is the most interesting and indeed we have given it a special name, the "proper time" ($\Delta\tau$; "delta tau") between the events. Note that, for this observer the events occur at the same place. Since the squared interval is the same in all inertial frames of reference, we therefore have:

$$0^2 - c^2(\Delta\tau)^2 = \Delta x^2 - c^2\Delta t^2.$$

Solving this equation, we find that we can calculate the proper time $\Delta\tau$ in terms of the distance Δx and time Δt in any inertial frame using:

$$\Delta\tau = \sqrt{\Delta t^2 - \Delta x^2/c^2} = \Delta t\sqrt{1 - \frac{\Delta x^2}{c^2\Delta t^2}} = \Delta t\sqrt{1 - v^2/c^2}.$$

As before, we see that $\Delta\tau \le \Delta t$.

spacelike separation: Similarly, if the events are spacelike separated, there is an inertial frame in which the two are simultaneous – that is, in which $\Delta t = 0$. The distance between two events measured in such a reference frame is called the proper distance d. Much as above,

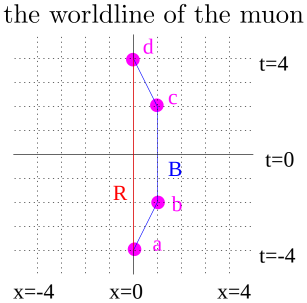
$$d = \sqrt{\Delta x^2 - c^2 \Delta t^2} \leq \Delta x.$$

Note that this seems to "go the opposite way" from the length contraction effect we derived in section 3.6. That is because here we consider the proper distance *between two particular events.* In contrast, in measuring the *length of an object*, different observers do *NOT* use the same pair of events to determine length. Do you remember our previous discussion of this issue?

lightlike separation: Two events that are along the same light ray satisfy $\Delta x = \pm c \Delta t$. It follows that they are separated by zero *interval* in all reference frames. One can say that they are separated by both zero proper time and zero proper distance.

4.1.2 Curved lines and accelerated objects

Thinking of things in terms of proper time and proper distance makes it easier to deal with, say, accelerated objects. Suppose we want to compute, for example, the amount of time experienced by a clock that is not in an inertial frame. Perhaps it quickly changes from one inertial frame to another, shown in the blue worldline (marked B) below. This blue worldline (B) is similar in nature to the worldline of the muon in part (b) of problem 3-7.

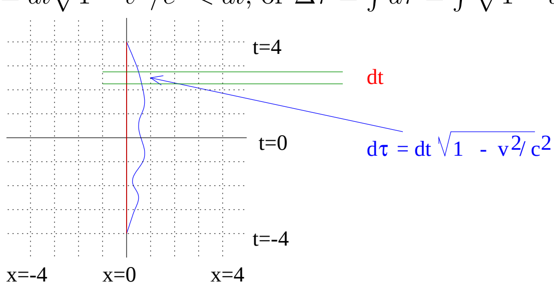


Note that the time experienced by the blue clock between events (a) and (b) is equal to the proper time between these events since, on that segment, the clock could be in an inertial frame. Surely the time measured by an ideal clock between (a) and (b) cannot depend on what it was doing before (a) or on what it does after (b).

Similarly, the time experienced by the blue clock between events (b) and (c) should be the same as that experienced by a truly inertial clock moving between these events; i.e. the proper time between these events. Thus, we can find the total proper time experienced by the clock by adding the proper time between (a) and (b) to the proper time between (b) and (c) and between (c) and (d). We also refer to this as the total proper time along the clock's worldline between (a) and (d).

A red observer (R) is also shown above moving between events $(0, -4)$ and $(0, +4)$. Let $\Delta\tau^R_{ad}$ and $\Delta\tau^B_{ad}$ be the proper time experienced by the red and the blue observer respectively between times $t = -4$ and $t = +4$; that is, between events (a) and (d). Note that $\Delta\tau^R_{a,d} > \Delta\tau^B_{a,d}$ and similarly for the other time intervals. Thus we see that the proper time along the broken line is less than the proper time along the straight line.

Since proper time (i.e., the interval) is analogous to distance in Euclidean geometry, we also talk about the total proper time along a curved worldline in much the same way that we talk about the length of a curved line in space. We obtain this total proper time much as we did for the blue worldline above by adding up the proper times associated with each short piece of the curve. This is just the usual calculus trick in which we approximate a curved line by a sequence of lines made entirely from straight line segments. One simply replaces any Δx (or Δt) denoting a difference between two points with dx or dt which denotes the difference between two infinitesimally close points. The rationale here, of course, is that if you look at a small enough (infinitesimal) piece of a curve, then that piece actually looks like a straight line segment. Thus we have $d\tau = dt\sqrt{1 - v^2/c^2} < dt$, or $\Delta\tau = \int d\tau = \int \sqrt{1 - v^2/c^2}dt < \Delta t$.

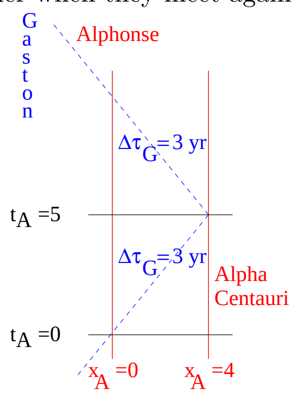


Again we see that a straight (inertial) line in spacetime has the *longest* proper time between two events. In other words, in Minkowskian geometry the *longest* line between two events is a straight line.

4.2 The Twin paradox

That's enough technical stuff for the moment. We're now going to use the language and results from the previous section to discuss a relativity classic: "The twin paradox." Using the notions of proper time and proper distance turns out to simplify the discussion significantly compared to what we would have had to go through before section 4.1.

Let's think about two identical twins who, for obscure historical reasons are named Alphonse and Gaston. Alphonse is in an inertial reference frame floating in space somewhere near our solar system. Gaston, on the other hand, will travel to the nearest star (Alpha Centauri) and back at .8c. Alpha Centauri is (more or less) at rest relative to our solar system and is four light years away. During the trip, Alphonse finds Gaston to be aging slowly because he is traveling at .8c. On the other hand, Gaston finds Alphonse to be aging slowly because, relative to Gaston, *Alphonse* is traveling at .8c. During the trip out there is no blatant contradiction, since we have seen that the twins will not agree on which event (birthday) on Gaston's worldline they should compare with which event (birthday) on Alphonse's worldline in order to decide who is older. But, who is older when they meet again and Alphonse returns to earth?



The above diagram shows the trip in a spacetime diagram in Alphonse's frame of reference. Let's work out the proper time experienced by each observer. For Alphonse, $\Delta x = 0$. How about Δt? Well, the amount of time that passes is long enough for Gaston to travel 8 light-years (there and back) at .8c. That is, $\Delta t = 8lyr/(.8c) = 10yr$. So, the *proper* time $\Delta\tau_A$ experienced by Alphonse is ten years.

On the other hand, we see that on the first half of his trip Gaston travels 4 light-years in 5 years (according to Alphonse's frame). Thus, Gaston experiences a proper time of $\sqrt{5^2 - 4^2} = 3years$. The same occurs on the trip back. So, the total proper time experienced by Gaston is $\Delta\tau_G = 6years$.

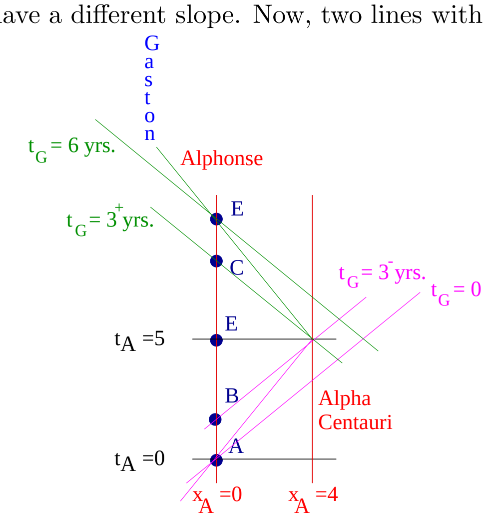
Is Gaston really younger then when they get back together? Couldn't we draw the same picture in Gaston's frame of reference and reach the opposite conclusion? *NO, we cannot.* The reason is that Gaston's frame of reference is not an inertial frame! Gaston does not always move in a straight line at constant speed with respect to Alphonse. In order to turn around and come back, Gaston must experience some force which makes him non-inertial. Most importantly, Gaston knows this! When, say, his rocket engine fires, he will feel the force acting on him and he will know that he is no longer in an inertial reference frame.

⋆⋆ The point here is *not* that the process is impossioble to describe in Gaston's frame of reference. Gaston experiences what he experiences, so there must be such a description. The point is, however, that *so far* we have not worked out the rules to understand frames of reference that are not inertial. Therefore, we cannot simply blindly apply the time dilation/length contraction rules for inertial frames to Gaston's frame of reference. Thus, we should not expect our results so far to directly explain what is happening from Gaston's point of view.

But, you might say, Gaston is *almost* always in an inertial reference frame. He is in one inertial frame on the trip out, and he is in another inertial frame on the trip back. What happens if we just put these two frames of reference together?

Let's do this, but we must do it carefully since we are now treading new ground. First, we should draw in Gaston's lines of simultaneity on Alphonse's spacetime diagram above. His lines of simultaneity will match[2] simultaneity in one inertial frame during the trip out, but they will match those of a *different* frame during the trip back. Then, I will use those lines of simultaneity to help me draw a diagram in Gaston's not-quite-inertial frame of reference, much as we have done in the past in going from one inertial frame to another.

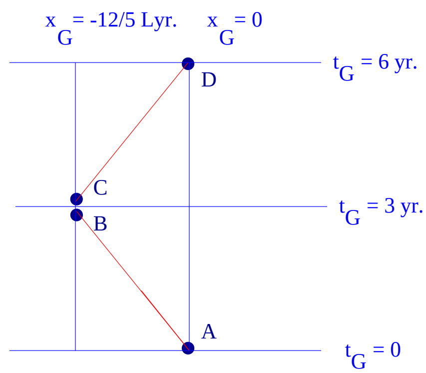
Since Gaston is in a different inertial reference frame on the way out than on the way back, I will have to draw two sets of lines of simultaneity and each set will have a different slope. Now, two lines with different slopes must intersect......



Here, I have marked several interesting events on the diagram, and I have also labeled the lines of simultaneity with Gaston's proper time at the events where he crosses those lines. Note that there are *two* lines of simultaneity marked $t_G = 3 years!$. I have marked one of these 3^- (which is "just before" Gaston turns around) and I have marked one 3^+ (which is "just after" Gaston turns around).

If I simply knit together Gaston's lines of simultaneity and copy the events from the diagram above, I get the following diagram in Gaston's frame of reference. By the way, it is safe to use the standard length contraction result to find that *in the inertial frame of Gaston on his trip out* and **in the inertial frame of Gaston on his way back** the distance between Alphonse and Alpha Centauri is $4Lyr\sqrt{1-(4/5)^2} = (12/5)Lyr$.

[2]Strictly speaking, we have defined lines of simultaneity only for observers who remain inertial for all time. However, for an observer following a segment of an inertial worldline, it is natural to introduce lines of simultaneity which match the lines of simultaneity in the corresponding fully inertial frame.



There are a couple of weird things here. For example, what happened to event E? In fact, what happened to all of the events between B and C? By the way, how old is Alphonse at event B? In Gaston's frame of reference (which *is* inertial before $t_G = 3$, so we can safely calculate things that are confined to this region of time), Alphonse has traveled $(12/5)Lyr.$ in three years. So, Alphonse must experience a proper time of $\sqrt{3^2 - (12/5)^2} = \sqrt{9 - 144/25} = \sqrt{81/25} = (9/5)years$. Similarly, Alphonse experiences $(9/5)years$ between events C and D. This means that there are $10 - 18/5 = (32/5)years$ of Alphonse's life missing from the diagram. **(Oooops!)**

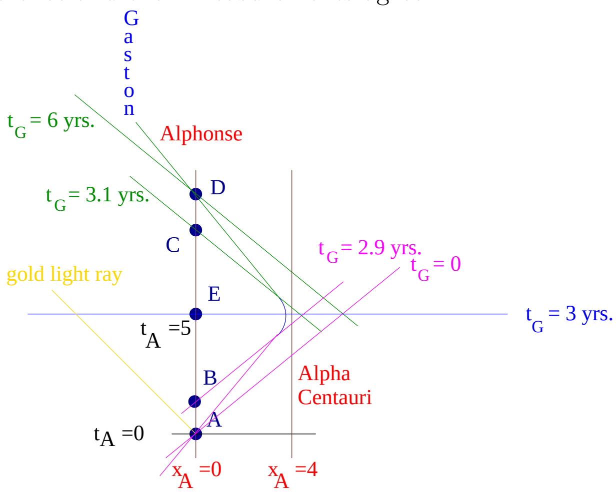
It turns out that part of our problem is the sharp corner in Gaston's worldline. The corner means that Gaston's acceleration is infinite there, since he changes velocity in zero time. Let's smooth it out a little and see what happens.

Suppose that Gaston still turns around quickly, but not so quickly that we cannot see this process on the diagram. If the turn-around is short, this should not change any of our proper times very much (proper time is a continuous function of the curve!!!), so Gaston will still experience roughly 6 years over the whole trip, and roughly 3 years over half. Let's say that he begins to slow down (and therefore ceases to be inertial) after 2.9 years so that after 3 years he is momentarily at rest with respect to Alphonse. Then, his acceleration begins to send him back home. A tenth of a year later (3.1 years into the trip) he reaches .8 c, his rockets shut off, and he coasts home as an inertial observer.

We have already worked out what is going on during the periods where Gaston is inertial. But, what about during the acceleration? Note that, at each instant, Gaston is in fact at rest in *some* inertial frame – it is just that he keeps changing from one inertial frame to another. One way to draw a spacetime diagram for Gaston is try to use, at each time, the inertial frame with respect to which he is at rest. This means that we would use the inertial frames to draw in more of Gaston's lines of simultaneity on Alphonse's diagram, at which point we can again copy things to Gaston's diagram.

A line that is particularly easy to draw is Gaston's $t_G = 3year$ line. This is because, at $t_G = 3years$, Gaston is momentarily at rest relative to Alphonse. This means that Gaston and Alphonse share a line of simultaneity! Of course, they label it differently. For Alphonse, it it $t_A = 5years$. For Gaston, it is

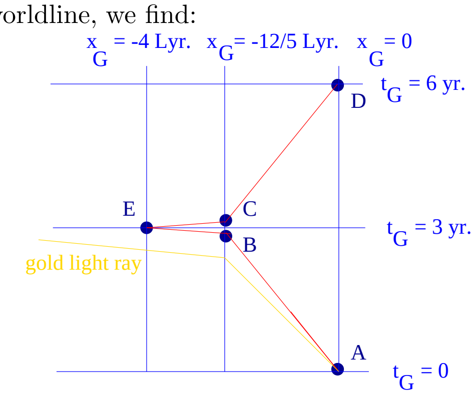
$t_G = 3years$. On that line, Alphonse and Gaston have a common frame of reference and their measurements agree.



Note that we finally have a line of simultaneity for Gaston that passes through event E!!! So, event E really does belong on Gaston's $t_G = 3year$ line after all. By the way, just "for fun" I have added to our diagram an light ray moving to the left from the origin.

We are almost ready to copy the events onto Gaston's diagram. But, to properly place event R, we must figure out just *where* it is in Gaston's frame. In other words, how far away is it from Gaston along the line $t_G = 3years$? Recall that, along that particular line of simultaneity, Gaston and Alphonse measure things in the same way. Therefore they agree that, along that line, event E is four light years away from Alphonse.

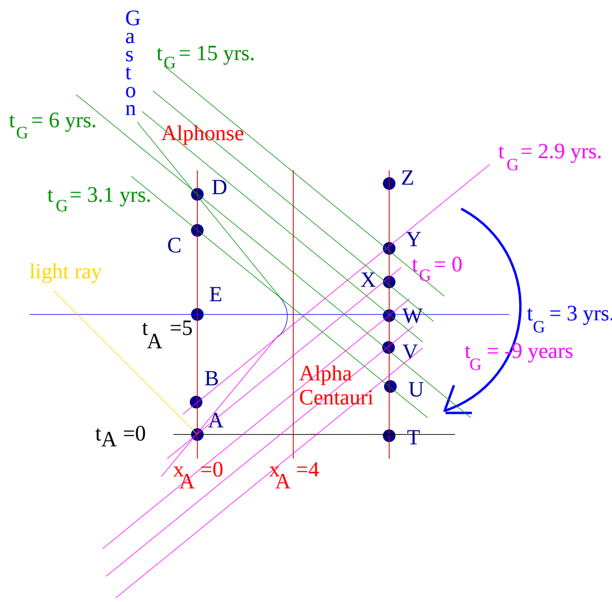
Placing event E onto Gaston's diagram connecting the dots to get Alphonse's worldline, we find:



There is something interesting about Alphonse's worldline between B and E. It is almost horizontal, and has speed much greater than one light-year per year!!! What is happening?

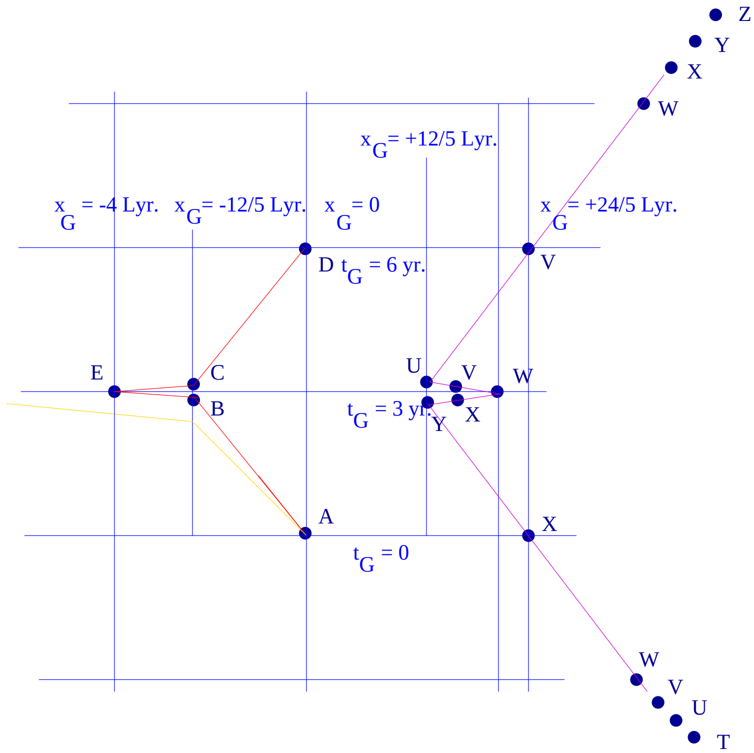
Notice that I have also drawn in the gold light ray from above, and that it too moves at more than one light-year per year in this frame. We see that Alphonse is in fact moving more slowly than the light ray, which is good. However, we also see that the speed of a light ray is not in general equal to c in an accelerated reference frame! In fact, it is not event constant since the gold light ray appears 'bent' on Gaston's diagram. Thus, it is only in inertial frames that light moves at a constant speed of 3×10^8 meters per second. This is one reason to avoid drawing diagrams in non-inertial frames whenever you can.

Actually, though, things are even worse than they may seem at first glance.... Suppose, for example, that Alphonse has a friend Zelda who in an inertial observer at rest with respect to Alphonse, but located four light years on the other side of Alpha Centauri. We can then draw the following diagram in Alphonse's frame of reference:



Once again, we simply can use Gaston's lines of simultaneity to mark the events (T,U,V,W,X,Y,Z) in Zelda's life on Gaston's diagram. In doing so, however, we find that some of Zelda's events appear on *TWO* of Gaston's lines of simultaneity – a (magenta) one from before the turnaround and a (green) one from after the turnaround! In fact, many of them (like event W) appear on **three** lines of simultaneity, as they are caught by a third 'during' the turnaround when Gaston's line of simultaneity sweeps downward from the magenta $t = 2.9$ to the green $t = 3.1$ as indicated by the big blue arrow!

Marking all of these events on Gaston's diagram (taking the time to first calculate the corresponding positions) yields something like this:



The events T, U, V, W at the very bottom and W, X, Y, Z are not drawn to scale, but they indicate that Zelda's worldline is reproduced in that region of the diagram in a more or less normal fashion.

Let us quickly run though Gaston's description of Zelda's life: Zelda merrily experiences events T, U, V, W, X, and Y. Then, Zelda is described as "moving backwards in time" through events Y, X, W, V, and U. During most of this period she is also described as moving faster than one light-year per year. After Gaston's $t_G = 3.1 year$ line, Zelda is again described as moving forward in time (at a speed of 4 light-years per 5 years), experiencing events V, W, X, Y, for the third time and finally experiencing event Z.

The moral here is that non-inertial reference frames are "all screwed up." Observers in such reference frames are likely to describe the world in a very funny way. To figure out what happens to them, it is certainly best to work in an *inertial* frame of reference and use it to carefully construct the non-inertial spacetime diagram.

⋆⋆ By the way, there is also the issue of what Gaston would *see* if he watched Alphonse and Zelda through a telescope. This has to do with the sequence in which light rays reach him, and with the rate at which they reach him. This is also interesting to explore, but I will leave it for the homework (see problem 4).

4.3 More on Minkowskian Geometry

Now that we've ironed out the twin paradox, I think it's time to talk more about Minkowskian Geometry (a.k.a. "why you should like relativity"). We will shortly see that understanding this geometry makes relativity much simpler. Or, perhaps it is better to say that relativity is in fact simple but that we so far been viewing it through a confusing "filter" of trying to separate space and time. Understanding Minkowskian geometry removes this filter, as we realize that space and time are really part of the same object. (I am sure that, somewhere, there is a very appropriate Buddhist quote that should go here.)

4.3.1 Drawing proper time and proper distance

Recall that in section 4.1 we introduced the notion of the spacetime *interval*. The interval was a quantity built from both time and space, but which had the interesting property of being the same in all reference frames. We write it as:

$$(interval)^2 = \Delta x^2 - c^2 \Delta t^2. \tag{4.3}$$

Recall also that this quantity has two different manifestations[3]: proper time, and proper distance. In essence these are much the same concept. However, it is convenient to use one term (proper time) when the squared interval is negative and another (proper distance) when the squared interval is positive.

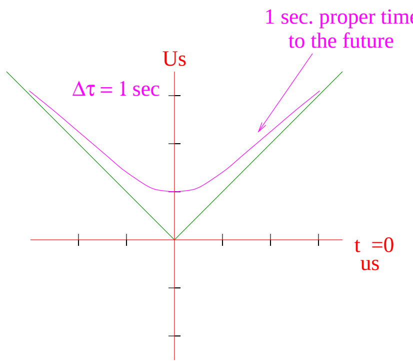
Let's draw some pictures to better understand these concepts. Suppose I want to draw (in an inertial frame) the set of all events that are one second of *proper* time ($\Delta\tau = 1sec$) to the future of some event (x_0, t_0). We have

$$-(1sec)^2 = -\Delta\tau^2 = \Delta t^2 - \Delta x^2/c^2.$$

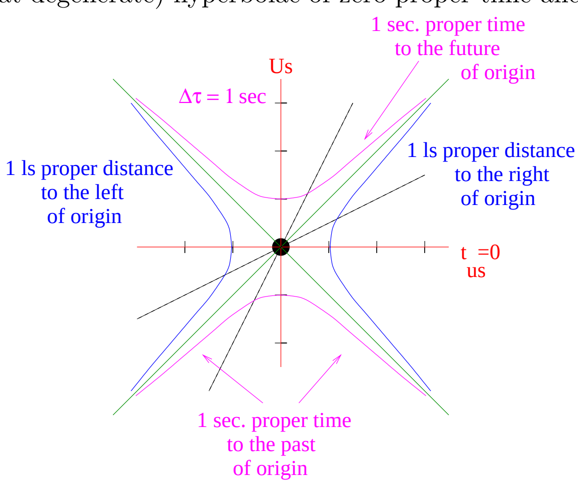
Suppose that we take $x_0 = 0, t_0 = 0$ for simplicity. Then we have just $x^2/c^2 - t^2 = -(1sec)^2$.

You may recognize this as the equation of a hyperbola with focus at the origin and asymptotes $x = \pm ct$. In other words, the hyperbola asymptotes to the light cone. Since we want the events one second of proper time to the future, we draw just the top branch of this hyperbola:

[3]Insert appropriate Hindu quote here.



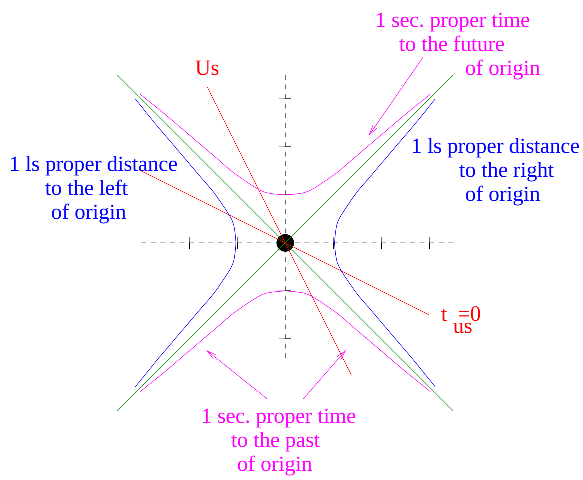
There are similar hyperbolae representing the events one second of proper time in the past, and the events one light-second of proper distance to the left and right. We should also note in passing that the light light rays form the (somewhat degenerate) hyperbolae of zero proper time and zero proper distance.



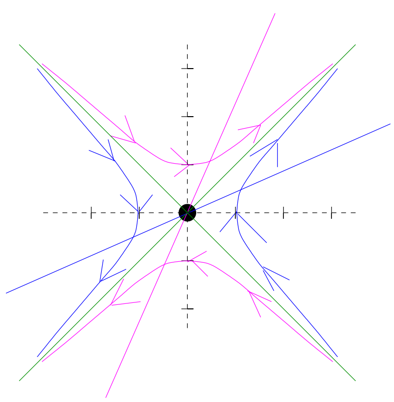
4.3.2 Changing Reference Frames

Note that on the diagram above I have drawn in the worldline and a line of simultaneity for a second inertial observer moving at half the speed of light relative to the first. How would the curves of constant proper time and proper distance look if we re-drew the diagram in this new inertial frame? **Stop reading and think about this for a minute.**

Because the separation of two events in **proper** time and **proper** distance is invariant (i.e., independent of reference frame), these curves must look exactly the same in the new frame. That is, any event which is one second of proper time to the future of some event A (say, the origin in the diagram above) in one inertial frame is also one second of proper time to the future of that event in any other inertial frame and therefore must lie on the same hyperbola $x^2 - c^2t^2 = -(1sec)^2$. The same thing holds for the other proper time and proper distance hyperbolae.

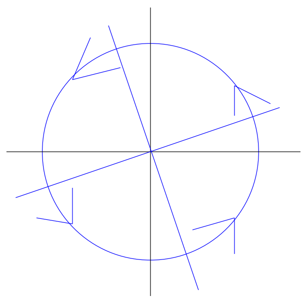


> We see that changing the inertial reference frame simply slides events along a given hyperbola of constant time or constant distance, but does not move events from one hyperbola to another.



Remember our Euclidean geometry analogue from last time? The above observation is exactly analogous to what happens when we rotate an object[4]. The points of the object move along circles of constant radius from the axis, but do not hop from circle to circle.

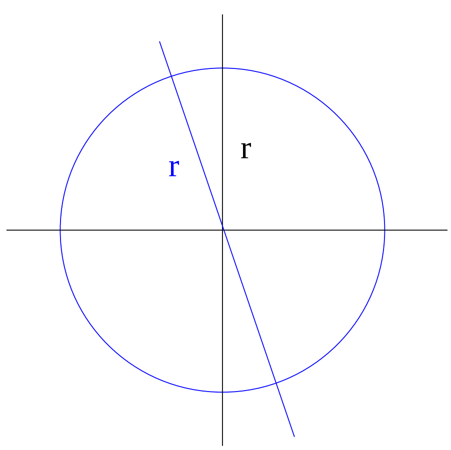
[4] Actually, it is analogous to what happens when we rotate but the object stays in place. This is known as the different between an 'active' and a 'passive' rotation. It seemed to me, however, that the main idea would be easier to digest if I did not make a big deal out of this subtlety.



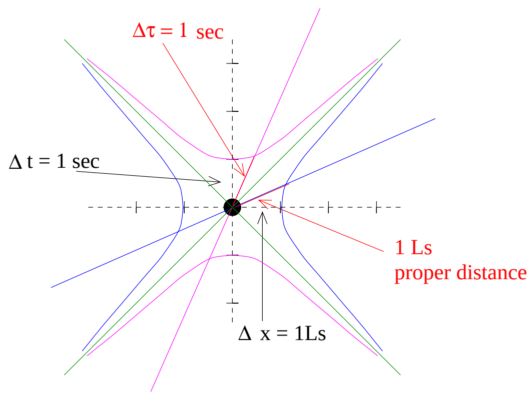
By the way, the transformation that changes reference frames is called a 'boost.' (Think of an object being "boosted" up to a higher level or strength (speed), or think of a "booster" stage on a rocket.) So, what I mean is that "boosts are analogous to rotations."

4.3.3 Hyperbolae, again

In order to extract the most from our diagrams, let's hit the analogy with circles one last time. If I draw an arbitrary straight line through the center of a circle, it always intersects the circle a given distance from the center.



What happens if I draw an arbitrary straight line through the origin of our hyperbolae?



If it is a timelike line, it could represent the worldline of some inertial observer. Suppose that the observer's clock reads zero at the origin. Then the worldline intersects the future $\Delta\tau = 1sec$ hyperbola at the event where that observers clock reads one second.

Similarly, since a spacelike line is the line of simultaneity of some inertial observer. It intersects the $d = 1Ls$ curve at what that observer measures to be a distance of $1Ls$ from the origin.

What we have seen is that these hyperbolae encode the Minkowskian geometry of spacetime. The hyperbolae of proper time and proper distance (which are different manifestations of the same interval) are the right way to think about how events are related in spacetime and make things much simpler than trying to think about time and space separately.

4.3.4 Boost Parameters and Hyperbolic Trigonometry

I keep claiming that this Minkowskian geometry will simplify things. So, you might rightfully ask, what exactly can we do with this new way of looking at things? Let's go back and look at how velocities combine in relativity. This is the question of "why don't velocities just add?" Or, if I am going at 1/2 c relative to Alice, and Charlie is going at 1/2 c relative to me, how fast is Charlie going relative to Alice? Deriving the answer is part of your homework in problem 3-10. As you have already seen in Einstein's book, the formula looks like

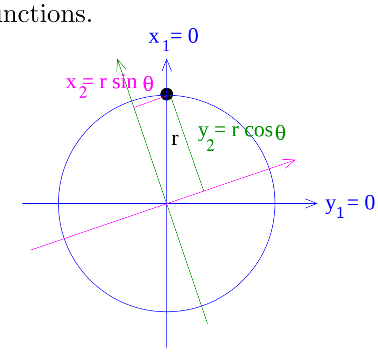
$$v_{AC}/c = \frac{v_{AB}/c + v_{BC}/c}{1 + v_{AB}v_{BC}/c^2}. \tag{4.4}$$

It is interesting to remark here that this odd effect was actually observed *experimentally* by Fizeau in the 1850's. He managed to get an effect big enough to see by looking at light moving through a moving fluid (say, a stream of water). The point is that, when it is moving through water, light does not in fact travel at speed c. Instead, it travels *relative to the water* at a speed c/n where n is around 1.5. The quantity n is known as the 'index of refraction' of water.

Thus, it is still moving at a good fraction of "the speed of light." Anyway, if the water is also flowing (say, toward us) at a fast rate, then the speed of the light toward us is given by the above expression in which the velocities do not just add together. This is just what Fizeau found[5], though he had no idea why it should be true!

Now, the above formula looks like a mess. Why in the world should the composition of two velocities be such an awful thing? As with many questions, the answer is that the awfulness is not in the composition rule itself, but in the *filter* (the notion of velocity) through which we view it. We will now see that, when this filter is removed and we view it in terms native to Minkowskian geometry, the result is quite simple indeed.

Recall the analogy between boosts and rotations. How do we describe rotations? We use an angle θ. Recall that rotations mix x and y through the sine and cosine functions.



$$\begin{aligned} x_2 &= r\sin\theta, \\ y_2 &= r\cos\theta. \end{aligned} \tag{4.5}$$

Now, one of the basic facts associated with the relation of sine and cosine to circles is the relation:

$$\sin^2\theta + \cos^2\theta = 1. \tag{4.6}$$

Similarly, there are other natural mathematical functions called *hyperbolic* sine (sinh) and *hyperbolic* cosine (cosh) that satisfy

$$\cosh^2\theta - \sinh^2\theta = 1, \tag{4.7}$$

so that they are related to hyperbolae.

These functions can be defined in terms of the exponential function, e^x:

$$\sinh\theta = \frac{e^\theta - e^{-\theta}}{2}$$

[5]Recall that Fizeau's experiments were one of the motivations for Michelson and Morely. Thus, we now understand the results not only of Michelson and Morely's experiment, but also of the experiments that prompted their work.

$$\cosh\theta = \frac{e^{\theta}+e^{-\theta}}{2}. \tag{4.8}$$

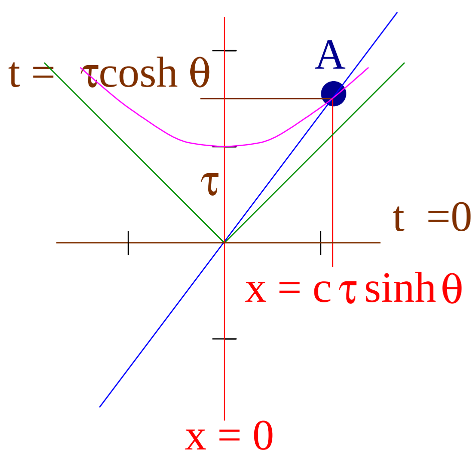
You can do the algebra to check for yourself that these satisfy relation (4.7) above. By the way, although you may not recognize this form, these functions are actually very close to the usual sine and cosine functions. Introducing $i = \sqrt{-1}$, one can write sine and cosine as[6].

$$\begin{aligned} \sin\theta &= \frac{e^{i\theta}-e^{-i\theta}}{i2} \\ \cos\theta &= \frac{e^{i\theta}+e^{-i\theta}}{2}. \end{aligned} \tag{4.9}$$

Thus, the two sets of functions differ only by factors of i which, as you can imagine, are related to the minus sign that appears in the formula for the squared interval.

Now, consider any event (A) on the hyperbola that is a proper time τ to the future of the origin. Due to the relation 4.7, we can write the coordinates t, x of this event as:

$$\begin{aligned} t &= \tau\cosh\theta, \\ x &= c\tau\sinh\theta. \end{aligned} \tag{4.10}$$



On the diagram above I have drawn the worldline of an inertial observer that passes through both the origin and event A. Note that the parameter θ gives some notion of how different the two inertial frames (that of the moving observer

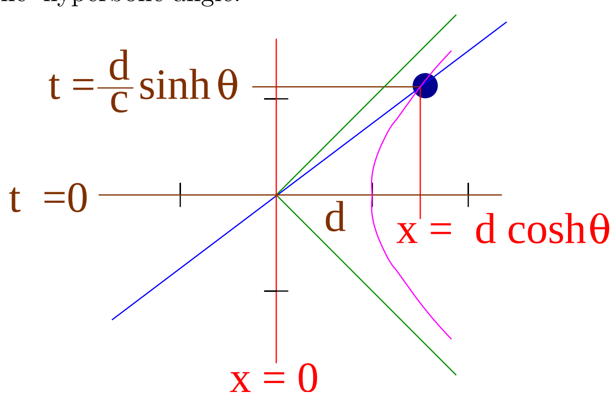
[6]These representations of sine and cosine may be new to you. That they are correct may be inferred from the following observations: 1) Both expressions are real. 2) If we square both and add them together we get 1. Thus, they represent the path of an object moving around the unit circle. 3) They satisfy $\frac{d}{d\theta}\sin\theta = \cos\theta$ and $\frac{d}{d\theta}\cos\theta = -\sin\theta$. As a result, for $\theta = \omega t$ they represent an object moving around the unit circle at angular velocity ω.

and that of the stationary observer) actually are. For $\theta = 0$, event A is at $x = 0$ and the two frames are the same, while for large θ event A is far up the hyperbola and the two frames are very different.

We can parameterize the points that are a proper distance d from the origin in a similar way, though we need to 'flip x and t.'

$$\begin{aligned} t &= d/c \sinh\theta, \\ x &= d \cosh\theta. \end{aligned} \tag{4.11}$$

If we choose the same value of θ, then we do in fact just interchange x and t, "flipping things about the light cone." Note that this will take the worldline of the above inertial observer into the corresponding line of simultaneity. In other words, a given worldline and the corresponding line of simultaneity have the same 'hyperbolic angle.'



Again, we see that θ is really a measure of the separation of the two reference frames. In this context, we also refer to θ as the *boost parameter* relating the two frames. The boost parameter is another way to encode the information present in the relative velocity, and in particular it is a very natural way to do so from the viewpoint of Minkowskian geometry.

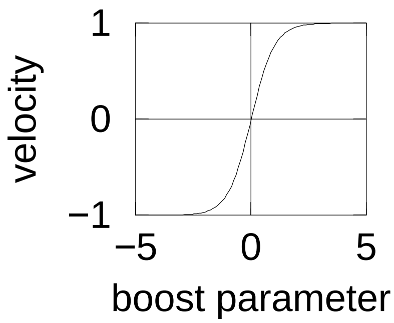
In what way is the relative velocity v of the reference frames related to the boost parameter θ? Let us again consider the inertial observer passing from the origin through event A on the hyperbola of constant proper *time*. This observer moves at speed:

$$v = \frac{x}{t} = \frac{c\tau \sinh\theta}{\tau\cosh\theta} = c\frac{\sinh\theta}{\cosh\theta} = c\tanh\theta, \tag{4.12}$$

and we have the desired relation. Here, we have introduced the hyperbolic tangent function in direct analogy to the more familiar tangent function of trigonometry. Note that we may also write this function as

$$\tanh\theta = \frac{e^\theta - e^{-\theta}}{e^\theta + e^{-\theta}}.$$

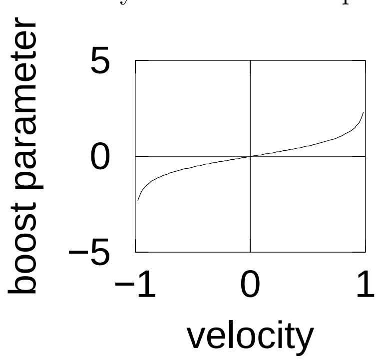
The hyperbolic tangent function may seem a little weird, but we can get a better feel for it by drawing a graph like the one below. The vertical axis is $\tanh\theta$ and the horizontal axis is θ.



To go from velocity v to boost parameter θ, we just invert the relationship:

$$\theta = \tanh^{-1}(v/c).$$

Here, $\tanh^{-1}$ is the function such that $\tanh^{-1}(\tanh\theta) = \tanh(\tanh^{-1}\theta) = 1$. This one is difficult to write in terms of more elementary functions (though it can be done). However, we can draw a nice graph simply by 'turning the above picture on its side.' The horizontal axis on the graph below is x and the vertical axis is $\tanh^{-1} x$. Note that two reference frames that differ by the speed of light in fact differ by an infinite boost parameter.



Now for the magic: Let's consider three inertial reference frames, Alice, Bob, and Charlie. Let Bob have boost parameter $\theta_{BC} = \tanh^{-1}(v_{BC}/c)$ relative to Charlie, and let Alice have boost parameter $\theta_{AB} = \tanh^{-1}(v_{AB}/c)$ relative to Bob. Then the relative velocity of Alice and Charlie is

$$v_{AC}/c = \frac{v_{AB}/c + v_{BC}/c}{1 + v_{AB}v_{BC}/c^2}. \tag{4.13}$$

Let's write this in terms of the boost parameter:

$$v_{AC}/c = \frac{\tanh(\theta_{AB}) + \tanh(\theta_{BC})}{1 + \tanh(\theta_{AB})\tanh(\theta_{BC}/c^2)}. \tag{4.14}$$

After a little algebra (which I have saved for your next homework), one can show that this is in fact:

$$\boxed{v_{AC}/c = \tanh(\theta_{AB} + \theta_{BC}). \quad (4.15)}$$

In other words, the boost parameter θ_{AC} relating Alice to Charlie is just the sum of the boost parameters θ_{AB} and θ_{BC}.

Boost parameters add:

$$\theta_{AC} = \theta_{AB} + \theta_{BC}!! \tag{4.16}$$

Because boost parameters are part of the native Minkowskian geometry of spacetime, they allow us to see the rule for combining boosts in a simple form. In particular, they allow us to avoid the confusion created by first splitting things into space and time and introducing the notion of "velocity."

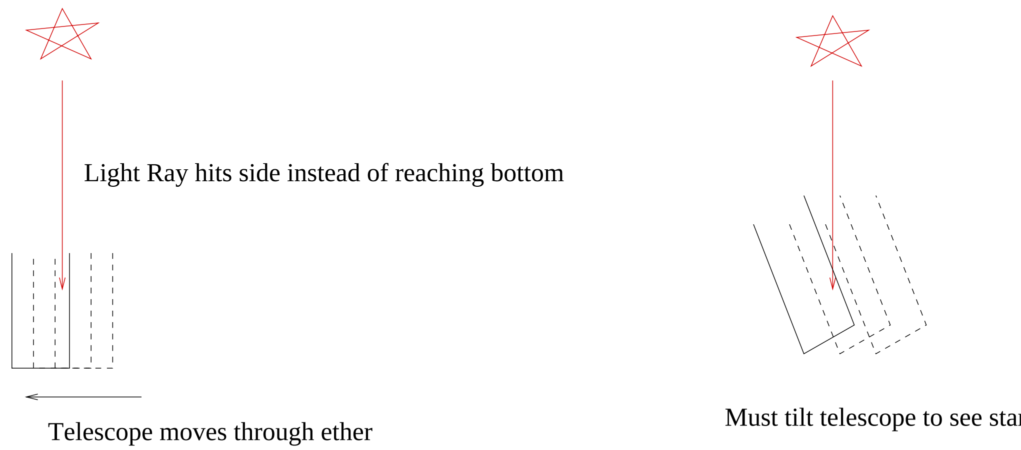
4.4 2+1 and higher dimensional effects: A return to Aberration

So, we are beginning to understand how this relativity stuff works, and how it can be self-consistent. In the last section we even saw that relativity looks pretty when viewed it is viewed in the right way! However, we are still missing something....

Although we now 'understand' the fact that the speed of light is the same in all inertial reference frames (and thus the Michelson-Morely experiment), recall that it was not just the Michelson-Morely experiment that compelled us to abandon the ether and to move to this new point of view. Another very important set of experiments involved stellar aberration (the tilting telescopes) – a subject to which we need to return. One might think that assuming the speed of light to be constant in all reference frames would remove *all* effects of relative motion on light, in which case the stellar aberration experiments would contradict relativity. However, we will now see that this is not so.

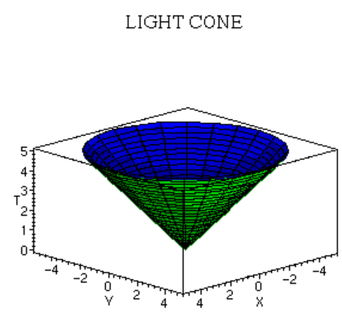
4.4.1 Aberration in Relativity

Recall the basic setup of the aberration experiments. Starlight hits the earth from the side, but the earth is "moving forward" so this somehow means that astronomers can't point their telescopes straight toward the star if they actually want to see it. This is shown in the diagram below.

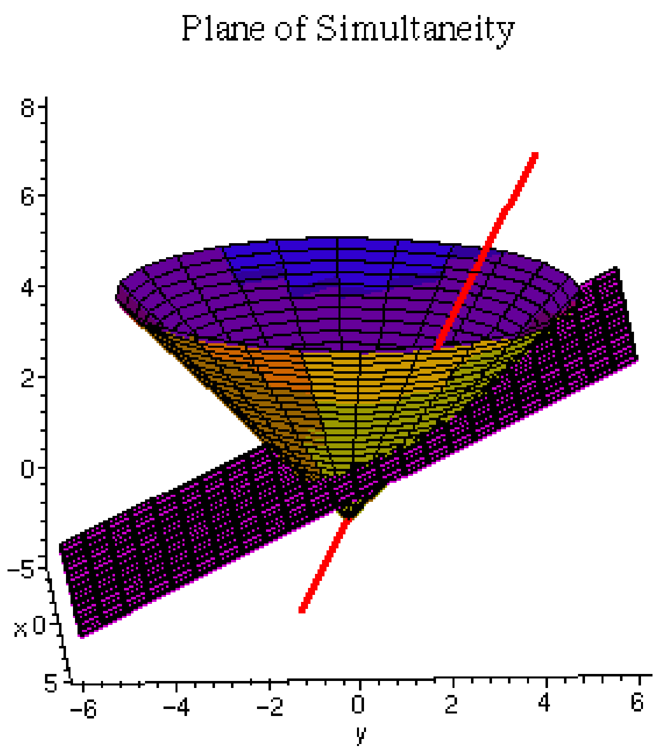


To reanalyze the situation using our new understanding of relativity we will have to deal the fact that the star light comes in from the side while the earth travels forward (relative to the star). Thus, we will need to use a spacetime diagram having three dimensions – two space, and one time. One often calls such diagrams "2+1 dimensional." These are harder to draw than the 1+1 dimensional diagrams that we have been using so far, but are really not so much different. After all, we have already talked a little bit about the fact that, under a boost, things behave reasonably simply in the direction perpendicular to the action of the boost: neither simultaneity nor lengths are affected in that direction.

We'll try to draw 2+1 dimensional spacetime diagrams using our standard conventions: all light rays move at 45 degrees to the vertical. Thus, a light cone looks like this:

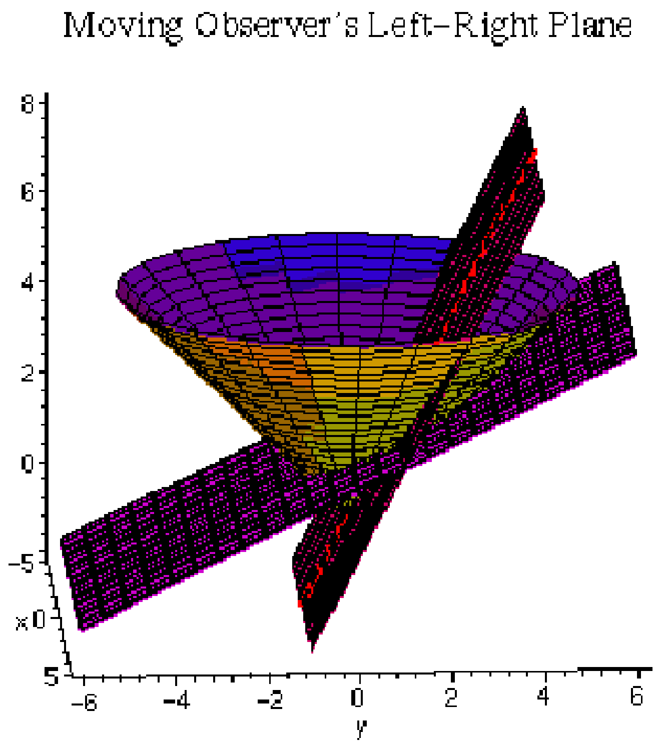


We can also draw an observer and their *plane* of simultaneity.



In the direction of the boost, this plane of simultaneity acts just like the lines of simultaneity that we have been drawing. However, in the direction perpendicular to the boost direction, the boosted plane of simultaneity is not tilted. This is the statement that simultaneity is not affected in this direction.

Now, let's put this all together. I also want to display the moving observer's idea of "right and left," so I have drawn the plane of events that the moving observer finds to be straight to her right or to their left (and not at all in front of or behind he). Here, the observer is moving across the paper, so her "right and left" are more or less into and out of the paper.



Of course, we would like to know how this all looks when redrawn in the moving observer's reference frame. One thing that we know is that every ray of light must still be drawn along some line at 45 degrees from the vertical. Thus, it will remain on the light cone. However, it may not be located at the same **place** on this light cone. In particular, note that the light rays direct straight into and out of the page as seen in the original reference frame are 'left behind' by the motion of the moving observer. That is to say that our friend is moving away from the plane containing these light rays. Thus, in the moving reference frame these two light rays do not travel straight into and out of the page, but instead move somewhat in the "backwards" direction!

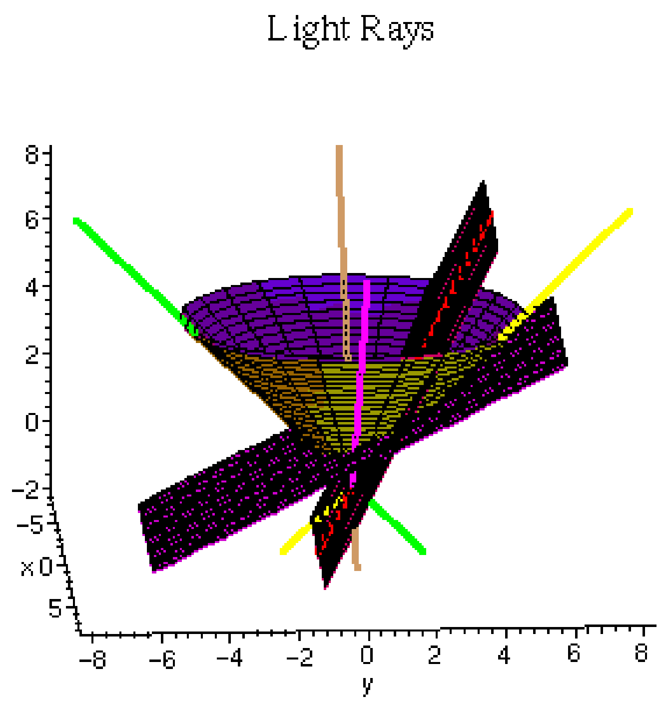
This is how the aberration effect is described in relativity. Suppose that, in the reference frame of our sun, the star being viewed through the telescope is "straight into the page." Then, in the reference frame of the sun, the light from the star is a light ray coming straight out from the page. However, in the "moving" reference frame of the earth, this light ray appears to be moving a bit "backwards." Thus, astronomers must point their telescopes a bit forward in order to catch this light ray.

Qualitatively, the aberration effect is actually quite similar in Newtonian and post-Einstein physics. However, you may recall that I mentioned before that the actual *amount* of the aberration effect observed in the 1800's made no sense to physicists of the time. This is because, at the quantitative level, the Newtonian and post-Einstein aberration effects are quite different. As usual, the post-Einstein version gets the numbers exactly correct, finally tying up the loose ends of 19th century observations. Einstein's idea that the speed of light is in fact the same in all inertial reference frames wins again.

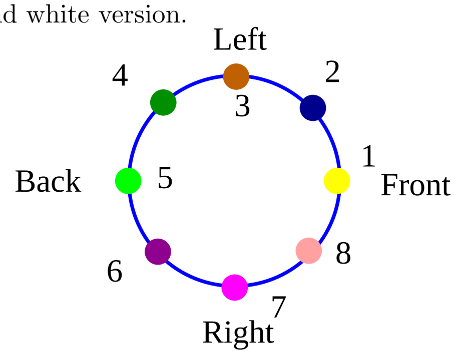
4.4.2 More on boosts and the 2+1 light cone: the headlight effect

It is interesting to explore the effect of boosts on 2+1 light cones in more detail, as this turns out to uncover two more new effects. Instead of investigating this by drawing lots of three-dimensional pictures, it is useful to find a way to encode the information in terms of a two-dimensional picture that is easily drawn on the blackboard or on paper. We can do this by realizing that the light *cone* above can be thought of as being made up of a collection of light **rays** arrayed in a **circle**.

To see what I mean, consider some inertial reference frame, perhaps the one in which one of the above diagrams is drawn. That observer finds that light from an "explosion" at the origin moves outward along various rays of light. One light ray travels straight forward, one travels straight to the observer's left (into the page), one travels straight to the observer's right (out of the page), and one travels straight backward. These light rays have been drawn below (front = yellow, back = green, left (into page) =tan, right (out of page) = magenta in the color version).

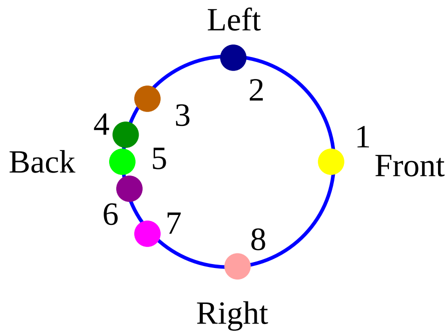


There is one light ray traveling outward in each direction, and of course the set of all directions (in two space dimensions) forms a circle. Thus, we may talk about the circle of light rays. It us convenient to dispense with all of the other parts of the diagram and just draw this circle of light rays. The picture below depicts the circle of light rays in the same reference frame used to draw the above diagram and uses the corresponding colored dots to depict the front, back, left, and right light rays. I have also added a few more light rays for future use and I have numbered the light rays to help keep track of them in the black and white version.



Light Circle in the Original Frame

Now let's draw the corresponding circle of light rays from the moving observer's perspective. As we saw before, a given light ray from one reference frame is still some light ray in the new reference frame. Therefore, the effect of the boost on the light cone can be described by simply moving the various dots to appropriate new locations on the circle. For example, the light rays that originally traveled straight into and out of the page now fall a bit 'behind' the moving observer. So, they are now moved a bit toward the back. Front, back, left, and right now refer to the new reference frame.



Light Circle in the Second Frame

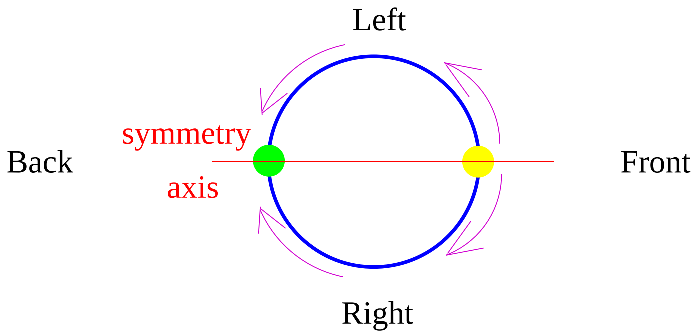
Note that most of the dots have fallen toward our current observer's back side – the side which represents (in the current reference frame) the direction of motion of the first observer! Suppose then that the first observer were actually, say, a star like the sun. In it's own rest frame, a star shines more or less equally brightly in all directions – in other words, it emits the same number of rays of light in all directions. So, if we drew those rays as dots on a corresponding light-circle in the star's frame of reference, they would all be equally spread out as in the *first* light circle we drew above.

$\star\star$ What we see, therefore, is that in another reference frame (with respect to which the star is moving) the light rays do not radiate symmetrically from the star. Instead, most of the light rays come out in one particular direction! In particular, they tend to come out in the direction that the star is moving. Thus, in this reference frame, the light emitted by the star is bright in the direction of motion and dim in the opposite direction and the star shines like a beacon in the direction it is moving. For this reason, this is known as the "headlight" effect.

By the way, this effect is seen all the time in high energy particle accelerators and has important applications in materials science and medicine. Charged particles whizzing around the accelerator emit radiation in all directions as described in their own rest frame. However, in the frame of reference of the laboratory, the radiation comes out in a tightly focussed beam in the direction of the particles' motion. This means that the radiation can be directed very precisely at materials to be studied or tumors to be destroyed.

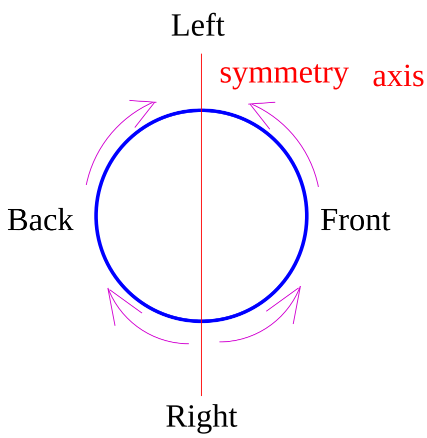
4.4.3 Multiple boosts in 2+1 dimensions

Cool, eh? But wait! We're not done yet. I want you to look back at the above two circles of light rays and notice that there is a certain symmetry about the direction of motion. So, suppose you are given a circle of light rays marked with dots which show, as above, the direction of motion of light rays in your reference frame. Suppose also that these light rays were emitted by a star, or by any other source that emits equally in all directions in its rest frame. Then you can tell which direction the star is moving relative to you by identifying the symmetry axis in the circle! There must always be such a symmetry axis. The result of the boost was to make the dots flow as shown below:

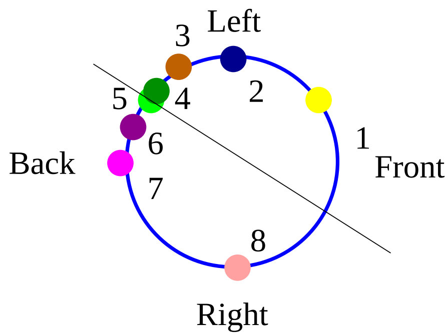


The yellow and green (font and back) dots are on the symmetry axis, and so do not move at all.

So, just for fun, let's take the case above and consider *another* observer who is moving not in the forward/backward direction, but instead is moving in the direction that is "left/right" relative to the "moving" observer above. To find out what the dots looks like in the new frame of references, we just rotate the flow shown above by 90 degrees as shown below



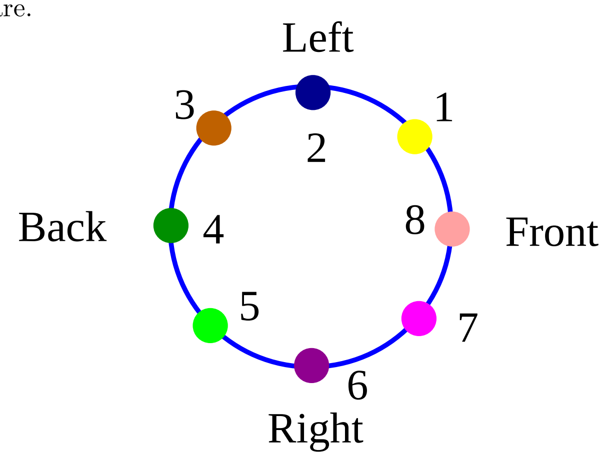
and apply it to the dots in the second frame. The result looks something like this:



The new symmetry axis is shown above. Thus, with respect to the original observer, this new observer is *not* moving along a line straight to the right. Instead, the new observer is moving somewhat in the forward direction as well.

But wait.... something else interesting is going on here.... the light rays don't line up right. Note that if we copied the above symmetry axis onto the light circle in the original frame, it would sit exactly on top of rays 4 and 8. However, in the figure above the symmetry axis sits half-way *between* 1 and 8 and 4 and 5. This is the equivalent of having first rotated the light circle in the original frame by 1/16 of a revolution before performing a boost along the new symmetry axis! The new observer differs from the original one not just by a boost, but by a rotation as well!

In fact, by considering two further boost transformations as above (one acting only backward, and then one acting to the right), one can obtain the following circle of light rays, which are again evenly distributed around the circle. You should work through this for yourself, pushing the dots around the circle with care.



Thus, by a series of boosts, one can arrive at a frame of reference which, while it is not moving with respect to the original fame, is in fact *rotated* with respect to the original frame. By applying only boost transformations, we have managed to turn our observer by 45 degrees in space. This just goes to show again that time and space are completely mixed together in relativity, and that boost transformations are even more closely related to rotations than you might have thought.

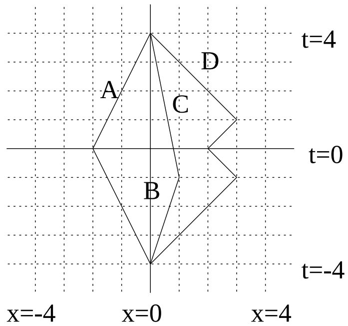
A boost transformation can often be thought of as a "rotation of time into space." In this sense the above effect may be more familiar: Consider three perpendicular axes, x, y, and z. By performing only rotations about the x and y axes, one can achieve the same result as any rotation about the z axes. In the above discussion, the boosts are analogous to rotations about the x and y axes, while the rotation is indeed a rotation 'about the t axis.'

4.4.4 Other effects

Boosts in 3+1 dimensions and higher works pretty much like it they do in 2+1 dimensions, which as we have seen has only a few new effects beyond the 1+1 case on which we spent most of our time. There is really only one other interesting effect in 2+1 or higher dimensions that we have not discussed. This has to do with how rapidly moving objects actually *look*; that is, they have to do with how light rays actually reach your eyes to be processed by your brain. This is discussed reasonably well in section 4.9 of *Inside Relativity* Mook and Vargish, which used to be a required textbook for this class. This section is probably the greatest loss due to no longer using this text as it is quite well done. It used to be that I would save time by not talking about this effect directly and simply asking students to read that section. At the time of writing these notes, I had not decided for sure whether I would discuss this material this time, but probably time will simply not allow it. For anyone who is interested, I suggest you find a copy of Mook and Vargish (I think there are several in the Physics Library) and read 4.9. If you are interested in reading even more, the references in their footnote 13 on page 117 are a good place to start. One of those papers refers to another paper by Penrose, which is probably the standard reference on the subject. By the way, this can be an excellent topic for a course project, especially if you are artistically inclined or if you like to do computer graphics.

4.5 Homework Problems

4-1. The diagram below is drawn in some inertial reference frame using units of seconds for time and light-seconds for distance. Calculate the total time experienced by a clock carried along each of the four worldlines (A,B,C,D) shown below. Each of the four worldlines starts at $(t = -4, x = 0)$ and ends at $(t = +4, x = 0)$. Path B runs straight up the t-axis. Remember that the proper time for a path that is not straight can be found by breaking it up into straight pieces and adding up the proper times for each piece.



4-2. This problem is to give you some practice putting everything together. Once again, we and our friends, Alice and Bob, are inertial observers who all meet at a single event. At this event, our clock, Alice's clock, and Bob's clock all read zero and a firecracker explodes. Alice moves to our right at $c/2$ and Bob moves to our left at $c/2$.

Draw a single spacetime diagram in our reference frame showing all of the following:

(a) Alice, Bob, and the outgoing light from the explosion.

(b) The curve representing all events that are a *proper* time of one second to the future of the explosion. Also draw in the curves representing the events that are: i) one second of proper time to the past of the explosion, ii) one light second of proper distance to the left of the explosion, and iii) one light second of proper distance to the right of the explosion.

(c) The *events* A, U, and B where Alice's clock reads one second, where our clock reads one second, and where Bob's clock reads one second.

(d) Finally, suppose that we (but not Alice or Bob) are holding the *middle* of a stick that is *two* light seconds long (and which is at rest relative to us). Draw in the worldlines of both ends of that stick. Also mark the *events* X and Y occupied by the ends of that stick on the line $t_{us} = 0$.

4-3. Redraw everything in problem (4-2) using Alice's frame of reference.

4-4. Let's get one more perspective on the twin paradox. It is always interesting to ask what each twin *sees* during the trip. Now, note that what you actually *see* has to do with light rays, and with when a bit of light happens to reach your eye. So, to study this question, we should study light rays sent from one twin to the other. We will again have Gaston go off to Alpha Centauri (4 light-years away) and back at $.8c$ while Alphonse stays at home.

(a) Let's first think about what Gaston (the traveling twin) sees. Start by drawing a spacetime diagram for the trip in any inertial frame (it is easiest to use Alphonse's frame of reference). Now, suppose that Alphonse emits one light ray every year (according to his own proper time). Draw these light rays on the diagram. How many of these light rays does Gaston see on his way out? [Hint: You should be able to read this off of your graph.] How many does Gaston see on his way back? What does this tell you about what Gaston actually *sees* if he watches Alphonse through a telescope?

(b) Now let's figure out what Alphonse sees. Draw another spacetime diagram for the trip in some inertial frame (again, Alphonse's frame is the easiest one to use), but this time suppose that Gaston emits one light ray every year (according to *his* own proper time – you may have to calculate this). What does this diagram tell you about what Alphonse sees if he watches Gaston through a telescope during the trip??

4-5. In relativity, it is always nice to look at things from several different reference frames. Let's look at the same trip of Gaston to Alpha Centauri and back, with Alphonse staying at home. This time, though, draw the spacetime diagram using the inertial reference frame that Gaston had *on his outward trip.* Using this frame of reference, calculate the total proper time that elapses for both Alphonse and Gaston between the event where Gaston leaves Alphonse and the event where they rejoin.

4-6. How about some practice working with hyperbolic trig functions? Recall that

$$\sinh\theta = \frac{e^{\theta} - e^{-\theta}}{2},$$

$$\cosh\theta = \frac{e^{\theta} + e^{-\theta}}{2},$$

and

$$\tanh\theta = \frac{\sinh\theta}{\cosh\theta}$$

.

(a) Verify that $\cosh^2\theta - \sinh^2\theta = 1$.

(b) Verify that

$$\frac{\tanh\theta_1 + \tanh\theta_2}{1 + \tanh\theta_1 \tanh\theta_2} = \tanh(\theta_1 + \theta_2),$$

so that the law of composition of velocities from last week just means that "boost parameters add."

[Hint: Note that

$$\frac{\tanh\theta_1 + \tanh\theta_2}{1 + \tanh\theta_1 \tanh\theta_2} = \frac{(2\cosh\theta_1)(2\sinh\theta_2) + (2\sinh\theta_1)(2\cosh\theta_2)}{(2\cosh\theta_1)(2\cosh\theta_2) + (2\sinh\theta_1)(2\sinh\theta_2)}.$$

Try evaluating the numerator and denominator separately.]

(c) Sketch a graph of $\cosh\theta$ vs. θ, letting θ range from large negative values to large positive values. [Hint: Note that for large positive θ, $e^{-\theta}$ is very small ($e^{-\theta} \approx 0$).]

(d) Sketch a graph of $\sinh\theta$ vs. θ, letting θ range from large negative values to large positive values. [Hint: Note that for large positive θ, $e^{-\theta}$ is very small ($e^{-\theta} \approx 0$).]

(e) Sketch a graph of v/c vs. θ for $v/c = \tanh\theta$, letting θ range from large negative values to large positive values. [Hint: Note that for large positive θ, $e^{-\theta}$ is very small ($e^{-\theta} \approx 0$).]

(f) Let $\tau = 1sec$ and let θ range from large negative values to large positive values. Sketch the curve in spacetime described by:

$$t = \tau\cosh\theta,$$
$$x = (c\tau)\sinh\theta.$$

(g) Let $d = 1Ls$ and let θ range from large negative values to large positive values. Sketch the curve in spacetime described by:

$$t = (d/c)\sinh\theta,$$
$$x = d\cosh\theta.$$

4-7. Suppose that Alice is moving to your right at speed $c/2$ (relative to you) and Bob is moving to Alice's right at speed $c/2$ relative to her.

(a) What is the boost parameter between you and Alice?

(b) Between Alice and Bob?

(c) Between you and Bob? Use this to calculate the relative velocity between you and Bob. [Hint: Your calculator almost certainly[7] has a built-in function called $\tanh^{-1}$ or arctanh. The relation between θ and v can be written as $\theta = \tanh^{-1}(v/c) = \text{arctanh}(v/c)$.]

[7] On some modern calculators this function is hidden inside various menus. If you really can't find it you may want to use the fact that $\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$.

Chapter 5

Accelerating Reference Frames in Special Relativity

We have now reached an important point in our study of relativity. Although I know that many of you are still absorbing it, we have learned the basic structure of the new ideas about spacetime, how they developed, and how they fit with the various pieces of experimental data. We have also finished all of the material in Einstein's *Relativity* (and in fact in most introductions) associated with so-called 'special relativity.' You may well be wondering, "What's next?"

One important subject with which we have not yet dealt is that of "dynamics," or, "what replaces Newton's Laws in post-Einstein physics?" I would like to discuss this in some depth, both for its own sake and because it will provide a natural transition to our study of General Relativity and gravity. However, there is something else that we must discuss first. Recall, for example, that Newton's second Law (F=ma), the centerpiece of pre-relativistic physics, involves acceleration. Although we have to some extent been able to deal with accelerations in special relativity (as in the twin paradox), we have seen that accelerations produce further unexpected effects. We need to study these more carefully before continuing onward. So, for most of this chapter we are going to carefully investigate the simple but illustrative special case known as 'uniform' acceleration. We'll save true discussion of dynamics (forces and such) for chapter 6.

5.1 The Uniformly Accelerating Worldline

Now, what do I mean by 'uniform' acceleration? One might at first think that this means that the acceleration $a = dv/dt$ of some object is constant, as measured in some inertial frame. However, this would imply that the velocity (relative to that frame) as a function of time is of the form $v = v_0 + at$. One notes that this eventually exceed the speed of light. Given our experience to date, this would seem to be a bit odd.

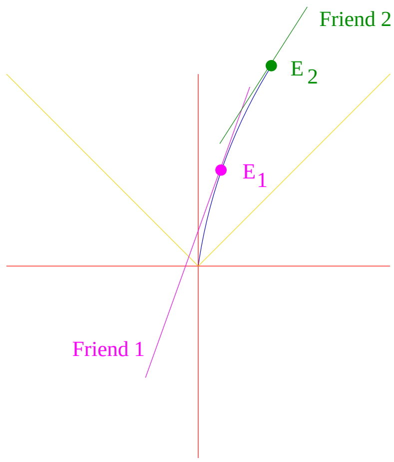
Also, on further reflection, one realizes that this notion of acceleration depends strongly on the choice of inertial frame. The dv part of a involves subtracting velocities, and we have seen that plain old subtraction does not in fact give the relative velocity between two inertial frames. Also, the dt part involves time measurements, which we know to vary greatly between reference frames. Thus, there is no guarantee that a constant acceleration a as measured in some inertial frame will be constant in any other inertial frame, or that it will in any way "feel" constant to the object that is being accelerated.

5.1.1 Defining uniform acceleration

What we have in mind for uniform acceleration is something that *does* in fact feel constant to the object being accelerated. In fact, we will take this as a definition of "uniform acceleration." Recall that we can in fact feel accelerations directly: when an airplane takes off, a car goes around a corner, or an elevator begins to move upward we feel the forces associated with this acceleration (as in Newton's law F=ma). To get the idea of uniform acceleration, picture a large rocket in deep space that burns fuel at a constant rate. Here we have in mind that this rate should be constant *as measured by a clock in the rocket ship.* Presumably the astronauts on this rocket experience the same force at all times.

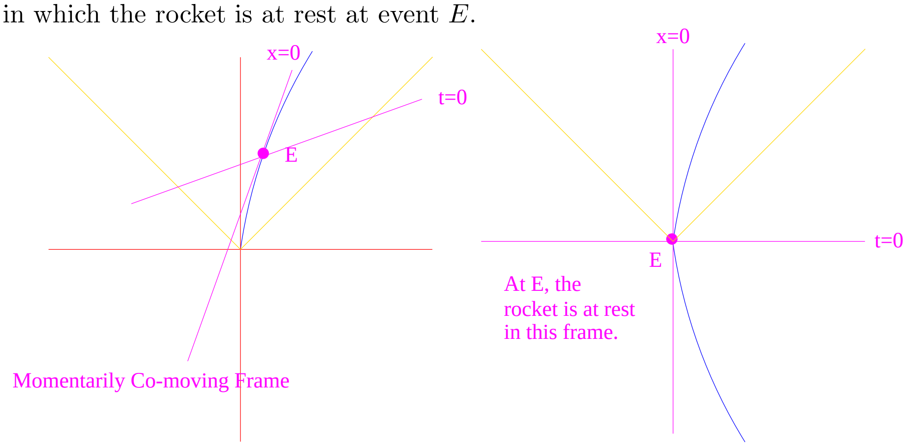
Now, I admitted a few minutes ago that Newton's laws will need to be modified in relativity. However, we know that Newton's laws hold for objects small velocities (much less than the speed of light) relative to us. So, it *is* OK to use them for slowly moving objects. We will suppose that these laws are precisely correct in the limit of zero relative velocity.

So, how can we keep the rocket "moving slowly" relative to us as it continues to accelerate? We can do so *by continuously changing our own reference frame.* Perhaps a better way to say this is that we should arrange for many of our friends to be inertial observers, but with a wide range of velocities relative to us. During the short time that the rocket moves slowly relative to us, we use our reference frame to describe the motion. Then, at event E_1 (after the rocket has sped up a bit), we'll use the reference frame of one of our inertial friends whose velocity relative to us matches that of the rocket at event E_1. Then the rocket will be at rest relative to our friend. Our friend's reference frame is known as the *momentarily co-moving inertial frame* at event E_1. A bit later (at event E_2), we will switch to another friend, and so on.



In fact, to do this properly, we should switch friends (and reference frames) fast enough so that we are always using a reference frame in which the rocket is moving only infinitesimally slowly. Then the relativistic effects will be of zero size. In other words, we wish to borrow techniques from calculus and take the *limit* in which we switch reference frames continually, always using the momentarily co-moving inertial frame. So, we'll have lots of fun with calculus in this chapter.

Anyway, the thing that we want to be constant in uniform acceleration is called the "proper acceleration." Of course, it can change along the rocket's worldline (depending on how fast the rocket decides to burn fuel), so we should talk about the proper acceleration 'at some event (E) on the rocket's worldline.' To find the proper acceleration (α) at event E, first consider an inertial reference frame in which the rocket is at rest at event E.



> The proper acceleration $\alpha(E)$ at event E is just the acceleration of the rocket at event E as computed in this momentarily co-moving reference frame.

Thus we have

$$\alpha(E) = dv_E/dt_E, \tag{5.1}$$

where the E-subscripts remind us that this is to be computed in the momentarily co-moving inertial frame at event E. Notice the analogy with the definition of proper time along a worldline, which says that the proper time is the time as measured in a co-moving inertial frame (i.e., a frame in which the worldline is at rest).

An important point is that, although our computation of $\alpha(E)$ involves a discussion of certain reference frames, $\alpha(E)$ is a quantity that is intrinsic to the motion of the rocket and does *not* depend on choosing of some particular inertial frame from which to measure it. Thus, it is not necessary to specify an inertial frame in which $\alpha(E)$ is measured, or to talk about $\alpha(E)$ "relative" to some frame. As with proper time, we use a Greek letter (α) to distinguish proper acceleration from the more familiar frame-dependent acceleration a.

We should also point out that the notion of proper acceleration is also just how the *rocket* would naturally measure its own acceleration (relative to inertial frames). For example, a person in the rocket might decide to drop a rock out the window at event E. If the rock is gently released at event E, it will initially have no velocity relative to the rocket – its frame of reference will be the momentarily co-moving inertial frame at event E. Of course, the rock will then begin to be left behind. If the observer in the rocket measures the relative acceleration between the rock and the rocket, this will be the same size (though in the opposite direction) as the acceleration of the rocket as measured by the (inertial and momentarily co-moving) rock. In other words, it will be the proper acceleration α of the rocket.

5.1.2 Uniform Acceleration and Boost Parameters

So, now we know what we mean by uniform acceleration. But, it would be useful to know how to draw this kind of motion on a spacetime diagram (in some inertial frame). In other words, we'd like to know what sort of worldline this rocket actually follows through spacetime.

There are several ways to approach this question, but I want to use some of the tools that we've been developing. As we have seen, uniform acceleration is a very natural notion that is not tied to any particular reference frame. We also know that, in some sense, it involves a change in velocity and a change in time. One might expect the discussion to be simplest if we measure each of these in the most natural way possible, without referring to any particular reference frame. What do you think is the natural measure of velocity (and the change in velocity)? By natural, I mean something associated with the basic structure (geometry!) of spacetime. What do you think is a natural measure of the passage time? **Stop reading for a moment and think about this.**

As we discussed last week, the natural way to describe velocity (in terms of, say, Minkowskian geometry) is in terms of the associated boost parameter θ. Recall that boost parameters really do add together (eq. 4.16) in the simple, natural way. This means that when we consider a difference of two boost parameters (like, say, in $\Delta\theta$ or $d\theta$), this *difference* is in fact independent of the reference frame in which it is computed. The boost parameter of the reference frame itself just cancels out.

What about measuring time? Well, as we have discussed, the 'natural' measure of time along a worldline is the proper time. The proper time is again independent of any choice of reference frame.

OK, so how does this relate to our above discussion? Let's again think about computing the proper acceleration $\alpha(E)$ at some event E using the momentarily co-moving inertial frame. We have

$$\alpha(E) = \frac{dv_E}{dt_E}. \tag{5.2}$$

What we want to do is to write dv_E and dt_E in terms of the boost parameter (θ) and the proper time (τ).

Let's start with the time part. Recall that the proper time τ along the rocket's worldline is just the time that is measured by a clock *on the rocket.* Thus, the question is just "How would a small time interval $d\tau$ measured by this clock (at event E) compare to the corresponding time interval dt_E measured in the momentarily co-moving inertial frame?" But we are interested only in the infinitesimal time around event E where there is negligible relative velocity between these two clocks. Clocks with no relative velocity measure time intervals in exactly the same way. So, we have $dt_E = d\tau$.

Now let's work in the boost parameter, using $d\theta$ to replace the dv_E in equation (5.1). Recall that the boost parameter θ is just a function of the velocity $v/c = \tanh\theta$. So, let's try to compute dv_E/dt_E using the chain rule. You can use the definition of tanh to check that

$$\frac{dv}{d\theta} = \frac{c}{\cosh^2\theta}.$$

Thus, we have

$$\frac{dv}{d\tau} = \frac{dv}{d\theta}\frac{d\theta}{d\tau} = \frac{c}{\cosh^2\theta}\frac{d\theta}{d\tau}. \tag{5.3}$$

Finally, note that at event E, the boost parameter θ of the rocket relative to the momentarily co-moving inertial frame is zero. So, if we want $\frac{dv_E}{d\tau}$ we should substitute $\theta = 0$ into the above equation[1]:

$$\alpha = \frac{dv_E}{d\tau} = \frac{c}{\cosh^2\theta}\bigg|_{\theta=0}\frac{d\theta}{d\tau} = c\frac{d\theta}{d\tau}. \tag{5.4}$$

In other words,

$$\frac{d\theta}{d\tau} = \alpha/c. \tag{5.5}$$

[1]Note that this means that our chain rule calculation (5.3) has in fact shown that, for small velocities, we have approximately $\theta \approx v/c$. This may make you feel even better about using boost parameters as a measure of velocity.

As we have discussed, $d\theta$ and $d\tau$ do *not* in fact depend on a choice of inertial reference frame. As a result, the relation (5.5) holds whether or not we are in the momentarily co-moving inertial frame.

If we translate equation (5.5) into words, it will come as no surprise: "An object that experiences uniform acceleration gains the same amount of boost parameter for every second of proper time; that is, for every second of time measured by a clock on the rocket."

It will be useful to solve (5.5) for the case of uniform α and in which the boost parameter (and thus the relative velocity) vanishes at $\tau = 0$. For this case, (5.5) yields the relation:

$$\theta = \alpha\tau/c. \tag{5.6}$$

This statement encodes a particularly deep bit of physics. In particular, it turns out to answer the question "Why can't an object go faster than the speed of light?" to which I promised we would return. Here, we have considered the simple case of a rocket that tries to continually accelerate by burning fuel at a constant rate. What we see is that it gains equal boost parameter in every interval of proper time. So, will it ever reach the speed of light? No. After a very long (but finite) proper time has elapsed the rocket will merely have a large (but finite) boost parameter. Since any finite boost parameter (no matter how large) corresponds to some v less than c, the rocket never reaches the speed of light.

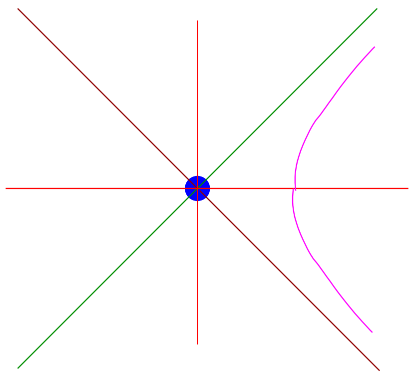
Similarly, it turns out that whether or not the acceleration is uniform, any rocket must burn an infinite amount of fuel to reach the speed of light. Thus, the speed of light (infinite boost parameter) plays the same role in relativity that was played by *infinite* velocity in Newtonian physics.

5.1.3 Finding the Worldline

In the preceding section we worked out the relation between the proper acceleration α of an object, the boost parameter θ that describes the object's motion, and the proper time τ along the object's worldline. This relation was encoded in equation (5.5), $\frac{d\theta}{d\tau} = \alpha/c$.

This results told us quite a bit, and in particular let to insight into the "why things don't go faster than light" issue. However, we still don't know exactly what worldline a uniformly accelerating object actually follows in some inertial frame. This means that we don't yet really know how to draw the uniformly accelerating object on a spacetime diagram, so that we cannot yet apply our powerful diagrammatic tools to understanding the physics of uniform acceleration.

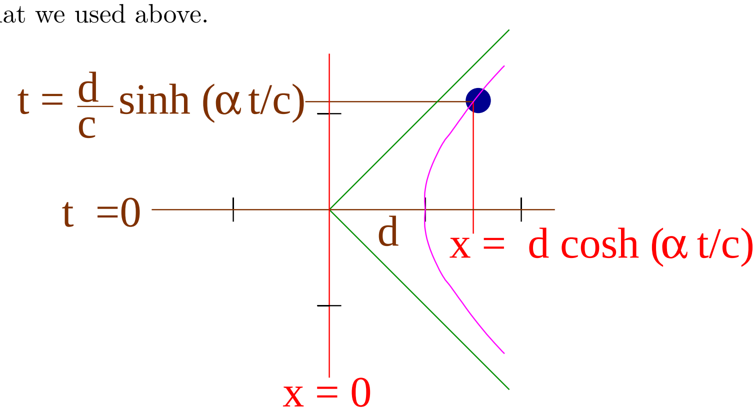
We will work out the exact shape of this worldline in this section. Let's start by drawing the rough *qualitative* shape of the worldline on a spacetime diagram. The worldline will have $v = 0$ at $t = 0$, but the velocity will grow with t. The velocity will thus be nearly $+c$ for large positive t and it will be nearly $-c$ for large negative t. As a result, the diagram will look something like this:



Next, let us recall that uniform acceleration is in some sense *invariant*. When the uniformly accelerated rocket enters our frame of reference (i.e., when $v = 0!$), I find its acceleration to be α no matter what inertial frame we are in! Thus, the curve should in some sense 'look the same' in every inertial frame.

So, any guesses? Can you think of a curve that looks something like the figure above that is 'the same' in all inertial frames?

How about the constant proper distance curve $x = d\cosh\theta$ from section (4.3.4)? Since θ was a boost angle there, it is natural to *guess* that it is the same $\theta = \alpha t/c$ that we used above.



Let us check our guess to see that it is in fact correct. What we will do is to simply take the curve $x = d\cosh(\alpha\tau/c), t = (d/c)\sinh(\alpha\tau/c)$ and show that, for the proper choice of the distance d, its velocity is $v = c\tanh(\alpha\tau/c)$, where τ is the proper time along the curve. But we have seen that this relation between time is the defining property of a uniformly accelerated worldline with proper acceleration α, so this will indeed check our guess.

First, we simply calculate:

$$\begin{aligned} dx &= \frac{\alpha d}{c}\sinh(\alpha\tau)d\tau \\ dt &= \frac{\alpha d}{c^2}\cosh(\alpha\tau)d\tau. \end{aligned} \tag{5.7}$$

Dividing these two equations we have

$$v = \frac{dx}{dt} = c\tanh(\alpha\tau). \tag{5.8}$$

Now, we must show that τ is the proper time along the curve. But

$$d\text{propertime}^2 = dt^2 - \frac{1}{c^2}dx^2 = \frac{\alpha^2 d^2}{c^4}d\tau^2. \tag{5.9}$$

So, we need only choose d such that $\alpha d/c^2 = 1$ and we are done. Thus, $d = c^2/\alpha$. In summary,

> If we start a uniformly accelerated object in the right place (c^2/α away from the origin), it follows a worldline that remains a constant proper distance (c^2/α) from the origin.

For a general choice of starting location (say, x_0), it follows a worldline that remains a constant proper distance $\frac{c^2}{\alpha}$ from some other event. Since it is sometimes useful to have this more general equation, let us write it down here:

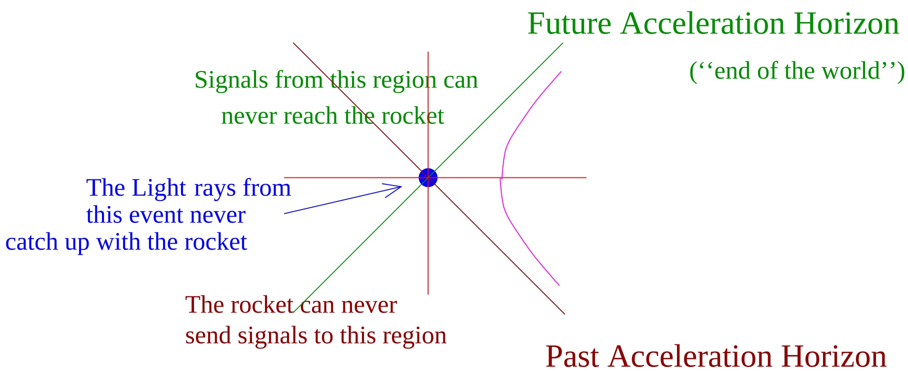
$$x - x_0 = \frac{c^2}{\alpha}\left(\cosh\left(\frac{\alpha\tau}{c}\right) - 1\right). \tag{5.10}$$

5.2 Exploring the uniformly accelerated reference frame

We have now found that a uniformly accelerating observer with proper acceleration α follows a worldline that remains a constant proper distance c^2/α away from some event. Just which event this is depends on where and when the observer began to accelerate. For simplicity, let us consider the case where this special event is the origin. Let us now look more closely at the geometry of the situation.

5.2.1 Horizons and Simultaneity

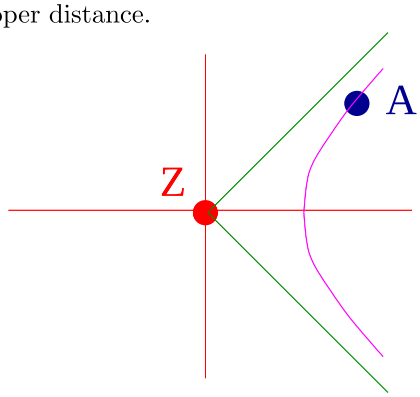
The diagram below shows the uniformly accelerating worldline together with a few important light rays.



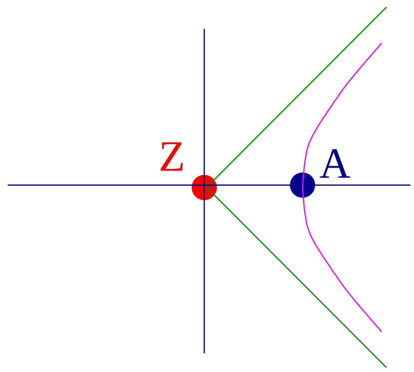
Note the existence of the light ray marked "future acceleration horizon." It marks the boundary of the region of spacetime from which the uniformly accelerated observer can receive signals, since such signals cannot travel faster than c. This is an interesting phenomenon in and of itself: merely by undergoing uniform acceleration, the rocket ship has cut itself off from communication with a large part of the spacetime. In general, the term 'horizon' is used whenever an object is cut off in this way. On the diagram above there is a light ray marked "past acceleration horizon" which is the boundary of the region of spacetime *to* which the uniformly accelerated observer can *send* signals.

When considering inertial observers, we found it very useful to know how to draw their lines of simultaneity and their lines of constant position. Presumably, we will learn equally interesting things from working this out for the uniformly accelerating rocket.

But, what notion of simultaneity should the rocket use? Let us define the rocket's lines of simultaneity to be those of the associated momentarily co-moving inertial frames. It turns out that these are easy to draw. Let us simply pick any event A on the uniformly accelerated worldline as shown below. I have also marked with a Z the event from which the worldline maintains a constant proper distance.



Recall that a boost transformation simply slides the events along the hyperbola. This means that we can find an inertial frame in which the above picture looks like this:

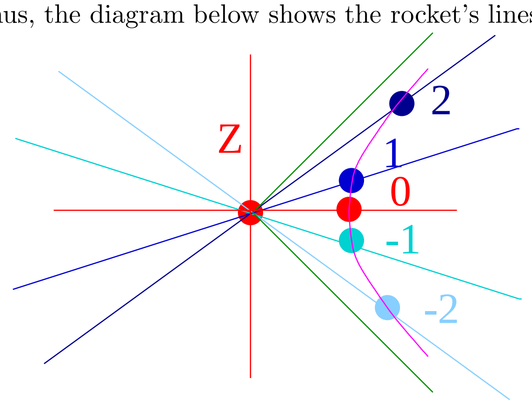


In the new frame of reference, the rocket is at rest at event A. Therefore, the rocket's line of simultaneity through A is a horizontal line. Note that this line passes through event Z.

This makes the line of simultaneity easy to draw on the *original* diagram. What we have just seen is that:

> Given a uniformly accelerating observer, there is an event Z from which it maintains proper distance. The observer's line of simultaneity through any event A on her worldline is the line that connects event A to event Z.

Thus, the diagram below shows the rocket's lines of simultaneity.



Let me quickly make one comment here on the passage of time. Suppose that events $-2, -1$ above are separated by the same sized boost as events $-1, 0$, events $0, 1$, and events $1, 2$. From the relation $\theta = \alpha\tau/c^2$ it follows that each such pair of events is also separated by the same interval of proper time along the worldline.

$\star\star$ But now on to the more interesting features of the diagram above! Note that the acceleration horizons divide the spacetime into four regions. In the right-most region, the lines of simultaneity look more or less normal. However, in the top and bottom regions, there are no lines of simultaneity at all! The rocket's lines of simultaneity simply do not penetrate into these regions. Finally, in the left-most region things again look more or less normal except that the labels on the lines of simultaneity seem to go the wrong way, 'moving backward in time.'

And, of course, all of the lines of simultaneity pass through event Z where the horizons cross. These strange-sounding features of the diagram should remind you of the weird effects we found associated with Gaston's acceleration in our discussion of the twin paradox in section 4.2.

As with Gaston, one is tempted to ask "How can the rocket see things running backward in time in the left-most region?" In fact, the rocket does not see, or even know about, *anything* in this region. As we mentioned above, no signal of any kind from any event in this region can ever catch up to the rocket. As a result, this phenomenon of finding things to run backwards in time is a pure mathematical artifact and is not directly related to anything that observers on the rocket actually notice.

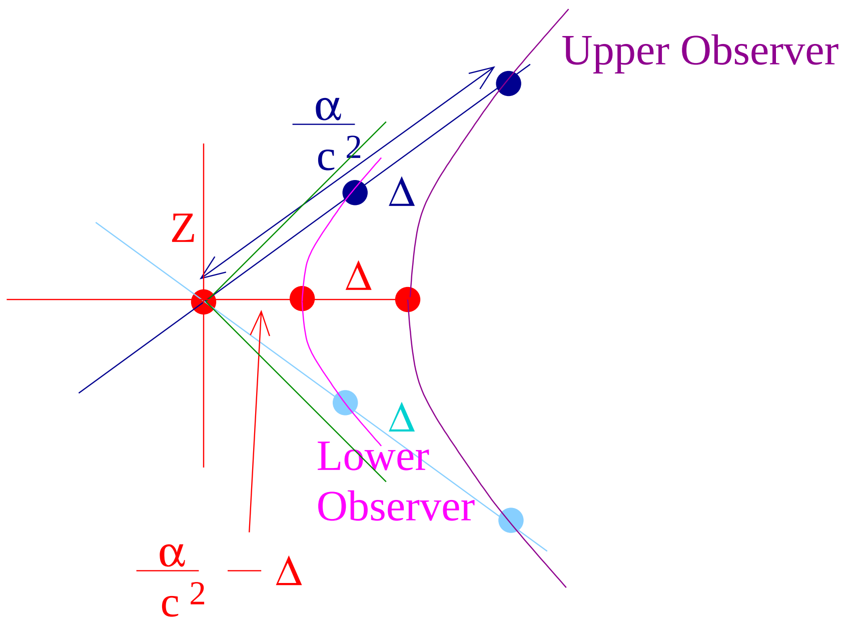
5.2.2 Friends on a Rope

In the last section we uncovered some odd effects associated with the the acceleration horizons. In particular, we found that there was a region in which the lines of simultaneity seemed to run backward. However, we also found that the rocket could neither signal this region nor receive a signal from it. As a result, the fact that the lines of simultaneity run backward here is purely a mathematical artifact.

Despite our discussion above, you might wonder if that funny part of the rocket's reference frame might somehow still be meaningful. It turns out to be productive to get another perspective on this, so let's think a bit about how we might actually construct a reference frame for the rocket.

Suppose, for example, that I sit in the nose (the front) of the rocket. I would probably like to use our usual trick of asking some of my friends (or the students in class) to sit at a constant distance from me in either direction. I would then try to have them observe nearby events and tell me which ones happen where. We would like to know what happens to the ones that lie below the horizon. Let us begin by asking the question: what worldlines do these fellow observers follow?

Let's see.... Consider a friend who remains a constant distance Δ below us as measured by us; that is, as measured in the momentarily co-moving frame of reference. This means that this distance is measured along our line of simultaneity. But look at what this means on the diagram below:



Recall that a distance (measured in some inertial frame) between two events on a given a line of simultaneity (associated with that same inertial frame) is in fact the proper distance between those events. Thus, on each line of simultaneity the proper distance between us and our friend is Δ. But, along each of these lines the proper distance between us and event Z is α/c^2. Thus, along each of these lines, the proper distance between our friend and event Z is $\alpha/c^2 - \Delta$. In other words, the proper distance between our friend and event Z is again a constant and our friend's worldline must *also* be a hyperbola!

Note, however, that the proper distance between our friend and event Z is less than the proper distance c^2/α between *us* and event Z. This means that our friend is again a uniformly accelerated observer, *but with a different proper acceleration!*

We can use the relations from section 5.1.3 to find the proper acceleration α_L of our lower friend. The result is

$$\frac{c^2}{\alpha_L} = proper\ distance\ between\ Z\ and\ lower\ friend = \frac{c^2}{\alpha} - \Delta, \tag{5.11}$$

or

$$\frac{\alpha_L}{c^2} = \frac{1}{\frac{c^2}{\alpha} - \Delta} = \frac{\alpha}{c^2 - \Delta\alpha}, \tag{5.12}$$

so that our friend's proper acceleration is *larger* than our own.

In particular, let's look at what happens when our friend is sufficiently far below us that they reach the acceleration horizons. This is $\Delta = \frac{c^2}{\alpha}$. At this value, we find $\alpha_L = \infty$!! Note that this fits with the fact that they would have to travel right along a pair of light rays and switch between one ray and the other in zero time....

⋆⋆ So then, suppose that we hung someone below us on a rope and slowly lowered them toward the horizon. The proper acceleration of the person (and thus the force that the rope must exert on them) becomes infinite as they get near the horizon. Similarly (by Newton's 3rd law) the force that they exert on the rope will become infinite as they near the horizon. Thus, no matter what it is made out of, the rope must break (or begin to stretch, or somehow fail to remain rigid such that the person falls away, never to be seen by us again) before the person is lowered across the horizon.

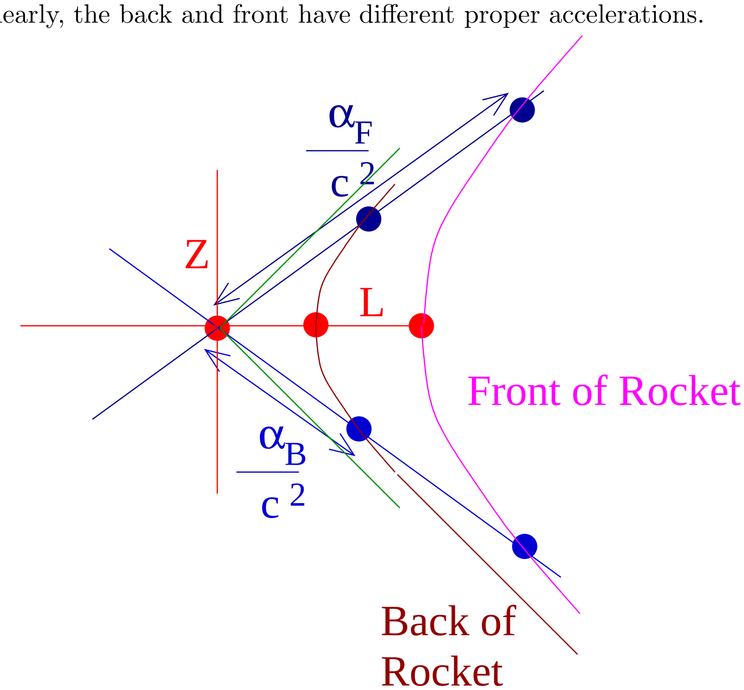
Again we see that, in the region beyond the horizon, the reference frame of a uniformly accelerating object is "unphysical" and could never in fact be constructed. There is no way to make one of our friends move along a worldline below the horizon that remains at a constant proper distance from us.

5.2.3 The Long Rocket

Suppose now that our rocket is long enough that we should draw separate worldlines for its front and back. If the rocket is 'rigid,' it will remain a constant proper length Δ as time passes. This is just like our 'friend on a rope' example. Thus, the back of the rocket also follows a uniformly accelerated worldline with a proper acceleration α_B which is related to the proper acceleration α_F of the front by:

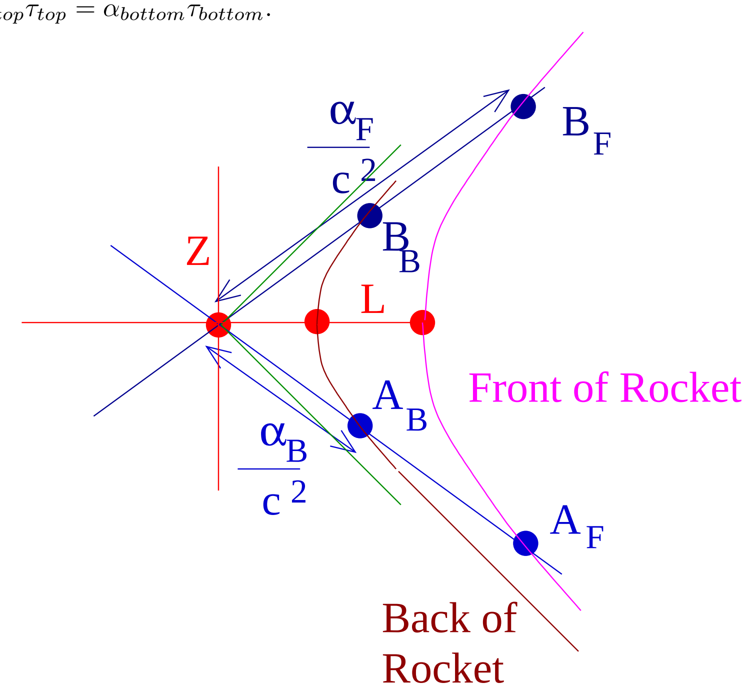
$$\alpha_B = \frac{\alpha_F c^2}{c^2 - \Delta\alpha_F}. \tag{5.13}$$

Clearly, the back and front have different proper accelerations.



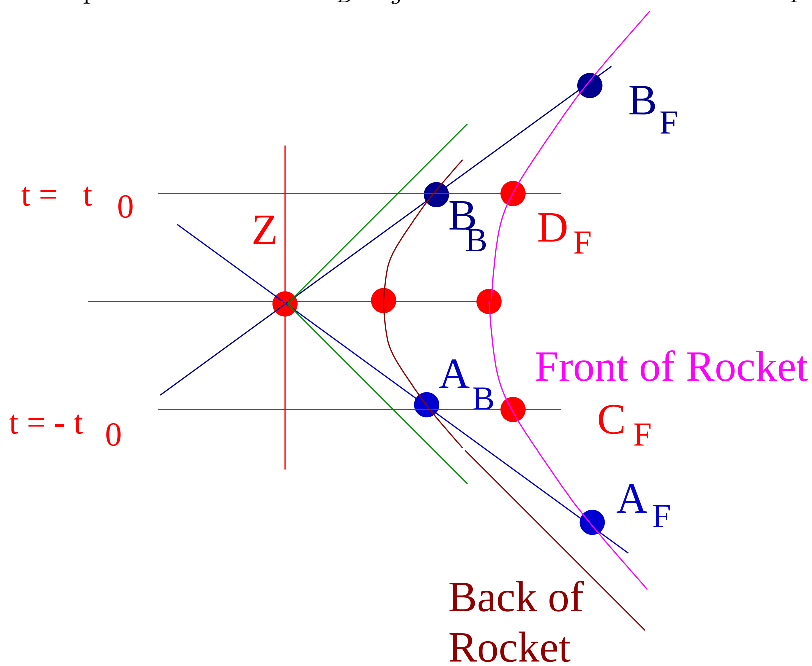
Note that the front and back of the rocket do in fact have the same lines of simultaneity, so that they agree on which events happen "at the same time." But do they agree on how much time passes between events that are not simultaneous?

Since they agree about lines of simultaneity it must be that, along any such line, both ends of the rocket have the same speed v and the same boost parameter θ. However, because the proper acceleration of the back is greater than that of the front, the relation $\theta = \alpha\tau/c^2$ then tells us that more proper time τ passes at the front of the rocket than at the back. In other words, there is more proper time between the events A_F, B_F below than between events A_B, B_B. In fact, $\alpha_{top}\tau_{top} = \alpha_{bottom}\tau_{bottom}$.



⋆⋆ Here it is important to note that, since they use the same lines of simultaneity, both ends of the rocket *agree* that the front (top) clock runs faster! Thus, this effect is of a somewhat different nature than the time dilation associated with inertial observers. This, of course, is because all accelerated observers are not equivalent – some are more accelerated than others.

By the way, we could have read off the fact that $\Delta\tau_{Front}$ is bigger than $\Delta\tau_{Back}$ directly from our diagram without doing any calculations. (This way of doing things is useful for certain similar homework problems.) To see this, note that between the two lines ($t = \pm t_0$) of simultaneity (for the inertial frame!!) drawn below, the back of the rocket is moving faster (relative to the inertial frame in which the diagram is drawn) than is the front of the rocket. You can see this from the fact that the front and back have the same line of simultaneity (and therefore the **same** speed) at events B_F, B_B and at events A_F, A_B. This means that the speed of the back at B_B is *greater* than that of the front at D_F and that the speed of the back at A_B is *greater* than that of the front at C_F.



Thus, relative to the inertial frame in which the diagram is drawn, the back of the rocket experiences more time dilation in the interval $(-t_0, t_0)$ and it's clock runs more slowly. Thus, the proper time along the back's worldline between events A_B and B_B is less than the proper time along the front's worldline between events C_F and D_F. ⋆⋆ We now combine this with the fact that the proper time along the front's worldline between A_F and B_F is even greater than that between C_F and D_F. Thus, we see that the front clock records much more proper time between A_F and B_F than does the back clock between A_B and B_B.

5.3 Homework Problems

5-1. Suppose that you are in a (small) rocket and that you make the following trip: You start in a rocket in our solar system (at rest with respect to the Sun). You then point your rocket toward the center of the galaxy and accelerate uniformly for ten years of your (i.e., proper) time with a proper acceleration of $1g = 10m/s^2$. Then, you decelerate uniformly for ten years (of proper time) at $1g$, so that you are again at rest relative to the sun:

(a) Show that $1g$ is very close to $1 light - year/year^2$. Use this value for g in the problems below. [Note: This part is just a unit-conversion problem.]

(b) Draw a spacetime diagram showing your worldline (and that of the Sun) in the Sun's reference frame.

(c) As measured in the Sun's reference frame, how far have you traveled?

(d) As measured in the Sun's reference frame, how long did it take you to get there?

(e) Relative to the Sun, what were your boost parameter (θ) and your velocity v at the event where you switched from accelerating to decelerating?

5-2. Suppose that you are at the *top* of a (rigid) rocket which is half a light year tall. If the rocket is accelerating such that *your* proper acceleration is $1g$,

(a) What is the proper acceleration at the bottom of the rocket? (Also, how heavy would you feel if you were at the bottom?)

(b) For every second that passes for a clock at the bottom of the rocket, how much time passes for you?

(c) Draw a spacetime diagram (using an inertial frame of reference) that shows the front of the rocket, the back of the rocket, and at least two of the rocket's lines of simultaneity.

(d) What is the proper acceleration of the bottom if the rocket is 3/4 of a light year tall?

(e) What is the proper acceleration of the bottom if the rocket is 99/100 of a light year tall?

5-3. Suppose that you are tossed out of a (short) uniformly accelerating rocket.

a) Sketch a spacetime diagram (in an inertial frame) showing the acceleration horizons of the accelerating rocket.

b) Draw your worldline on this diagram.

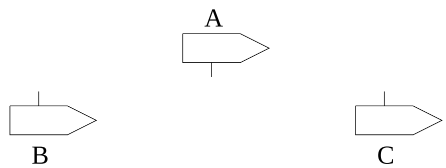
c) Suppose that you send out light rays at regular intervals to signal the rocket. Draw several of these light rays on your diagram.

d) The people on the rocket watch the light rays you send out through a telescope. Describe what they see. Do they see you age slowly or quickly? When do they see you cross the horizon?

e) Suppose now that the rocket sends out light rays toward you at regular intervals. Describe what you see if you watch these light rays through a telescope. Do you see people in the rocket age slowly or quickly? Do you see anything special when you cross the horizon?

5-4. If you like, you may assume that all accelerations are uniform in this problem.

Three small rocket ships A, B, and C drift freely in a region of space far from all other objects:



The rockets are not rotating and have no relative motion. Rocket A is equidistant from B and C (i.e., it is the same distance from each). Rocket A sends out a light pulse and, when the rockets B and C receive this pulse, each starts its engine. (The things sticking out of the rockets in the picture above are the antennas that emit and receive the light pulse.) Rockets B and C are identical, and their guidance computers are programmed in exactly the same way. As a result, as reckoned by A, rockets B and C will have the same velocity at every instant of time and so remain a constant distance apart (again as measured by A). Rocket A remains in the same inertial frame the entire time. Rockets A, B, and C carry clocks of identical construction, all of which reach $t = 0$ when the light signal arrives at B and C.

(a) Using the reference frame of Rocket A, draw a spacetime diagram showing the worldlines of rockets B and C.

(b) Suppose that, at some time after rockets B and C begin to accelerate, an observer in the inertial rocket (A) takes readings of the clocks in rockets B and C. As usual, A does this by using various friends with the same reference or by otherwise ensuring that light travel time is not an issue. How do the clocks A, B, and C compare? (Just say which is running fastest, slowest, etc. I don't need a quantitative answer.)

(c) What happens if an observer in rocket B compares the clocks? Hint: Recall that since the speed of B (relative to A) continues to increase, the associated time dilation factor will not be constant. As a result, the answer (at least for comparing B's clock to A's) will depend on *when* B makes this comparison. For simplicity, I suggest you think about what happens at a very late time, long after B has passed A, when the relative speed of A and B is nearly the speed of light.

(d) What happens if an observer in rocket C compares the clocks? Suppose that B and C begin very close together and consider only what happens at a very late time. Note: A complete analysis covering all cases is more complicated. It turns out that whether C sees A's clock run faster or B's clock run faster depends on the initial separation between B and C.

(e) [Optional] Does the answer to D depend on whether rockets A and C start off close together or far apart?

Preliminary Comments for problems 5 and 6: The problems below will give you a feel for how to answer more complicated questions in special relativity. In problem 1, you calculated the proper time along a number of worldlines made up of straight line segments. Using a little calculus, much the same method gives the proper time along continuous curves as well. Remember that, if t and x are coordinates in some inertial frame, the proper time is given by

$$\Delta\tau^2 = \Delta t^2 - \frac{\Delta x^2}{c^2}.$$

For an infinitesimal piece of worldline spanning a time dt, a distance dx, and a proper time $d\tau$, we have

$$d\tau^2 = dt^2 - \frac{dx^2}{c^2},$$

or,

$$d\tau = \sqrt{dt^2 - \frac{dx^2}{c^2}} = \sqrt{1 - \frac{dx}{dt}\frac{1}{c^2}}dt.$$

Thus, if we know the velocity $v(t)$ of some worldline, we can find the associated proper time by using:

$$\boxed{\Delta\tau = \int_{t_0}^{t} dt\ \sqrt{1 - [v(t)]^2/c^2}.}$$

5-5. Suppose that we are in an inertial reference frame, and that some (small) rocket is moving relative to us. Suppose that the rocket is at rest relative to us until $t = t_0$, after which it moves relative to us with velocity $v(t) = c\sqrt{1 - \frac{t_0^2}{t^2}}$. Finally, let τ be the reading on a clock in that rocket and suppose that this clock is set to read $\tau = 0$ when $t = t_0$.

(a) Calculate the reading τ on this clock as a function of t (after $t = t_0$).

(b) Use equation (5.5) to calculate the proper acceleration α of the rocket as a function of τ. Note that (5.5) involves $\frac{d\theta}{d\tau}$ and not $\frac{d\theta}{dt}$. As a result, you may need to use the chain rule:

$$\frac{d\theta}{d\tau} = \frac{d\theta}{dt}\frac{dt}{d\tau}. \tag{5.14}$$

(c) What happens to α as $t \to \infty$?

Hint: $\frac{d}{dx}\tanh^{-1}(x) = \frac{1}{1-x^2}$.

5-6. Suppose that we are in an inertial reference frame, and that some (small) rocket is moving relative to us. Suppose that the rocket is at rest relative to us until $t = t_0$, after which it moves with velocity $v(t) = c\sqrt{1 - \frac{t_0^4}{t^4}}$. Let τ be the reading on a clock in that rocket and suppose that this clock reads $\tau = 0$ when $t = t_0$.

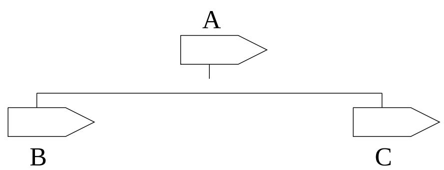
(a) What is the reading τ on this clock as a function of t (after $t = t_0$)?

(b) What happens to τ as $t \to \infty$?

(c) Finally, what is the proper acceleration along this worldline?

(d) What happens to α as $t \to \infty$?

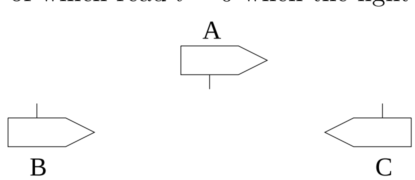
5-7. Suppose that, in problem 4 above, a fragile thread is strung between rockets B and C as shown below:



Initially, the thread is just barely long enough to cover the required distance. Now, as the rockets speed up, the distance between the rockets remains constant as measured by A. But, the string *ought* to contract, right? (Length contraction and all that ...) So, presumably, the string will break when the speed becomes large enough that the artificial prevention (from the fact that the string is tied to B and C) of the natural contraction imposes an intolerable stress on the string.

Is this really so? Draw a spacetime diagram and use it to argue your point.

5-8. Three small rocket ships A, B, and C drift freely in a region of space far from all other objects. The rockets are not rotating and have no relative motion. Rocket A is equidistant from B and C (i.e., it is the same distance from each). *This time, however, rockets B and C face each other so that they accelerate toward each other.* Rocket A sends out a light pulse and, when rockets B and C receive this pulse, each starts it's engine. Rockets B and C are identical and their guidance computers are programmed in exactly the same way. As a result, as reckoned by A, rockets B and C will have the same speed at every instant of time (although they will be moving in opposite directions). Rocket A remains in the same inertial frame the entire time. Rockets A, B, and C carry clocks of identical construction, all of which read $t = 0$ when the light signal arrives at B and C.



(a) Using the reference frame of Rocket A, draw a spacetime diagram showing the worldlines of rockets B and C.

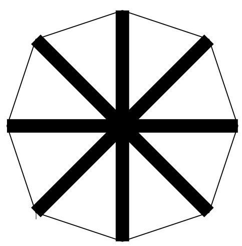
(b) Suppose that, at some time after rockets B and C begin to accelerate, an observer in the inertial rocket (A) measures the readings of the clocks in rockets B and C. How do the clocks A, B, and C compare? (Just say which is running fastest, slowest, etc. I don't need a quantitative answer.)

(c) What happens if an observer in rocket B compares the clocks?

(d) What happens if an observer in rocket C compares the clocks?

(e) Suppose that B and C make measurements of their separation from A along their lines of simultaneity. How 'fast' do they find A to be moving away from them? If they measure their separation from both the front and back of A, what happens to their measurements of the length of A as time passes?

5-9. Consider a wagon wheel of radius r with rigid spokes that starts at rest in some inertial frame but then begins to spin rapidly so that the outside of the wheel is moving at $.8c$ relative to the inertial frame.



Since length contraction occurs only in the direction of motion, the *radius* r of the wheel remains unchanged (as measured in the inertial frame).

(a) Suppose that you are in the original inertial frame. What value will you measure for the circumference of the wheel? (Assume here that the wheel does not break under the stress.)

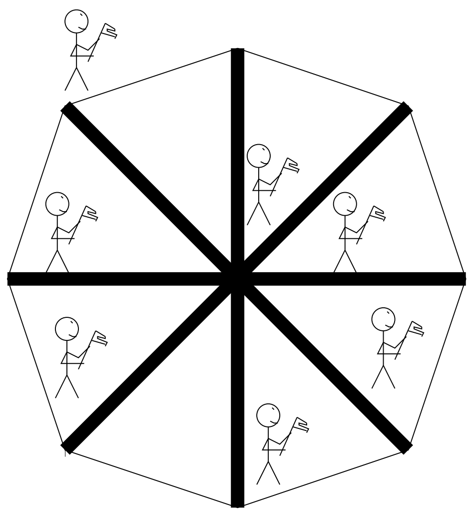
(b) Suppose that you measure the circumference by tacking down measuring rods around the outside of the spinning wheel (so that the rods spin with the wheel). What value of the circumference will this measurement produce?

(c) If the spokes of the wheel are connected by pieces of thread, what will happen to the thread while the wheel is getting up to speed?

(d) Suppose that identical clocks are placed at the end of each spoke and that the clocks are all synchronized in the original inertial frame *before* the wheel is spun up. As viewed from the inertial frame, do the clocks remain synchronized with each other after the wheel is spun up? Do they remain synchronized with a clock that remains inertial?

(e) Consider again the clocks in (D). If you are standing at the end of one of the spokes and make measurements of the clocks next to you, what do you find? Are they synchronized with yours? Do they run at the same rate?

5-10. Consider a bunch of spokes of length r on a hub, spinning relative to some inertial frame. We then send builders out onto the spokes to build a platform on top of the spokes *while it is spinning*. For simplicity, suppose that each builder is stationed at just one place on the wheel and does all of her work there. Suppose that the wheel is spinning 'quickly,' say, fast enough that the outside of the wheel is moving at $.8c$ relative to the inertial frame.



(a) Suppose that all of the builders at a given distance from the hub stretch a tape measure around to measure the circumference of the circle they make on the platform. How does this number compare with the distance they measure to the hub (the radius they measure)?

(b) Suppose that all of the builders were the same age in the morning before work. When they get together at the end of the day (where the end of the day is determined in the inertial frame), will they all still be the same age? If not, how will their ages differ?

(c) Suppose that, for some reason, we bring our merry-go-round to a stop after it is built. Will it still fit together without having to buckle or tear? If not, what will happen to it?

5-11. Consider a beacon on a light house. If a stationary observer holds up a screen (see below), he finds a bright spot of light on the screen sweeping past with a speed that increases with the distance from the light house. Are there observers for whom the spot sweeps past at a speed greater than c? Is this a problem for special relativity? Compare this with a very long steel bar rotating at the same rate.


Rotating
Light
House
Screen

Chapter 6

Dynamics: Energy and Momentum in Relativity

Read Einstein, ch. 15

Up until now, we have been concerned mostly with describing motion. We have asked how various situations appear in different reference frames, both inertial and accelerated. However, we have largely ignored the question of what would make an object follow a given worldline ('dynamics'). The one exception was when we studied the uniformly accelerated rocket and realized that it must burn equal amounts of fuel in equal amounts of proper time. This realization came through using Newton's second law in the regime where we expect it to hold true: in the limit in which v/c is vanishingly small.

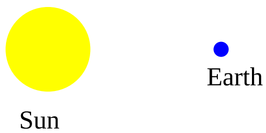
6.1 Dynamics, or, "Whatever happened to Forces?"

Recall that Newton's various laws used the old concepts of space and time. As a result, before we can apply them to situations with finite relative velocity, they will have to be at least rewritten and perhaps greatly modified to accommodate our new understanding of relativity. This was also true for our uniformly accelerating rocket. A constant thrust does not provide a constant acceleration as measured from a fixed inertial reference frame but, instead, it produces a constant *proper* acceleration.

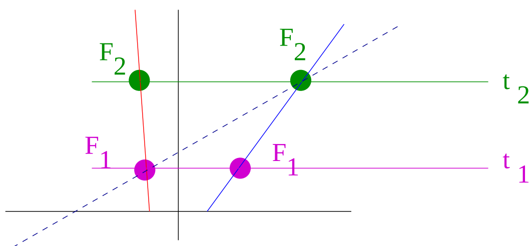
Now, a central feature of Newton's laws (of much of pre-Einstein physics) was the concept of force. It turns out that the concept of force is not as useful in relativistic physics. This has something to do with our discovery that acceleration is now a frame-dependent concept (so that a statement like $F = ma$ would be more complicated), but the main point actually involves Newton's *third* law:

> The third law of Newtonian Physics: When two objects (A and B) exert forces $F_{A\ on\ B}$ and $F_{B\ on\ A}$ on each other, these forces have the same size but act in opposite directions.

To understand why this is a problem, let's think about the gravitational forces between the Sun and the Earth.



Recall that Newton said that the force between the earth and the sun is given by an inverse square law: $F = \frac{GM_{earth}M_{sun}}{d^2}$ where d is the distance between them. In particular, the force between the earth and sun decreases if they move farther apart. Let's draw a spacetime diagram showing the two objects moving apart.



At some time t_1 when they are close together, there is some strong force F_1 acting on each object. Then, later, when they are farther apart, there is some weaker force F_2 acting on each object.

However, what happens if we consider this diagram in a moving reference frame? I have drawn in a line of simultaneity (the dashed line) for a different reference frame above, and we can see that it passes through one point marked F_1 and one point marked F_2! This shows that Newton's third law as stated above cannot possibly hold[1] in all reference frames.

So, Newton's third law has to go. But of course, Newton's third law is not **completely** wrong – it worked very well for several hundred years! So, as with the law of composition of velocities and Newton's second law, we may expect that it is an approximation to some *other* (more correct) law, with this approximation being valid only for velocities that are very small compared to c.

It turns out that this was not such a shock to Einstein, as there had been a bit of trouble with Newton's third law even before relativity itself was understood. Again, the culprit was electromagnetism.

[1]You might wonder if you could somehow save the third law by having the concept of force depend on which inertial frame you use to describe the system. Then in the moving frame, the forces would not be F_1 and F_2. However, the forces in the moving frame must still somehow be *determined* by F_1 and F_2. Thus, if F_1 and F_2 do not agree, neither can the forces in the moving frame.

6.2 Fields, Energy, and Momentum

To see the point, consider an electron in an electric field. We have said that it is really the *field* that exerts a force on the electron. Newton's third law would seem to say that the electron then exerts a force on the electric field. But what would this mean? Does an electric field have mass? Can it accelerate?

Luckily for Einstein, this problem had been solved. It was understood that the way out of this mess was to replace the notion of force with two somewhat more abstract notions: energy and momentum. Since not all of you are intimately familiar with these notions, let me say just a few words about them before we continue.

6.2.1 A word on Energy (E)

Actually, I don't need to say too much here. Most people have an intuitive concept of energy as "what comes out of a power plant" and this is almost good enough for our purposes. Anything which can *do* something[2] has energy: batteries, light, gasoline, wood, coal (these three can be burned), radioactive substances, food, and so. Also, any moving object has energy due to it's motion. For example, a moving bowling ball has energy that allows it to knock down bowling pins. By the way, in Newtonian physics, there in an energy $\frac{1}{2}mv^2$ (called 'kinetic energy' from the Greek word for motion) due to the motion of an object of mass m.

The most important thing about energy is that it cannot be created or destroyed; it can only be transformed from one form to another. As an example, in a power plant, coal is burned and electricity is generated. Burning coal is a process in which the chemical energy stored in the coal is turned into heat energy. This heat energy boils water and creates a rising column of steam (which has energy due to its motion). The column of steam then turns a crank which turns a wire in a magnetic field. This motion converts the mechanical energy of motion of the wire into electrical energy.

Physicists say that Energy is "conserved," which means that the total energy E in the world can not change.

6.2.2 A few words on Momentum (P)

Momentum is a bit less familiar, but it is like energy in that it cannot be created or destroyed: it can only be transferred from one object to another. Thus, momentum is also "conserved." Momentum is a quantity that describes in a certain sense "how much motion is taking place, and in what direction." If the total momentum is zero, we might say that there is "no net motion" of a system.

[2]Most physics books here say 'work' instead of 'something,' but that is another story.

Let's look back at the bowling ball example above. The energy of the bowling ball is a measure of how much mayhem the ball can cause when it strikes the pins. However, when the ball hits the pins, the pins do not fly about in an arbitrary way. In particular, the pins tend to fly away in more or less the same direction as the ball was moving originally. This is because, when the ball hits the pins, it gives up not only some of its energy to the pins, but also some of its momentum. The momentum is the thing that knows what direction the ball was traveling and makes the pins move in the same direction that the ball was going[3].

In Newtonian physics, the momentum p of a moving object is given by the formula $p = mv$. This says that an object that moves very fast has more momentum than one that moves slowly, and an object that has a large mass has more momentum than one with a small mass. This second bit is why it is easier to knock over a bowling pin with a bowling ball than with a ping-pong ball.

By the way, the fact that momentum is a type of object which points in some direction makes it something called a *vector*. A vector is something that you can visualize as an arrow. The length of the arrow tells you how big the vector is (how much momentum) and the direction of the arrow tells you the direction of the momentum.

Now, in Newtonian physics, momentum conservation is closely associated with Newton's third law. One way to understand this is to realize that both rules (Newton's third law and momentum conservation) guarantee that an isolated system (say, a closed box off in deep space) that begins at rest cannot ever start to move. In terms of Newton's third law, this is because, if we add up all of the forces between things inside the box, they will cancel in pairs: $F_{A\ on\ B} + F_{B\ on\ A} = 0$. In terms of momentum conservation, it is because the box at rest has zero momentum, whereas a moving box has a nonzero momentum. Momentum conservation says that the total momentum of the box cannot change from zero to non-zero.

In fact, in Newtonian physics, Newton's third law is *equivalent* to momentum conservation. To see this, consider two objects, A and B, with momenta $p_A = mv_A$ and $p_B = mv_B$. Suppose for simplicity that the only forces on these objects are caused by each other. Note what happens when we take a time derivative:

$$\begin{aligned} \frac{dp_A}{dt} &= m_A a_A = F_{B\ on\ A}, \\ \frac{dp_B}{dt} &= m_B a_B = F_{A\ on\ B}. \end{aligned} \tag{6.1}$$

[3]Some pins will of course fly off somewhat to one side or another. This does not reflect a failure of momentum conservation. Instead, the amount of 'leftward' momentum carried by some of the objects will be equal and opposite to the amount of 'rightward' momentum carried by the others, so that these bit exactly cancel out. An important difference between Energy and momentum is that energy is always positive while momentum can have either sign (in fact, it is a vector). Thus, while two energies never cancel against each other, two momenta sometimes do.

The total momentum is $P_{total} = p_A + p_B$. We have

$$\frac{dP_{total}}{dt} = F_{B\ on\ A} + F_{A\ on\ B} = 0. \tag{6.2}$$

Thus, Newton's 3rd law is equivalent to momentum conservation. One holds if and only if the other does.

Anyway, physicists in the 1800's had understood that there was a problem with Newton's third law when one considered electric fields. It did not really seem to make sense to talk about an electron exerting a force on an electric field. However, it turns out that one could meaningfully talk about momentum carried by an electromagnetic field, and one could even compute the momentum of such a field – say, for the field representing a light wave or a radio wave. Furthermore, if one adds the momentum of the electro-magnetic field to the momentum of all other objects, Maxwell's equations tell us that the resulting total momentum is in fact conserved. In this way, physicists had discovered that momentum conservation was a slightly more abstract principle that held true more generally than did Newton's third law. In relativity, too, it turns out to be a good idea to think in terms of momentum and momentum conservation instead of thinking in terms of Newton's third law.

6.3 On to relativity

Now, while the concepts of momentum and energy can make sense in relativistic physics, the detailed expressions for them in terms of mass and velocity should be somewhat different than in the Newtonian versions. However, as usual we expect that the Newtonian versions are correct in the particular limit in which all velocities are small compared to the speed of light.

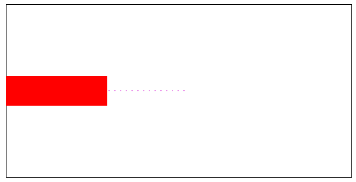
There are a number of ways to figure out what the correct relativistic expressions are. A very nice (and self-contained[4]) way is described below in section 6.6. However, that way of getting at the answer is a bit technical. So, for the moment, we're going to approach the question from a different standpoint.

You see, Einstein noticed that, even within electromagnetism, there was still something funny going on. Momentum *was* conserved, but this did not necessarily seem to keep isolated boxes (initially at rest) from running away! The example he had in mind was connected with the observation that light can exert pressure. This was well known in Einstein's time and could even be measured. The measurements were made as early as 1900, while Einstein published his theory of special relativity in 1905. It was known, for example, that pressure caused by light from the sun was responsible for the long and lovely tails on comets: light pressure (also called radiation pressure or solar wind) pushed droplets of water and bits of dust and ice backwards from the comet making a long and highly reflective tail. Nowadays, we can use lasers to lift grains of sand, or to smash things together to induce nuclear fusion.

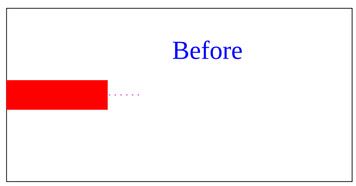
[4]Our treatment in this section will rely on Maxwell's equations, which we will not solve ourselves.

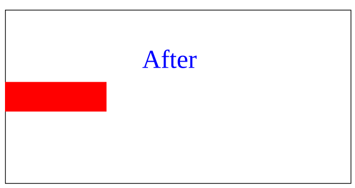
6.3.1 Lasers in a box

Anyway, suppose that we start with a box having a powerful laser[5] at one end.



When the laser fires a pulse of light, the light is near the left end and pressure from this light pushes the box to the left. The box moves to the left while the pulse is traveling to the right. Then, when the pulse hits the far wall, its pressure stops the motion of the box. The light itself is absorbed by the wall and disappears.





Now, momentum conservation says that the total momentum is always zero. Nevertheless, the entire box seems to have moved a bit to the left. With a large enough battery to power the laser, we could repeat this experiment many times and make the box end up very far to the left of where it started. Or, perhaps we do not even need a large battery: we can imagine recycling the energy used the laser. If we could catch the energy at the right end and then put it back in the battery, we would only need a battery tiny enough for a single pulse. By simply recycling the energy many times, we could still move the box very far to the left. This is what really worried our friend Mr. Einstein.

6.3.2 Center of Mass

The moving laser box worried him because of something called the center of mass. Here's the idea: Imagine yourself in a canoe on a lake. You stand at one end of the canoe and then walk forward. However, while you walk forward, the canoe will slide backward a bit. A massive canoe slides only a little bit, but

[5]Of course, lasers did not exist when Einstein was working on this. He just used a regular light source but, if lasers had been around, that's what he would have used in his example.

a light canoe will slide a lot. It turns out that in non-relativistic physics the position (technically known as the 'center of mass') does not move.

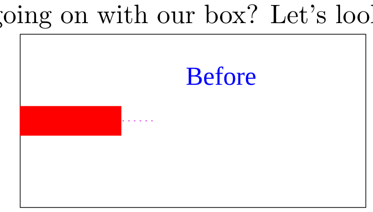
This follows from Newton's third law and momentum conservation. To understand the point suppose that in the above experiment we throw rocks from left to right instead of firing the laser beam. While most of the box would shift a bit to the left (due to the recoil) with each rock thrown, the rock in flight would travel quite a bit to the *right*. In this case, a sort of average location of all of the things in the box (including the rock) does not move.

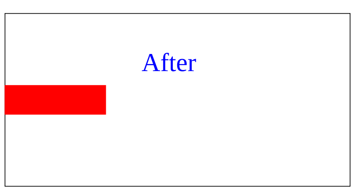
Suppose now that we want to recycle the rock, taking it back to the left to be thrown again. We might, for example, try to throw it back. But this would make the rest of the box shift back to the right, just where it was before. It turns out that any other method of moving the rock back to the left side has the same effect.

To make a long story short, since the average position cannot change, a box can never move itself more than one box-length in any direction, and this can only be done by piling everything inside the box on one side. In fact, when there are no forces from outside the box, the center of mass of the stuff in the box does not accelerate at all! In general, it is the center of mass that responds directly to outside forces.

6.3.3 Mass vs. Energy

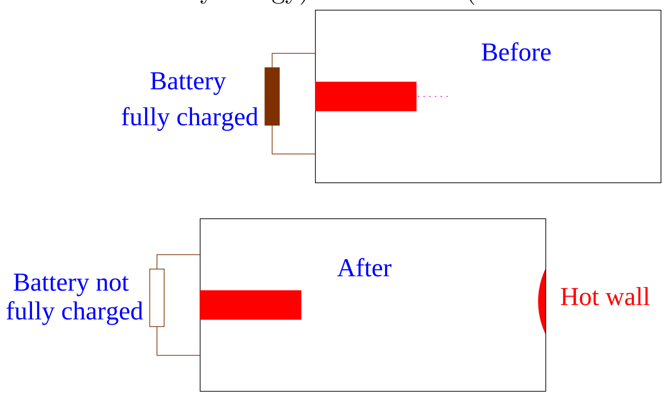
So, what's going on with our box? Let's look at the experiment more carefully.





After the experiment, it is clear that the box has moved, and in fact that every single atoms in the box has slid to the left. So, the center of mass seems to have moved! But, Einstein asked, might something else have changed during the experiment which we need to take into account? Is the box after the experiment really identical to the one before the experiment began?

The answer is: "not quite." Before, the experiment, the battery that powers the laser is fully charged. After the experiment, the battery is not fully charged. What happened to the associated energy? It traveled across the box as a pulse of light. It was then absorbed by the right wall, causing the wall to become hot. The net result is that energy has been transported from one end of the box (where it was battery energy) to the other (where it became heat).



So, Einstein said, "perhaps we should think about something like the center of *energy* as opposed to the center of *mass*." But, of course, the mass must also contribute to the center of energy... so is mass a form of energy?

Anyway, the relevant question here is "Suppose we want to calculate the center of mass/energy. Just how much mass is a given amount of energy worth?" Or, said another way, how much energy is a given amount of mass worth?

Well, from Maxwell's equations, Einstein could figure out the energy transported. He could also figure out the pressure exerted on the box so that he knew how far all of the atoms would slide. Assuming that the center of mass-energy did not move, this allowed him to figure out how much energy the mass of the box was in fact worth. The computation is a bit complicated, so we won't do it here[6]. However, the result is that an object of mass m which is at rest is worth the energy:

$$\boxed{E = mc^2 \qquad (6.3)}$$

Note that, since $c^2 = 9 \times 10^{16} m^2/s^2$ is a big number, a small mass is worth a lot of energy. Or, a 'reasonable amount' of energy is in fact worth very little mass. This is why the contribution of the energy to the 'center of mass-energy' had not been noticed in pre-Einstein experiments. Let's look at a few. We buy electricity in 'kilowatt-hours' (kWh) – roughly the amount of energy it takes to run a house for an hour. The mass equivalent of 1 kilowatt-hour is

$$m = \frac{1kWh}{c^2} = \frac{1kWh}{c^2}\frac{3600sec}{hr.}\frac{1000W}{1kW} = \frac{3.6 \times 10^6}{9 \times 10^{16}} = 4 \times 10^{-10} kg. \qquad (6.4)$$

In other words, not much.

By the way, one might ask whether the fact that both mass and energy contribute to the 'center of mass-energy' really means that mass and energy are

[6]Perhaps some senior physics major would like to take it up as a course project?

convertible into one another. Let's think about what this really means. We have a fair idea of what energy is, but what is mass? We have not really talked about this yet in this course, but what Newtonian physicists meant by mass might be better known as 'inertia.' In other words, mass is defined through its presence in the formula $F = ma$ which tells us that the mass is what governs how difficult an object is to accelerate.

6.3.4 Spacetime Diagrams

So, then, what we really want to know is whether adding energy to an object increases its inertia. That is, is it harder to move a hot wall than a cold wall?

To get some perspective on this, recall that one way to add energy to an object is to speed it up. But we have already seen that rapidly moving objects are indeed hard to accelerate (e.g., a uniformly accelerating object never accelerates past the speed of light). Another way to add energy is to heat things up. But, this just means that you make the various atoms speed up and move around very fast in random ways. So, this example is really a lot like our uniformly accelerating rocket.

Another strong argument in favor of energy having inertia is that, in Newtonian physics, it is exactly the things that contribute to the center of mass which, together, make up the *total* mass of the box and govern it's resistance to acceleration by outside forces. Thus, on this basis we expect that adding energy to a system (say, charging a battery) does in fact give it more inertia; i.e, more mass.

By the way, this explains something rather odd that became known through experiments in the 1920's and 30's, a while after Einstein published his theory of relativity (in 1905). As you know, atomic nuclei are made out of protons and neutrons. An example is the Helium nucleus (also called an α particle) which contains two neutrons and two protons. However, the masses of these objects are:

Proton mass: $1.675 \times 10^{-27} kg$.

Neutron mass: $1.673 \times 10^{-27} kg$.

α **particle mass:** $6.648 \times 10^{-27} kg$.

So, if we check carefully, we see that an α particle has *less* mass than the mass of two protons plus the mass of two neutrons. The difference is $m_\alpha - 2m_p - 2m_n = -.0477 \times 10^{-27} kg$.

Why should this be the case? First note that, since these when these four particles stick together, they must have less energy when they are close together than when they are far apart. That is, it takes energy to rip them apart. But, if energy has inertia, this means exactly that the object you get by sticking them together (the α particle) will have less inertia (mass) than $2m_p + 2m_n$.

This, by the way, is how nuclear fusion works as a power source. For example, inside the sun, it often happens that two neutrons and two protons will be pressed close together. If they bind together to form an α particle then this releases an extra $.0477 \times 10^{-27} kg$ of energy that becomes heat and light.

Again, it is useful to have a look at the numbers. This amount of mass is worth an energy of $E = mc^2 = 5 \times 10^{-12} Watt - seconds \approx 1.4 \times 10^{-15} kWh$. This may not seem like much, but we were talking about just 2 protons and 2 neutrons. What if we did this for one gram[7] worth of stuff? Since four particles, each of which is about 1 Amu of mass, give the above result, one gram would produce the above energy multiplied by $\frac{1}{4}$ of Avagadro's number. In other words, we should multiply by 1.5×10^{23}. This yields roughly $2 \times 10^8 kWh = 5kW \cdot years!$. In other words, fusion energy from 1 gram of material could power 5 houses for one year! Nuclear fission yields comparable results.

By the way, when we consider any other form of power generation (like burning coal or gasoline), the mass of the end products (the burned stuff) is again less than the mass of the stuff we started with by an amount that is exactly c^{-2} times the energy released. However, for chemical processes this turns out to be an extremely tiny fraction of the total mass and is thus nearly impossible to detect.

6.4 More on Mass, Energy, and Momentum

In the last section we saw that what we used to call mass and energy can be converted into each other – and in fact are converted into each other all of the time. Does this mean that mass and energy really *are* the same thing? Well, that depends on exactly how one defines mass and energy.... the point is that, as with most things in physics, the old (Newtonian) notions of mass and energy will no longer be appropriate. So, we must extend both the old concept of mass and the old concept of energy before we can even start talking. There are various ways to extend these concepts. I'm going to use a more modern choice which is standard in the technical literature. Unfortunately, this modern choice seems to be less common in the popular literature and may therefore seem to contradict things you have read elsewhere.

6.4.1 Energy and Rest Mass

My notion of mass will be independent of reference frame. This is not the case for an older convention which a closer tie to the old $F = ma$. This older convention then defines a mass that changes with velocity. However, for the moment, let me skirt around this issue by talking about the "rest mass" (m_0) of an object, which is just the mass it has when it is at rest[8]. In particular, an object at rest has inertia $m_0 c^2$.

[7] About half of a paper clip.

[8] I suspect that the linguistic evolution is as follows: People first used the old convention where mass meant inertia. They then introduced this separate notion of 'rest mass.' As time

Recall that, in Newtonian physics, an object also has an energy $\frac{1}{2}m_0v^2$ due to its motion. Almost certainly, this expression will need to be modified in relativity, but it should be approximately correct for velocities small compared with the speed of light. Thus, for a slowly moving object we have

$$E = m_0c^2 + \frac{1}{2}m_0v^2 + \textit{ small corrections}. \tag{6.5}$$

Note that we can factor out an m_0c^2 to write this as:

$$E = m_0c^2(1 + \frac{1}{2}\frac{v^2}{c^2} + \textit{ small corrections}). \tag{6.6}$$

In section 6.6 we will derive the precise form of these small corrections. However, this derivation is somewhat technical and relies on a more in-depth knowledge of energy and momentum in Newtonian physics than some of you will have. Since I do not want it to obscure the main points of our discussion, I have relegated the derivation to a separate section (6.6) at the end of the chapter. For the moment, we will content ourselves with a well-motivated guess.

You may recall seeing an expression like (6.6) in some of your homework. It came up there because it gives the first few terms in the Taylor's series expansion of the time-dilation factor,

$$\frac{1}{\sqrt{1 - v^2/c^2}} = 1 + \frac{1}{2}\frac{v^2}{c^2} + \textit{ small corrections}, \tag{6.7}$$

a factor which has appeared in almost every equation we have due to its connection to the interval and Minkowskian geometry.

It is therefore natural to guess that the correct relativistic formula for the total energy of a moving object is

$$E = \frac{m_0c^2}{\sqrt{1 - v^2/c^2}} = m_0c^2 \cosh\theta. \tag{6.8}$$

This is exactly the formula[9] that will be derived in section 6.6.

6.4.2 Momentum and Mass

Momentum is a little trickier, since we only have one term in the expansion so far: $p = mv + $ *small corrections*. Based on the analogy with energy, we expect that this is the expansion of something native to Minkowskian geometry

passed, physicists found that the only time they ever used the word mass was in the phrase 'rest mass,' as the other concept was in fact better served by the term energy (which, as we have seen, carried inertia). After awhile, they got tired of inserting the word 'rest' and writing the subscript 0.

[9] The old convention that you will see in some books (but not in my class!!) is to define a 'velocity-dependent mass' $m(v)$ through $m(v) = \frac{m_0}{\sqrt{1-v^2/c^2}} = E/c^2$. This is an outdated convention and does not conform to the modern use of the term 'mass.'

– probably a hyperbolic trig function of the boost parameter θ. Unfortunately there are at least two natural candidates, $m_0 c \sinh\theta$ and $m_0 c \tanh\theta$ (which is of course just $m_0 v$). The detailed derivation is given in 6.6, but it should come as no surprise that the answer is the $\sinh\theta$ one that is simpler from the point of Minkowskian geometry and which is not the Newtonian answer. Thus the relativistic formula for momentum is:

$$p = \frac{m_0 v}{\sqrt{1 - v^2/c^2}} = m_0 c \sinh\theta. \tag{6.9}$$

If you don't really know what momentum is, don't worry too much about it. We will only touch on momentum briefly and the brief introduction in section 6.2.2 should suffice. I should mention, however, that the relativistic formulas for energy and momentum are very important for things you encounter everyday – like high resolution computer graphics! The light from your computer monitor[10] is generated by electrons traveling at $10 - 20\%$ of the speed of light and then hitting the screen. This is fast enough that, if engineers did not take into account the relativistic formula for momentum and tried to use just $p = mv$, the electrons would not land at the right places on the screen and the image would be all screwed up. There are some calculations about this in homework problem (5).

By the way, you may notice a certain similarity between the formulas for p and E in terms of rest mass m_0 and, say, the formulas (4.10) on page 104 for the position x and the coordinate time t relative to the origin for a moving inertial object in terms of it's own proper time τ and boost parameter θ. In particular, we have

$$\frac{pc}{E} = \frac{v}{c} = \tanh\theta \tag{6.10}$$

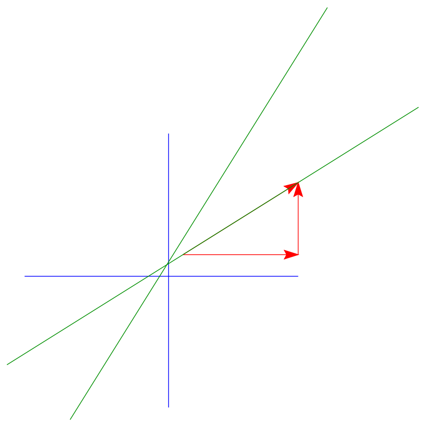
We also have

$$E^2 - c^2 p^2 = m_0 c^4 \left(\cosh^2\theta - \sinh^2\theta\right) = m_0^2 c^4. \tag{6.11}$$

Since m_0 does not depend on the reference frame, this is an *invariant* like, say, the interval. Hmm.... The above expression even looks kind of like the interval.... Perhaps it is a similar object?

⋆⋆⋆ Here is what is going on: a displacement (like Δx, or the position relative to an origin) in general defines a *vector* – an object that can be thought of like an arrow. Now, an arrow that you draw on a spacetime diagram can point in a timelike direction as much as in a spacelike direction. Furthermore, an arrow that points in a 'purely spatial' direction as seen in one frame of reference points in a direction that is not purely spatial as seen in another frame.

[10] So long as it is not an LCD display.



So, spacetime vectors have time parts (components) as well as space parts. A displacement in spacetime involves $c\Delta t$ as much as a Δx. The interval is actually something that computes the size of a given spacetime vector. For a displacement, it is $\Delta x^2 - c\Delta t^2$.

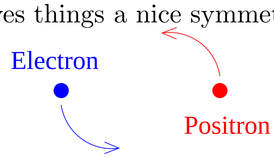
Together, the momentum and the energy form a *single* spacetime vector. The momentum is already a vector in space, so it forms the space part of this vector. It turns out that the energy forms the time part of this vector. So, the size of the energy-momentum vector is given by a formula much like the one above for displacements. This means that the rest mass m_0 is basically a measure of the size of the energy-momentum vector.

Furthermore, we see that this 'size' does not depend on the frame of reference and so does not depend on how fast the object is moving. However, for a rapidly moving object, both the time part (the energy) and the space part (the momentum) are large – it's just that the Minkowskian notion of the size of a vector involves a minus sign, and these two parts largely cancel against each other.

6.4.3 How about an example?

As with many topics, a concrete example is useful to understand certain details of what is going on. In this case, I would like to illustrate the point that while energy and momentum are both conserved, *mass* is not conserved.

Let's consider a simple system called 'positronium.' This consists of an electron (negatively charged) and a positron (positively charged) which orbit each other due to their electrical attraction. This system is nice because both particles have exactly the same rest mass which we can call m_e (e for electron). This gives things a nice symmetry.



A snapshot of the orbiting particles is shown above. Let us suppose that they are orbiting at speed $4c/5$, as measured in an inertial frame where the particles just go round and round each other. At the time shown, the electron is going straight upward and the positron is going straight downward. Each particle has a momentum

$$|p| = \frac{m_e v}{\sqrt{1 - v^2/c^2}} = \frac{4}{3} m_e c, \tag{6.12}$$

and an energy

$$E = \frac{m_e c^2}{\sqrt{1 - v^2/c^2}} = \frac{5}{3} m_e c^2. \tag{6.13}$$

Now, what is the energy and momentum of the positronium system as a whole? Well, the momenta are of the same size, but they are in opposite directions. So, they cancel out and the total momentum is zero. However, the energies are both positive (energy doesn't care about the direction of motion), so they add together. We find:

$$\begin{aligned} p_{positronium} &= 0, \\ E_{positronium} &= \frac{10}{3} m_e c^2. \end{aligned} \tag{6.14}$$

So, what is the rest mass of the positronium system?

$$E^2 - p^2 c^2 = \frac{100}{9} m_e^2 c^4. \tag{6.15}$$

So, the rest mass *of the positronium system* is given by dividing the right hand side by c^4. The result is $\frac{10}{3} m_e$, which is significantly greater than the rest mass of the electron plus the rest mass of the positron[11].

Similarly, two massless particles can in fact combine to make an object with a finite non-zero mass. For example, placing photons in a box adds to the mass of the box. We'll talk more about massless particles (and photons in particular) below.

6.5 Energy and Momentum for Light

At this point we have developed a good understanding of energy and momentum for objects. However, there has always been one other very important player in our discussions, which is of course light itself. In this section, we'll take a moment to explore the energy and momentum of light waves and to see what it has to teach us.

[11] I have cheated heavily here. A real positronium system must of course have *less* rest mass than would the particles separately. The difference is due to the electrical potential energy which is negative due to the attractive force between the particles. I have ignored this here, but it is the negative potential energy which makes real positronium hold together. More properly, one would say that the calculation here gives the rest mass of a system made by 'tying the two particles together with strong, light string and then spinning things up.

6.5.1 Light speed and massless objects

OK, let's look one more time at the question: "What happens if we try to get an object moving at a speed greater than c?" Let's look at the formulas for both energy and momentum. Notice that $E = \frac{m_0 c^2}{\sqrt{1-v^2/c^2}}$ becomes infinitely large as v approaches the speed of light. Similarly, an object (with finite rest mass m_0) requires an infinite momentum to move at the speed of light. Again this tells us is that, much as with our uniformly accelerating rocket from last week, no finite effort will ever be able to make any object (with $m_0 > 0$) move at speed c.

By the way, what happens if we try to talk about energy and momentum for light itself? Of course, many of our formulas (such as the one above) fail to make sense for $v = c$. However, some of them do. Consider, for example,

$$\frac{pc}{E} = \frac{v}{c}. \tag{6.16}$$

Since light moves at speed c through a vacuum, this would lead us to expect that for light we have $E = pc$. In fact, one can compute the energy and momentum of a light wave using Maxwell's equations. One finds that both the energy and the momentum of a light wave depend on several factors, like the wavelength and the size of the wave. However, in all cases the energy and momentum exactly satisfies the relation $E = pc$. As a result, we can consider a bit of light (a.k.a., a photon) with any energy E so long as we also assign it a corresponding momentum $p = E/c$. The energy and momentum of photons adds together in just the way that we saw in section 6.4.3 for massive particles.

So, what is the rest mass of light? Well, if we compute $m_0^2 c^2 = E^2 - p^2 c^2 = 0$, we find $m_0 = 0$. Thus, light has no mass. This to some extent shows how light can move at speed c and have finite energy. The zero rest mass 'cancels' against the infinite factor coming from $1 - v^2/c^2$ in our formulas above.

By the way, note that this also goes the other way: if $m_0 = 0$ then $E = \pm pc$ and so $\frac{v}{c} = \frac{pc}{E} = \pm 1$. Such an object has no choice but to always move at the speed of light.

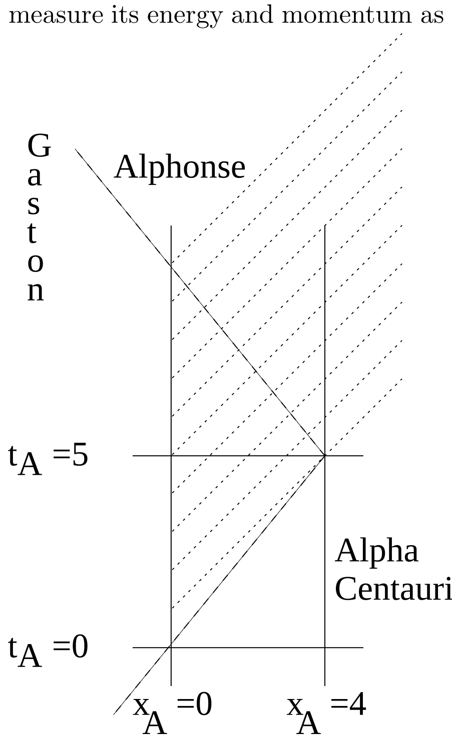
6.5.2 Another look at the Doppler effect

Recall that, for a massive particle (i.e., with $m_0 > 0$), if we are in a frame that is moving rapidly toward the object, the object has a large energy and momentum as measured by us. One might ask if the same is true for light.

The easy way to discover that it is in fact true for light as well is to use the fact (which we have not yet discussed, and which really belongs to a separate subject called 'quantum mechanics,' but what the heck...) that light actually comes in small chunks called 'photons.' The momentum and energy of a single photon are both proportional to its frequency f, which is the number of times that the corresponding wave shakes up and down every second.

Remember the Doppler effect from problem (3-13? This is also the effect seen in problem (4-4) when our twins Alphonse and Gaston sent pulses of light toward each other. The frequency with which the light was emitted in, say, Alphonse's frame of reference was not the same as the frequency at which the light was received in the other frame (Gaston's).

The result was that if Gaston was moving toward Alphonse, the frequency was higher in Gaston's frame of reference. Using the relation between frequency and energy (and momentum), we see that for this case the energy and momentum of the light is indeed higher in Gaston's frame of reference than in Alphonse's frame of reference. So, moving toward a ray of light has a similar effect on how we measure its energy and momentum as does moving toward a massive object.



6.6 Deriving the Relativistic expressions for Energy and Momentum

Due to its more technical nature and the fact that this discussion requires a more solid understanding of energy and momentum in Newtonian physics, I have saved this section for last. It is very unlikely that I will go over this in class and you may consider this section as optional reading. Still, if you're inclined to see just how far logical reasoning can take you in this subject, you're going to really enjoy this section.

It turns out that the easiest way to do the derivation is by focusing on momentum[12]. The energy part will then emerge as a pleasant surprise. The argument has four basic inputs:

1. We know that Newtonian physics is not exactly right, but it is a good approximation at small velocities. So, for an object that moves slowly, it's momentum is well approximated by $p = mv$.

2. We will assume that, whatever the formula for momentum is, momentum in relativity is still *conserved*. That is, the total momentum does not change with time.

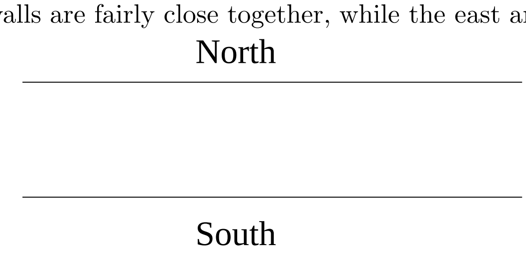
3. We will use the principle of relativity; i.e., the idea that the laws of physics are the same in any inertial frame of reference.

4. We choose a clever special case to study. We will look at a collision of two objects and we will assume that this collision is 'reversible.' That is, we will assume that it is possible for two objects to collide in such a way that, if we filmed the collision and played the resulting movie backwards, what we see on the screen could also be a real collision. In Newtonian physics, such collisions are called *elastic* because energy is conserved.

Let us begin with the observation that momentum is a vector. In Newtonian relativity, the momentum points in the same direction as the velocity vector. This follows just from symmetry considerations (in what other direction could it point?). As a result, it must also be true in relativistic physics. The only special direction is the one along the velocity vector.

It turns out that to make our argument we will have to work with at least two dimensions of space. This is sort of like how we needed to think about sticks held perpendicular to the direction of motion when we worked out the time dilation effect. There is just not enough information if we stay with only one dimension of space.

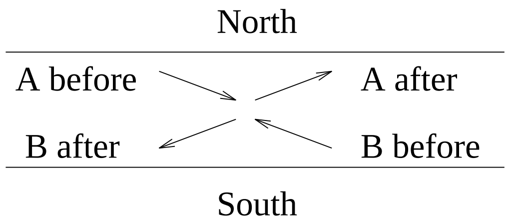
So, let us suppose that we are in a long, rectangular room. The north and south walls are fairly close together, while the east and west walls are far apart:



Now, suppose that we have two particles that have the same rest mass m_0, and which in fact are exactly the same when they are at rest. We will set things up so that the two particles are moving at the same speed relative to the room, but in opposite directions. We will also set things up so that they collide

[12] I first learned this argument by reading *Spacetime Physics* by Taylor and Wheeler.

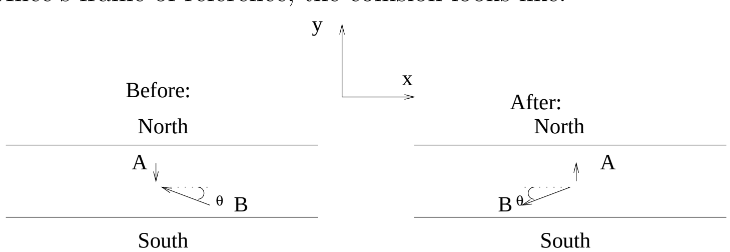
exactly in the middle of the room, but are not moving exactly along either the north-south axis or the east-west axis. Also, the particles will not quite collide head-on, so that one scatters to each side after the collision. In the reference frame of the room, the collision will look like this:



However, we will assume that the particles are *nearly* aligned with the east-west axis and that the collision is *nearly* head-on, so that their velocities in the north-south direction are small. Note that I have labeled one of the particles 'A' and one of them 'B.'

To proceed, we will analyze the collision in a different reference frame. Suppose that one of our friends (say, Alice) is moving rapidly to the east through the room. If she travels at the right speed she will find that, before the collision and relative to her, particle A does not move east or west but only moves north and south. We wish to set things up so that the motion of particle A in Alice's frame of reference is slow enough that we can use the Newtonian formula $p = mv$ for this particle in this frame of reference. For symmetry purposes, we will have another friend Bob who travels to the right fast enough that, relative to him, particle B only moves in the north-south direction.

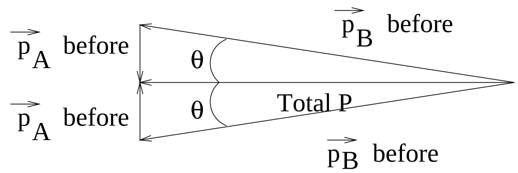
Now, suppose we set things up so that the collision is not only reversible, but in fact looks *exactly the same* if we run it in reverse. That is, we suppose that in Alice's frame of reference, the collision looks like:



where particle A has the same speed before as it does after, as does particle B. Also, the angle θ is the same both before and after. Such a symmetric situation must be possible unless there is an inherent breaking of symmetry in spacetime.

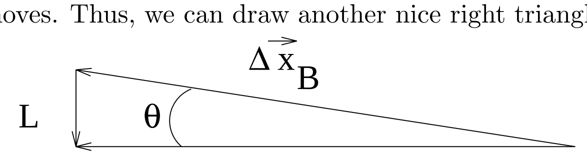
Now, the velocity of particle A in this frame is to be slow enough that its momentum is given by the Newtonian formula $p_A = m_0 v_A$. For convenience, I have indicated coordinate directions x and y on the diagram in Alice's reference frame. It's velocity in the x direction is zero, so its momentum in this direction must also be zero. Thus, particle A only has momentum in the y direction. As a result, the change in the momentum of particle A is $2m_0 dy/dt$, where dy/dt denotes the velocity in the y direction after the collision.

If momentum is to be conserved, the total vector momentum must be the same before as after. That is to say, if (in Alice's frame of reference) we add the arrows corresponding to the momentum before, and the momentum after, we must get the same result:



For the next part of the argument notice that if, after the collision, we observe particle B for awhile, it will eventually hit the south wall. Let us call this event Y, where B hits the south wall after the collision. Recall that the collision takes place in the middle of the box. Event Y and the collision will be separated by some period of time Δt_B (as measured by Alice) and some displacement vector $\Delta\vec{x}_B = (\Delta x_B, \Delta y_B)$ in space as measured by Alice. If the box has some length $2L$ in the north-south direction, then since the collision took place in the middle, $\Delta y_B = L$.

Also, if we trace particle B back in time before the collision, then there was some event before the collision when it was also at the south wall. Let us call this event X, when B was at the south wall before the collision. By symmetry, this event will be separated from the collision by the same Δt_B and by $-\Delta\vec{x}_B$. What I want you to notice, is that the displacement $\Delta\vec{x}_B$ points in the same direction as the momentum of particle B, since that is the direction in which B moves. Thus, we can draw another nice right triangle:



Note that this triangle has the same angle θ as the one drawn above. As a result, we have

$$\frac{L}{|\Delta\vec{x_B}|} = \sin\theta = \frac{p_A}{|\vec{p}_B|}. \tag{6.17}$$

Note that, since p_A has no x component, I have not bothered to represent it as a vector. If this notation bothers you, just replace all my p_A's with p_{Ay}. Here, $|\vec{p}_B|$ is the usual length of this vector, and similarly for $\vec{x_B}$.

Technically speaking, what we will do is to rearrange this formula as

$$\vec{p}_B = \frac{(p_A)(\Delta\vec{x_B})}{L}, \tag{6.18}$$

where now we have put the direction information back in. We will then compute $\vec{p}_B$ in the limit as the vertical velocity of particle A (and thus p_A, in Alice's frame) goes to zero. In other words, we will use the idea that a slowly moving particle (in Alice's frame) *could* have collided with particle B to determine particle B's momentum.

Let us now take a moment to calculate p_A. In the limit where the velocity of particle A is small, we should be able to use $p_A = m_0 dy/dt$ after the collision. Now, we can calculate dy/dt by using the time Δt_A that it takes particle A to go from the collision site (in the center of the box) to the north wall. In this time, it travels a distance L, so $p_A = m_0 L/\Delta t_A$. Again, the distance is L in Alice's frame, Bob's frame, or the Box's frame of reference since it refers to a direction perpendicular to the relative motion of the frames.

Thus we have

$$\vec{p}_B = \lim_{v_A \to 0} \frac{m_0(\Delta \vec{x_B})}{\Delta t_A}. \tag{6.19}$$

Now, what we want to do is in fact to derive a formula for the momentum of particle B. This formula should be the same whether or not the collision actually took place. Thus, we should be able to forget entirely about particle A and rewrite the above expression purely in terms of things having to do with particle B. We can do this by a clever observation.

Recall that we originally set things up in a way that was symmetric with respect to particles A and B. Thus, if we watched the collision from particle A's perspective, it would look just the same as if we watched it from particle B's perspective. In particular, we can see that the *proper* time $\Delta\tau$ between the collision and the event where particle A hits the north wall must be exactly the same as the *proper* time between the collision and the event where particle B hits the south wall.

Further, recall that we are interested in the formula above only in the limit of small v_A. However, in this limit Alice's reference frame coincides with that of particle A. As a result, the proper time $\Delta\tau$ is just the time Δt_A measured by Alice. Thus, we may replace Δt_A above with $\Delta\tau$, where $\Delta\tau$ is the proper time between the collision and the event where particle B hit the south wall. The expression no longer depends on particle A, so the limit is trivial. We have:

$$\vec{p}_B = \frac{m_0(\Delta \vec{x_B})}{\Delta \tau}. \tag{6.20}$$

Since the motion of B is uniform after the collision, we can replace this ratio with a derivative:

$$|\vec{p}_B| = m_0 \frac{d\vec{x_B}}{d\tau} = m_0 \frac{1}{\sqrt{1 - v^2/c^2}} \frac{d\vec{x_B}}{dt}. \tag{6.21}$$

Thus, we have derived

$$|\vec{p}| = m_0 \frac{\vec{v}}{\sqrt{1 - v^2/c^2}}, \tag{6.22}$$

the relativistic formula for momentum.

Now, the form of equation (6.21) is rather suggestive. It shows that the momentum forms the spatial components of a spacetime vector:

$$p = m_0 \frac{dx}{d\tau}, \tag{6.23}$$

where x represents all of the spacetime coordinates (t, x, y, z). One is tempted to ask, "What about the time component $m_0 dt/d\tau$ of this vector?"

Recall that we have assumed that the momentum is conserved, and that this must therefore hold in *every* inertial frame! If 3 components of a spacetime vector are conserved in every inertial frame, then it follows that the fourth one does as well. (Think about this from two different reference frames and you'll see why ...). So, this time component does represent *some* conserved quantity. We can get an idea of what it is by expanding the associated formula in a Taylor series for small velocity:

$$m_0 \frac{dt}{d\tau} = m_0 cosh\theta = m_0 \frac{1}{1 - v^2/c^2} = m_0(1 + \frac{1}{2}\frac{v^2}{c^2} + ...) = c^{-2}(m_0c^2 + \frac{1}{2}m_0v^2 + ...) \tag{6.24}$$

In Newtonian physics, the first term is just the mass, which is conserved separately. The second term is the kinetic energy. So, we identify this time component of the spacetime momentum as the (c^{-2} times) the energy:

$$E = c^2 p^t = \frac{m_0 c^2}{\sqrt{1 - v^2/c^2}}. \tag{6.25}$$

In relativity, mass and energy are not conserved separately. Mass and energy in some sense merge into a single concept 'mass-energy[13].' Also, we have seen that energy and momentum fit together into a single spacetime vector just as space and time displacements fit together into a 'spacetime displacement' vector. Thus, the concepts of momentum and energy also merge into a single 'energy-momentum vector.'

6.7 Homework Problems

Note: Energy can be measured in various units, like Joules (J) or kiloWatt-hours (kW-hrs., this is the unit that Niagara Mohawk uses on my electric bill). You can use any unit that you like. You may find the following relations between the various units useful.

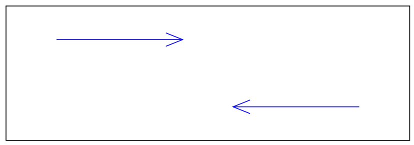
$$1kg(m^2/s^2) = 1Joule(J) = 1Watt - second(Ws) = \frac{1}{3.6} \times 10^{-6} kW - hrs.$$

6-1. How much energy would it take to accelerate you up to $.9c$?

[13]Usually just called 'energy' in modern terminology.

6-2. I pay Niagara Mohawk \$0.107 per kW-hr. How much would it cost me to accelerate you up to $.9c$?

6-3. Consider a box containing two photons traveling in opposite directions.



If the box has a rest mass m_0 and each photon has an energy E_0, what is the rest mass of the combined box-plus-photons system? **Hint:** How much energy and momentum does each of the three objects have?

6-4. In particle accelerators, one can collide an electron with a positron and (sometimes) they turn into a proton/anti-proton pair. The rest mass of an electron (or a positron) is $9.11 \times 10^{-31} kg$. The rest mass of a proton (or an anti-proton) is $1.673 \times 10^{-27} kg$. Suppose that the proton/anti-proton pair is created at rest and that the electron and positron had equal speed in opposite directions. How fast must the electron and positron have been moving for this reaction to be allowed by conservation of energy? Give the answer both in terms of speed v and boost parameter θ.

6-5. Here's a good calculation if you know a little physics. It has to do with how your TV and computer monitor work:

Particle physicists often use a unit of energy called the "electron-Volt" (eV). This amount of energy that an electron picks up when it accelerates across a potential of one Volt[14]. Since the charge on an electron is 1.6×10^{-19} Coulombs, one electron-Volt is $1.6 \times 10^{-19} J$.

(a) If an object at rest has a total energy of $1eV$, what is it's mass?

(b) The mass of an electron is $9.11 \times 10^{-31} kg$. What is the energy (in eV) of an electron at rest?

(c) In a standard CRT (Cathode Ray Tube, like the tube in your TV or computer monitor), electrons are accelerated through a potential difference of about 5000 volts. In other words, moving through that potential adds $5000 eV$ of energy to the electron. How fast is an electron going when it strikes your TV or computer screen?

(d) Consider the electron that is just about to strike your screen. What is it's momentum? If you used the old (and incorrect) formula $p = mv$ to calculate its momentum, how much would the answer be off?

[14] You can look at a PHY212 or 216 physics book for a definition of the Volt, but you won't actually need to know it for this problem.

Chapter 7

Relativity and the Gravitational Field

Einstein: Chapters XVIII-XX
Read Einstein, ch. 18-22

At the end of the last chapter, we finished the part of the course that is referred to as 'Special Relativity' (SR). Now, special relativity by itself was a real achievement. In addition to revolutionizing our conceptions of time and space, uncovering new phenomena, and dramatically changing our understanding of mass, energy, and momentum, Minkowskian geometry finally gave a good picture of how it can be that the speed of light (in a vacuum!) is the same in all frames of reference. However, in some sense there is still a large hole to be filled. We've talked about what happens when objects accelerate, but we have only begun to discuss *why* they accelerate, in terms of why and how various forces act on these objects.

We made some progress on this issue in the last chapter's discussion of energy and momentum. We have the relation $E = \frac{m_0 c^2}{\sqrt{1-v^2/c^2}}$ so we know that, when we feed an object a certain amount of energy it will speed up, and when we take energy away it will slow down. We can even use this formula to calculate exactly how much the object will speed up or slow down. But what we haven't talked about are the basic mechanisms that add and subtract energy – the 'forces' themselves. Of course, physicists already had some understanding of these forces when Einstein broke onto the scene. The important question, of course, is whether this understanding fit well with relativity or whether relativity would force some major change the understanding of the forces themselves.

Physicists in Einstein's time knew about many kinds of forces:

a) Electricity.

b) Magnetism.

c) Gravity.

d) Friction.

e) One object pushing another.

f) Pressure.

and so on..... Now, the first two of these forces are described by Maxwell's equations. As we have discussed, Maxwell's equations fit well with (and even led to!) relativity. Unlike Newton's laws, Maxwell's equations are fully compatible with relativity and require no modifications at all. Thus, we may set these forces aside as 'complete' and move on to the others.

Let's skip ahead to the last three forces. These all have to do in the end with atoms pushing and pulling on each other. In Einstein's time, such things we believed[1] to be governed by the electric forces between atoms. So, it was thought that this was also properly described by Maxwell's equations and would fit well with relativity.

You may have noticed that this leaves one force (gravity) as the odd one out. Einstein wondered: how hard can it be to make gravity consistent with relativity?

7.1 The Gravitational Field

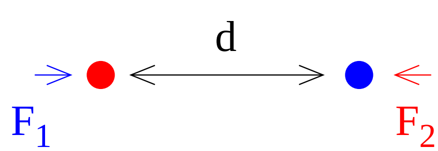
Let's begin by revisiting the pre-relativistic understanding of gravity. Perhaps we will get lucky and find that it too requires no modification.

7.1.1 Newtonian Gravity vs. relativity

Newton's understanding of gravity was as follows:

Newton's Universal Law of Gravity Any two objects of masses m_1 and m_2 exert 'gravitational' forces on each other of magnitude

$$F = G\frac{m_1 m_2}{d^2}, \tag{7.1}$$



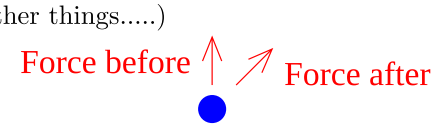
[1] This belief is basically false. A large part of such 'forces' comes from an effect that is not in fact described as a 'force' today. This effect is known as the 'Pauli exclusion principle' and states that no two electrons can occupy the same 'quantum state' (basically, that they cannot be stacked on top of each other). Today, we recognize this effect as coming from the fundamental quantum nature of the electron. (Protons and other 'fermions' behave similarly, while photons and other 'bosons' do not.) Quantum mechanics is another kettle of fish altogether, but in the end it does fit well with special relativity.

> directed toward each other, where $G = 6.67 \times 10^{-11} Nm^2/kg^2$ is called "Newton's Gravitational Constant." G is a kind of intrinsic measure of how strong the gravitational force is.

It turns out that this rule is *not* compatible with special relativity. In particular, having learned relativity we now believe that it should not be possible to send messages faster than the speed of light. However, Newton's rule above would allow us to do so using gravity. The point is that Newton said that the force depends on the separation between the objects *at this instant*[2].

Example: The earth is about eight light-minutes from the sun. This means that, at the speed of light, a message would take eight minutes to travel from the sun to the earth. *However,* suppose that, unbeknownst to us, some aliens are about to move the sun.

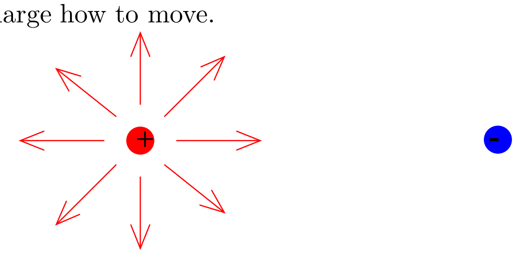
Then, based on our understanding of relativity, we would expect it to take eight minutes for us to find out! But Newton would have expected us to find out instantly because the force on the earth would shift (changing the tides and other things.....)



7.1.2 The importance of the field

Now, it is important to understand how Maxwell's equations get around this sort of problem. That is to say, what if the Sun were a positive electric charge, the earth were a big negative electric charge, and they were held together by an Electro-Magnetic field? We said that Maxwell's equations are consistent with relativity – so how what would they tell us happens when the aliens move the sun?

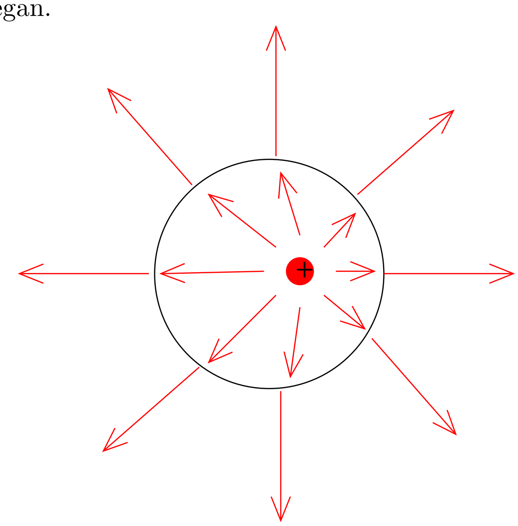
The point is that the positive charge does not act *directly* on the negative charge. Instead, the positive charge sets up an electric field which tells the negative charge how to move.



When the positive charge is moved, the electric field around it must change, but it turns out that the field does not change everywhere at the same time.

[2]Note that there is also an issue of simultaneity here. Which events on the two separated worldlines should one compare to compute the distance? Which notion of 'this instant' would one use to pick out these events?

Instead, the movement of the charge modifies the field only where the charge actually is. This makes a 'ripple' in the field which then moves outward at the speed of light. In the figure below, the black circle is centered on the original position of the charge and is of a size ct, where t is the time since the movement began.



Thus, the basic way that Maxwell's equations get around the problem of instant reaction is by having a *field* that will carry the message to the other charge (or, say, to the planet) at a finite speed. Oh, and remember that having a field that could carry momentum was also what allowed Maxwell's equations to fit with momentum conservation in relativity. What we see is that the field concept is the essential link that allows us to understand electric and magnetic forces in relativity.

Something like this must happen for gravity as well. Let's try to introduce a gravitational field by breaking Newton's law of gravity up into two parts. The idea will again be than an object should produce a gravitational field (g) in the spacetime around it, and that this gravitational field should then tell the other objects how to move through spacetime. Any information about the object *causing* the gravity should not reach the other objects directly, but should only be communicated through the field.

Old: $F = \frac{m_1 m_2 G}{d^2}$

New: $F_{on\ m_1} = m_1 g$,

$g = \frac{m_2 G}{d}$.

7.2 Some observations

I should mention that these notes will address our new topic (General Relativity) from a somewhat different point of view than your readings do. I do not mean to imply that my version is more accurate than the one in your readings (or vice versa) – the readings and I are simply stressing different aspects of the various thoughts that were rattling around inside Albert Einstein's head in the early 1900's. BTW, figuring out General Relativity was much harder than figuring out special relativity. Einstein worked out special relativity is about a year (and he did many other things in that year). In contrast, the development of general relativity required more or less continuous work from 1905 to 1916.

In fact, I'm going to stress several important ingredients, of which we have just seen the first. For future reference, they are:

a) Free fall and the gravitational field.

b) The question of whether light is affected by gravity.

c) Further reflection on inertial frames.

7.2.1 Free Fall

Before going on to the other important ingredients, let's take a moment to make a few observations about gravitational fields and to introduce some terminology.

Notice an important property of the gravitational field. The gravitational force on an object of mass m is given by $F = mg$. But, in Newtonian physics, we also have $F = ma$. Thus, we have

$$a = \frac{mg}{m} = g. \tag{7.2}$$

The result is that all objects in a given gravitational field accelerate at the same rate (if no other forces act on them). The condition where gravity is the only influence on an object is known as "free fall." So, the gravitational field g has a direct meaning: *it gives the acceleration of "freely falling" objects.*

A particularly impressive example of this is called the 'quarter and feather experiment.' Imagine taking all of the air out of a cylinder (to remove air resistance which would be an extra force), and then releasing a quarter and a feather at the same time. The feather would then then "drops like a rock." In particular, the quarter and the feather fall together in exactly the same way. I have put a video of this experiment (from when I did it live for my PHY211 class in fall 1999) on the PHY312 web site for you to check it out.

Now, people over the years have wondered if it was really true that all objects fall at exactly the same rate in a gravitational field, or if this was only approximately true. If it is exactly correct, they wondered *why* it should be so. It is certainly a striking fact.

For example, we have seen that energy is related to mass through $E = mc^2$. So, sometimes in order to figure out the exact mass of an object (like a hot wall that a laser has been shining on....) you have to include some things (like heat) that we used to count separately as 'energy' .... Does this E/c^2 have the same effect on gravity as the more familiar notion of mass?

In order to be able to talk about all of this without getting too confused, people invented two distinct terms for the following two distinct concepts:

1) Gravitational mass m_G. This is the kind of mass that interacts with the gravitational field. Thus, we have $F = m_G g$.

2) Inertial mass m_I. This is the kind of mass that goes into Newton's second law. So, we have $F = m_I a$.

Now, we can ask the question we have been thinking of in the clean form: is it always true that gravitational mass and inertial mass are the same? That is, do we always have $m_G = m_I$?

7.2.2 The 2nd ingredient: The effects of gravity on light

Let's leave aside for the moment further thought about fields as such and turn to another favorite question: to what extent is light affected by gravity?

Now, first, why do we care? Well, we built up our entire discussion of special relativity using light rays and we assumed in the process that light always traveled at a constant speed in straight lines! So, what if it happens that gravity can pull on light? If so, we may have to modify our thinking.

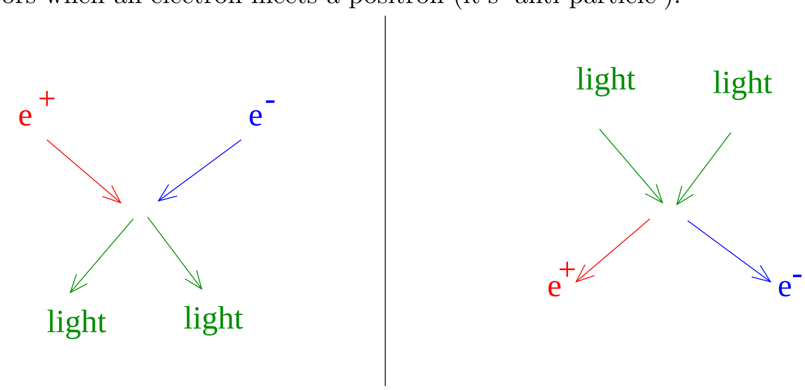
Clearly, there are two possible arguments:

i) No. Light has no mass ($m_{light} = 0$). So, gravity cannot exert a force on light and should not affect it.

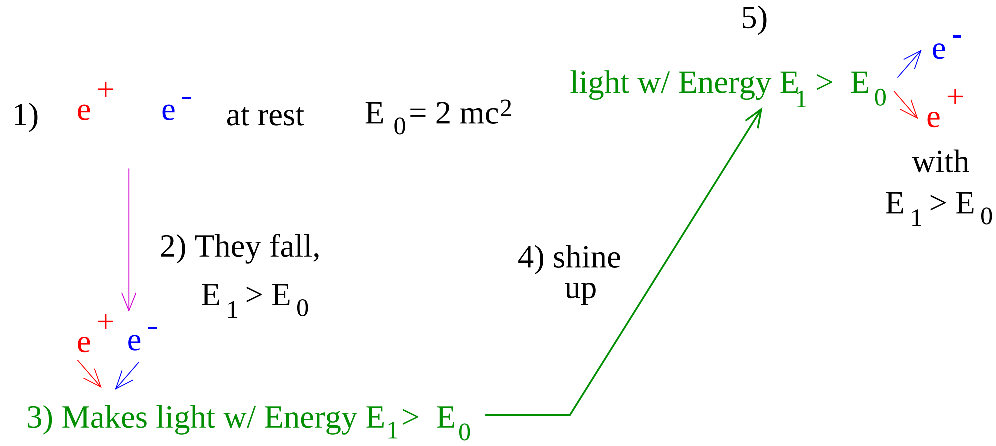
ii) Yes. After all, *all* things fall at the same rate in a gravitational field, even things with a very small mass. So, light should fall.

Well, we could go back and forth between these two points of view for quite awhile.... but let's proceed by introducing a third argument in order to break the tie. We'll do it by recalling that there is a certain equivalence between energy and mass.

In fact, in certain situations, "pure mass" can be converted into "pure energy" and vice versa. A nice example of this happens all the time in particle accelerators when an electron meets a positron (it's 'anti-particle').



Let us suppose that gravity does not effect light and consider the following process:



1) First, we start with an electron (mass m) and a positron (also mass m) at rest. Thus, we have a total energy of $E_0 = 2mc^2$.

2) Now, these particles fall a bit in a gravitational field. They speed up and gain energy. We have a new larger energy $E_1 > E_0$.

3) Suppose that these two particles now interact and turn into some light. By conservation of energy, this light has the same energy $E_1 > E_0$.

4) Let us take this light and shine it upwards, back to where the particles started. (This is not hard to do – one simply puts enough mirrors around the region where the light is created.) Since we have assumed that gravity does not affect the light, it must still have an energy $E_1 > E_0$.

5) Finally, let us suppose that this light interacts with itself to make an electron and a positron again. By energy conservation, these particles must have an energy of $E_1 > E_0$.

Now, at the end of the process, nothing has changed except that we have more energy than when we started. And, we can keep repeating this to make more and more energy out of nothing. Just think about what this would do, for example, to ideas about energy conservation!

We have seen some hard to believe things turn out to be true, but such an infinite free source of energy seems especially hard to believe. This strongly suggests that light is in fact affected by gravity in such a way that, when the light travels upwards though a gravitational field, it loses energy in much the same way as would a massive object.

7.2.3 Gravity, Light, Time, and all that

In the previous subsection we argued that light *is* in fact affected by gravity. In particular, when light travels upwards though a gravitational field, it looses just as much energy as would a massive object.

Now, what happens to light when it looses energy? Well, it happens that light comes in little packages called 'photons.' This was only beginning to be understood when Einstein started thinking about gravity, but it is now well established and it will be a convenient crutch for us to use in assembling our own understanding of gravity. The amount of energy in a beam of light depends on how many photons are in the beam, and on how much energy each photon has separately.

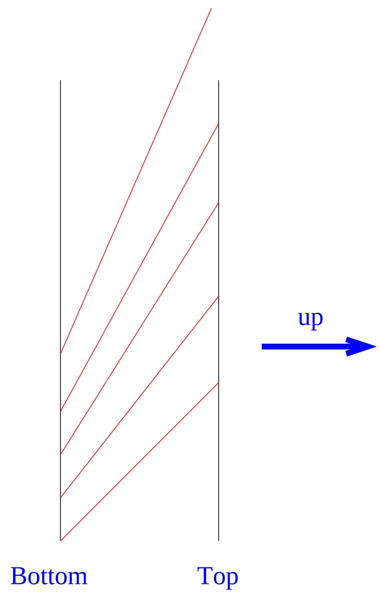
You can see that there are two ways for a beam of light to loose energy. It can either actually loose photons, or each photon separately can loose energy.

As the light travels up through the gravitational field, it should loose energy *continuously.* Loosing photons would not be a continuous, gradual process – it would happen in little discrete steps, one step each time a photon was lost. So, it is more likely that light looses energy in a gravitational field by each photon separately loosing energy.

How does this work? As we mentioned in section (6.5.1), the energy of a single photon depends on something called the *frequency* of the light. The frequency is just a measure of how fast the wave oscillates. The energy E is in fact proportional to the frequency f, through something called "Plank's constant" (h). In other words, $E = hf$ for a photon.

So, as it travels upwards in our gravitational field, this means that our light wave must loose energy by changing frequency and oscillating more slowly. It may please you to know that, long after this effect was suggested by Einstein, it was measured experimentally. The experiment was done by Pound and Rebke at Harvard in 1959.

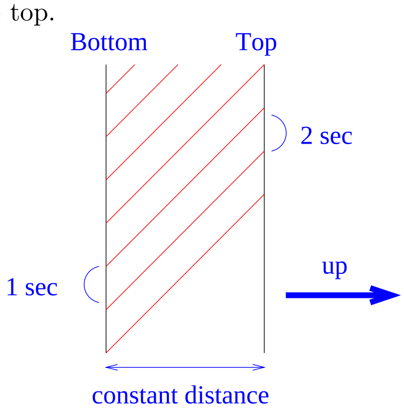
Now, a light wave is a bunch of wave crests and wave troughs that chase each other around through spacetime. Let's draw a spacetime diagram showing the motion of, say, a bunch of the wave crests. Note that, if the wave oscillates more slowly at the top, then the wave crests must be farther apart at the top than at the bottom.



But.... isn't each wave crest supposed to move at the same speed c in a vacuum? It looks like the speed of light gets faster and faster as time passes! Perhaps we have done something wrong? *By the way, do you remember any time before when we saw light doing weird stuff???*

Hmmmm.... something is definitely funny in the diagram above. Nothing is really changing with time, so each crest should act the same as the one before and move at the same speed, at least when the wave is at the same place. Let's choose to draw this speed as a 45^o line as usual. In that case, our diagram must look like the one below.

However, we *know* both from our argument above and from Pound and Rebke's experiment that the time between the wave crests is larger at the top. So, what looks like the same separation must actually represent a greater proper time at the top.



This may seem very odd. Should we believe that time passes at a faster rate higher up? Note that we are really comparing time as measured by two different clocks, one far above the other. Also note that these clocks have *no relative motion.*

In fact, this does really occur! The Pound and Rebke experiment is an observation of this kind, but it direct experimental verification was made by precise atomic clocks maintained by the National Bureau of Standards in the 1960's. They kept one clock in Washington D.C. (essentially at sea level) and one clock in Denver (much higher up). The one in Denver measured more time to pass (albeit only by a very small amount, one part in 10^{15}!).

7.2.4 Gravity and Accelerating Frames

Hmmmm.... so, we have clocks with *no relative motion* that run at different rates. Is this absurd? Well, no, and actually it should sound somewhat familiar. Do you recall seeing something like this before? (Hint: remember the accelerating rocket??)

Ah, yes. This sounds very much like the phenomenon that we saw in section (5.2.3) in which clocks at the front and back of a uniformly rocket ship experienced no relative motion but had clocks that ran at different rates. If one works out the math based on our discussion of energy and frequency above one finds that, at least over small distances, a gravitational field is not just qualitatively, but also quantitatively like an accelerating rocket ship with $a = g$!!

7.3 The Equivalence Principle

3) Finally, let's go back to an old issue: inertial frames.

Let's first see what we can recall. For example, are we in an inertial frame right now??? Can we estimate our acceleration relative to one?? If we were in a rocket ship out in space, what experiment could we perform to find out if we were accelerating?? Can we feel the forces acting on us?

Well, in the end we might say that our acceleration is very small because we can look over at the Sun and decide our acceleration relative to it is very small (much less than $10m/s^2$) ... and surely the Sun is in an inertial frame?

Or, even better, let us consider someone in China, on the opposite side of the earth from us. If we are accelerating away from the earth's center at $10m/s^2$, then they must be as well! *So, why aren't we getting farther apart???*

7.3.1 Gravity and Locality

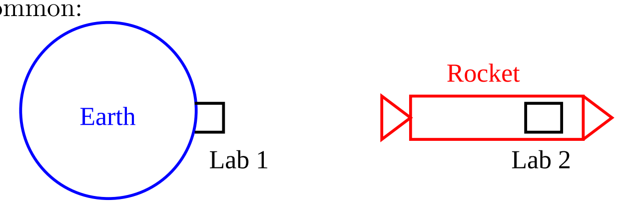
But, what if we were not allowed to look at the Sun? What if we were only allowed to make measurements here in this room? [Such measurements are called *local* measurements.] What objects in this room are in inertial frames?

How do we know? Should we drop a sequence of rocks, as we would in a rocket ship?

> If only local measurements are made, then it is the state of free-fall that is much like being in an inertial frame. In particular, a person in free-fall in a gravitational field **feels** just like an inertial observer!

Note how this fits with our observation about clocks higher up running faster than clocks lower down. We said that this exactly matches the results for an accelerating rocket with $a = g$. As a result, things that accelerate relative to the lab will behave like things that accelerate relative to the rocket. In particular, it is the freely falling frame that accelerates downward at g relative to the lab, while it is the inertial frame that accelerates downward at g relative to the rocket! Thus, clocks in a freely falling frame act like those in an inertial frame, and it is in the freely falling frame that clocks with no relative motion in fact run at the same rate!!

Similarly, a lab on the earth and a lab in a rocket (with it's engine on, and, say, accelerating at $10m/s^2$) are very similar. They have the following features in common:



1) Clocks farther up run faster in both cases, and by the same amount!

2) If the non-gravitational force on an object is zero, the object "falls" relative to the lab at a certain acceleration that does not depend on what the object is!

3) If you are standing in such a lab, you feel exactly the same in both cases.

Einstein's guess (insight?) was that, in fact:

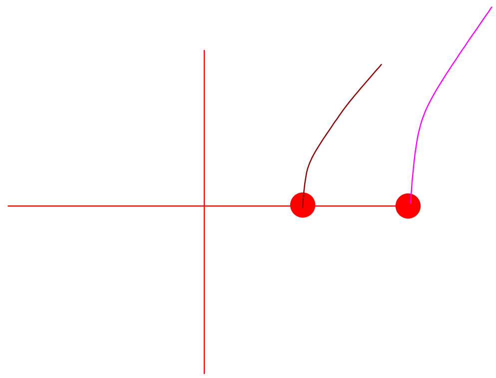
> Under local measurements, a gravitational field is *completely* equivalent to an acceleration.

This statement is known as **The Equivalence Principle**.

In particular, gravity has *NO* local effects in a freely falling reference frame. This ideas turns out to be useful even in answering non-relativistic problems. For example, what happens when I drop a hammer held horizontally? Does the heavy end hit first, or does the light end? Try using the equivalence principle idea to predict the answer.....

So then, what would be the best way to draw a spacetime diagram for a tower sitting on the earth? By "best," what I mean is: in what frame of reference do we most understand what is going on?

The answer of course is the frame that acts like an inertial frame. In this case, this is the freely falling reference frame. We have learned that, in such a reference frame, we can ignore gravity completely.



Now, how much sense does the above picture really make? Let's make this easy, and suppose that the earth were really big.... it turns out that, in this case, the earth's gravitational field would be nearly constant, and would weaken only very slowly as we go upward. Does this mesh with the diagram above?

Not really..... We said that the diagram above is effectively in an inertial frame. However, in this case we know that, if the distance between the bottom and top of the tower does not change, then the bottom must accelerate at a *faster* rate than the top does! But we just said that we want to consider a *constant* gravitational field! So, what's up?

Side note: No, it does not help to point out that the real earth's gravitational field is not constant. The point here is that the earth's gravitational field changes in a way that has nothing to do with the relationship $\alpha = c^2/l$ from the accelerated rocket.

7.3.2 How Local?

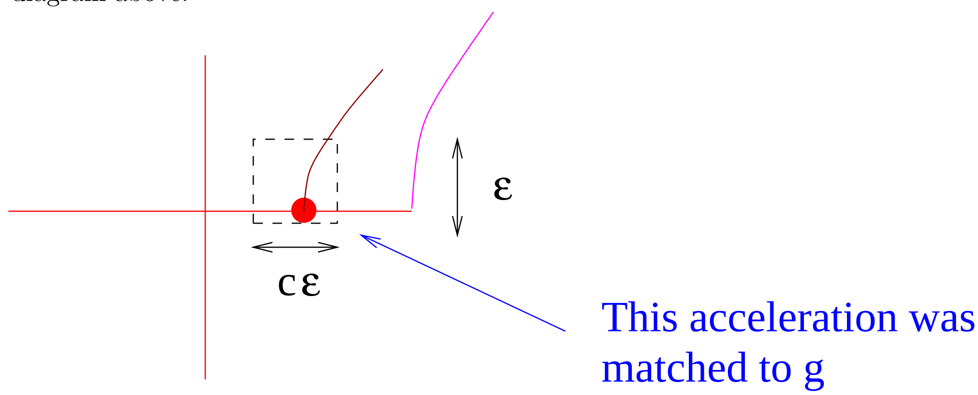
Well, we do have a way out of this: We realized before that the idea of freely falling frames being like inertial frames was not universally true. After all, freely falling objects on opposite side of the earth do accelerate towards each other. In contrast, any two inertial objects experience zero relative acceleration.

However, we did say that inertial and freely falling frames are the same 'locally.' Let's take a minute to refine that statement.

How local is local? Well, this is much like the question of "when is a velocity *small* compared to the speed of light?" What we found before was that Newtonian physics held true in the *limit* of small velocities. In the same way, our statement that inertial frames and freely-falling frames are similar is supposed to be true in the sense of a limit. This comparison becomes more and more valid the smaller a region of spacetime we use to compare the two.

Nevertheless, it is still meaningful to ask how accurate this comparison is. In other words, we will need to know just which things agree in the above limit, and we will need to understand better just what this limit is.

To understand Einstein's answer, let's consider a tiny box of spacetime from our diagram above.



For simplicity, consider a 'square' box of height ϵ and width $c\epsilon$. This square should contain the event at which we matched the "gravitational field g" to the acceleration of the rocket.

In this context, Einstein's proposal was that
Errors in *dimensionless* quantities like angles, v/c, and boost parameters frameboxshould be proportional to ϵ^2.

Let us motivate this proposal through the idea that the equivalence principle should work "as well as it possibly can." Suppose for example that the gravitational field is really *constant*, meaning that static observers at any position measure the same gravitational field g. We then have the following issue: when we match this gravitational field to an accelerating rocket in flat spacetime, do we choose a rocket with $\alpha_{top} = g$ or one with $\alpha_{bottom} = g$? Recall that any rigid rocket will have a different acceleration at the top than it does at the bottom. So, what we mean by saying that the equivalence principle should work 'as well as it possibly can' is that it should predict any quantity that does not depend on whether we match $\alpha = g$ at the top or at the bottom, but it will not directly predict any quantity that would depend on this choice.

To see how this translates to the ϵ^2 criterion above, let us consider a slightly simpler setting where we have only two freely falling observers. Again, we will study such observers inside a small box of spacetime of dimensions $\delta x = c\epsilon$, $\delta t = \epsilon$. Let's assume that they are located on opposite sides of the box, separated by a distance δx.

In general, we have seen that two freely falling observers will accelerate relative to each other. Let us write a Taylor's series expansion for this relative acceleration a as a function of the separation δx. In general, we have

$$a(\delta x) = a_0 + a_1 \delta x + O(\delta x^2). \tag{7.3}$$

But, we know that this acceleration vanishes in the limit $\delta x \to 0$ where the two observers have zero separation. As a result, $a_0 = 0$ and for small δx we have the approximation $a \approx a_1 \delta x$.

Now, if the spacetime were flat, there would be no relative acceleration and, if we start them at rest relative to each other, their relative velocity would always remain zero. This is an example of an error we would make if we tried to use the equivalence principle in too strong of a fashion. What is the correct answer? Well, the relative acceleration is $a_1 \delta x = a_1 c\epsilon$ and they accelerate away from each other for a time (within our box) $\delta t = \epsilon$. As a result, they attain a relative velocity of $v = a_1 c\epsilon^2$. But since our flat space model would have predicted $v = 0$, the error in v/c is also $a_1\epsilon^2$! Other examples turn out to work in much the same way, and this is why Einstein made the proposal above.

- To summarize: what we have found is that *locally* a freely falling reference frame is almost the same as an inertial frame. If we think about a freely falling reference frame as being *exactly* like an inertial frame, then we make a small error in computing things. The fractional error is proportional to ϵ^2, where ϵ is the size of the spacetime region needed to make the measurement.

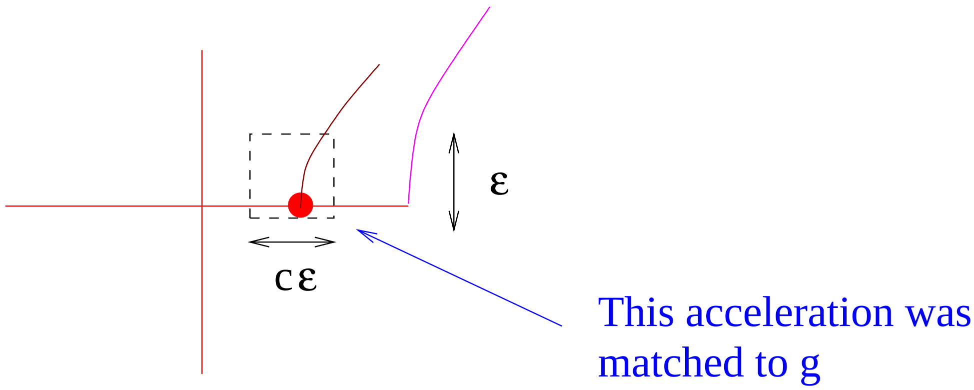
The factor of proportionality is called R after the mathematician Riemann, about whom we will say more in the next chapter. Note that R is *not* a radius. Since the error in an angle θ is $R\epsilon^2$, R has dimensions of $(length)^{-2}$.

7.4 Going beyond locality

Einstein: Chapters XX-XXII

In section (7.3.1) we talked about the fact that *locally* a freely falling frame in a gravitational field acts like an inertial frame does in the absence of gravity. However, we saw that freely falling frames and inertial frames are not exactly the same if they are compared over any bit of spacetime of finite size. No matter how small of a region of spacetime we consider, we always make some error if we interpret a freely falling frame as an inertial frame. So, since any real experiment requires a finite piece of spacetime, how can our local principle be useful in practice?

The answer lies in the fact that we were able to quantify the error that we make by pretending that a freely falling frame *is* an inertial frame. We found that if we consider a bit of spacetime of size ϵ, then the error in dimensionless quantities like angle or velocity measurements is ϵ^2. Note that *ratios* of lengths (L_1/L_2) or times (T_1/T_2) are also dimensionless. In fact, an angle is nothing but a ratio of an arc length to a radius ($\theta = s/r$)! So, this principle should also apply to *ratios* of lengths and/or times:

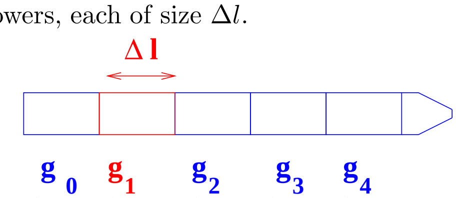


$$\delta \frac{T_1}{T_2} \propto \epsilon^2, \tag{7.4}$$

where δ denotes the error.

Let me pause here to say that the conceptual setup with which we have surrounded equation (7.4) is much like what we find in calculus. In calculus, we learned that *locally* any curve was essentially the same as a straight line. Of course, over a region of finite size, curves are generally not straight lines. However, the error we make by pretending a curve is straight over a small finite region is small. Calculus is the art of carefully controlling this error to build up curves out of lots of tiny pieces of straight lines[3]. Similarly, *the main idea of general relativity is to build up a gravitational field out of lots of tiny pieces of inertial frames.*

Suppose, for example, that we wish to compare clocks at the top and bottom of a tall tower. We begin by breaking up this tower into a larger number of short towers, each of size Δl.



If the tower is tall enough, the gravitational field may not be the same at the top and bottom – the top might be enough higher up that the gravitational field is measurably weaker. So, in general each little tower (0,1,2...) will have a different value of the gravitational field g (g_0, g_1, g_2....). If l is the distance of any given tower from the bottom, we might describe this by a function $g(l)$.

[3]In fact, in calculus we also have a result much like (7.4). When we match a straight line to a curve at $x = x_0$ we get the slope right, but miss the curvature. Thus, the straight line deviates from the curve by an amount proportional to $(x - x_0)^2$, a quadratic expression.

7.4.1 A Tiny Tower

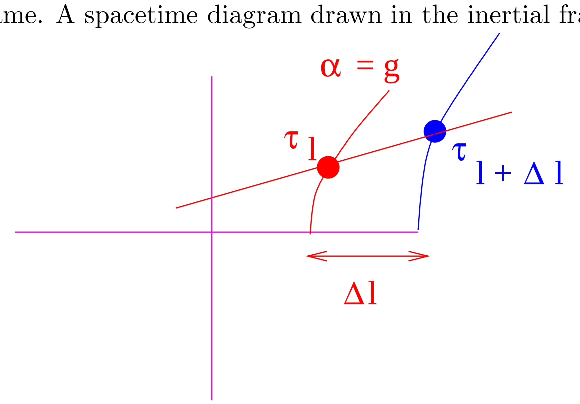
Let's compare the rates at which clocks run at the top and bottom of one of these tiny towers. We will try to do this by using the fact that a freely falling frame is much like an inertial frame. Of course, we will have to keep track of the error we make by doing this.

Recall that, in any accelerating rocket, the front and back actually do agree about simultaneity. As a result, all of our clocks in the towers will also agree about simultaneity. Thus, we can summarize all of the interesting information in a 'rate function' $\rho(l)$ which tells us how fast the clock at position l runs compared to the clock at position zero:

$$\rho(l) = \frac{\Delta\tau_l}{\Delta\tau_0}. \tag{7.5}$$

We wish to consider a gravitational field that does not change with time, so that ρ is indeed a function only of l and not of t.

So, let us model our tiny tower as a rigid rocket accelerating through an inertial frame. A spacetime diagram drawn in the inertial frame is shown below.



Now, the tiny tower had some acceleration g relative to freely falling frames. Let us suppose that we match this to the proper acceleration α of the *back* of the rocket. In this case, the back of the rocket will follow a worldline that remains a constant proper distance $d = c^2/\alpha$ from some fixed event.

Note that the top of the rocket remains a constant distance $d + \Delta l$ from this event. As a result, the top of the rocket has a proper acceleration $\alpha_{top} = \frac{c^2}{d+\Delta l}$. As we have learned, this means that the clocks at the top and bottom run at different rates:

$$\frac{\Delta\tau_{top}}{\Delta\tau_{bottom}} = \frac{\alpha_{bottom}}{\alpha_{top}} = \frac{d+\Delta l}{d} = 1 + \frac{\Delta l}{d}. \tag{7.6}$$

In terms of our rate function, this is just

$$\frac{\rho(l+\Delta l)}{\rho(l)} = \frac{\rho(l)+\Delta\rho}{\rho(l)} = 1 + \frac{\Delta\rho}{\rho(l)}. \tag{7.7}$$

Thus, we have

$$\frac{\Delta\rho}{\rho(l)} = \frac{\Delta l}{d} = \frac{\alpha \Delta l}{c^2}. \tag{7.8}$$

Now, how much of an error would we make if we use this expression for our tiny tower in the gravitational field? Well, the above *is* in fact a fractional change in a time measurement. So, the error must be of size Δl^2. So, for our tower case, we have

$$\frac{\Delta\rho}{\rho(l)} = \frac{g(l)\Delta l}{c^2} + k(\Delta l)^2 \tag{7.9}$$

for some number k. Here, we have replaced α with g, since we matched the acceleration $g(l)$ of our tower (relative to freely falling frames) to the proper acceleration α.

Actually, we might have figured out the error directly from expression (7.8) above. Recall that, after all, the error can be seen in the fact that the acceleration does not change in the same way from the bottom of the tower to the top of the tower as it did from the bottom of the rocket to the top of the rocket. The equivalence principle directly predicts only quantities that are independent of such matching details. So what would is the difference between these two options? Well, the difference in the accelerations is just $\Delta\alpha \approx \frac{d\alpha}{dl}\Delta l$. Note that α is already multiplied by Δl in the expression above. This means that, if we were to change the value if α by $\frac{d\alpha}{dl}\Delta l$, we would indeed create a term of the form $k(\Delta l)^2$! So, we see again that this term really does capture well all of the errors we might possibly make in matching a freely falling frame to an inertial frame[4].

Let us write our relation above as

$$\frac{\Delta\rho}{\Delta l} = (\frac{g}{c^2} + k\Delta l)\rho(l). \tag{7.10}$$

As in calculus, we wish to consider the limit as $\Delta l \to 0$. In this case, the left hand side becomes just the derivative $\frac{d\rho}{dl}$. On the right hand side, the first term does not depend on Δl at all, while the second term vanishes in this limit. Thus, we obtain

$$\frac{d\rho}{dl} = \frac{g(l)\rho(l)}{c^2}. \tag{7.11}$$

Note that the term containing k (which encodes our error) has disappeared entirely. This means that the relation (7.11) is *exact* and contains no error at all!

[4]A more thorough argument is to compare the clock we matched to an arbitrary worldline which is at rest a distance Δl away at $t = 0$. One can show that the result (7.9) holds for *any* such worldline and is thus independent of all other details of the matching. This follows directly from the fact that the line of simultaneity at τ_1 slopes upward to the right. Ask me for more details if you are curious.

We have managed to use our local matching of freely falling and inertial frames to make an exact statement not directly about $\rho(l)$, but about the derivative $\frac{d\rho}{dl}$.

7.4.2 The tall tower

Of course, we have still not answered the question about how the clocks actually run at different heights in the tower. To do so, we need to solve the equation (7.11) for $\rho(l)$. We can do this by multiplying both sides by dl and integrating:

$$\int_{\rho(0)}^{\rho} \frac{d\rho}{\rho} = \int_0^l \frac{g(l)}{c^2} dl. \tag{7.12}$$

Now, looking at our definition above we find that $\rho_0 = 1$. Thus, we have

$$\int_{\rho(0)}^{\rho} \frac{d\rho}{\rho} = \ln\rho - \ln 1 = \ln\rho \tag{7.13}$$

and so

$$\ln\rho = \int_0^l \frac{g(l)}{c^2} dl, \tag{7.14}$$

or,

$$\frac{\Delta\tau_l}{\Delta\tau_0} = \rho(l) = \exp(\int_0^l \frac{g(l)}{c^2} dl). \tag{7.15}$$

$\star\star$ Expression (7.15) is the exact relation relating clocks at different heights l in a gravitational field. One important property of this formula is that the factor inside the exponential is always positive. As a result, we find that clocks higher up in a gravitational field always run faster, regardless of whether the gravitational field is weaker or stronger higher up!

Note that, due to the properties of exponential functions, we can also write this as:

$$\frac{\Delta\tau_b}{\Delta\tau_a} = \rho(l) = \exp(\int_a^b \frac{g(l)}{c^2} dl). \tag{7.16}$$

7.4.3 Gravitational time dilation near the earth

The effect described in equation (7.15) is known as gravitational time dilation. There are a couple of interesting special cases of this effect that are worth investigating in detail. The first is a *uniform* gravitational field in which $g(l)$ is constant. Recall that this is not in fact the same as a rigid rocket accelerating through an inertial frame, as the acceleration is actually different at the top of the rigid rocket than at the bottom.

Still, in a uniform gravitational field with $g(l) = g$ the integral in (7.15) is easy to do and we get just:

$$\frac{\Delta\tau_l}{\Delta\tau_0} = e^{gl/c^2}. \tag{7.17}$$

In this case, the difference in clock rates grows *exponentially* with distance.

The other interesting case to consider is something that describes (to a good approximation) the gravitational field near the earth. We have seen that Newton's law of gravity is a pretty good description of gravity near the earth, so we should be able to use the Newtonian form of the gravitational field:

$$g = \frac{m_E G}{r^2}, \tag{7.18}$$

where r is the distance from the center of the earth. This means that we can use dr in place of dl in (7.15). Let us refer to the radius of the earth as r_0. For this case, it is convenient to compare the rate at which some clock runs at radius r to the rate at which a clock runs on the earth's surface (i.e., at $r = r_0$). Since $\int_{r_1}^{r_2} r^{-2} dr = r_1^{-1} - r_2^{-1}$, we have

$$\frac{\Delta\tau(r)}{\Delta\tau(r_0)} = \exp(\int_{r_0}^{r} \frac{m_E G}{c^2 r^2} dr) = \exp\left[\frac{m_E G}{c^2}\left(\frac{1}{r_0} - \frac{1}{r}\right)\right]. \tag{7.19}$$

Here, it is interesting to note that the r dependence drops out as $r \to \infty$, so that the gravitational time dilation factor between the earth's surface (at r_0) and infinity is actually finite. The result is

$$\frac{\Delta\tau(\infty)}{\Delta\tau(r_0)} = e^{\frac{m_E G}{r_0 c^2}}. \tag{7.20}$$

So, time is passing more slowly for us here on earth than it would be if we were living far out in space..... By how much? Well, we just need to put in some numbers for the earth. We have

$$\begin{aligned} m_E &= 6 \times 10^{24} kg, \\ G &= 6 \times 10^{-11} Nm^2/kg^1, \\ r_0 &= 6 \times 10^6 m. \end{aligned} \tag{7.21}$$

Putting all of this into the above formula gives a factor of about $e^{\frac{2}{3} \times 10^{-10}}$. Now, how big is this? Well, here it is useful to use the Taylor series expansion $e^x = 1 + x + \textit{small corrections}$ for small x. We then have

$$\frac{\Delta\tau(\infty)}{\Delta\tau(r_0)} \approx 1 + \frac{2}{3} \times 10^{-10}. \tag{7.22}$$

This means that time passes more slowly for us than it does far away by roughly one part in 10^{10}, or, one part in ten billion! This is an incredibly small amount

– one that can easily go unnoticed. However, as mentioned earlier, the national bureau of standards was in fact able to measure this back in the 1960's, by comparing very accurate clocks in Washington, D.C. with very accurate clocks in Denver! Their results were of just the right size to verify the prediction above.

In fact, there is an even more precise version of this experiment that is going on right now – constantly verifying Einstein's prediction every day! It is called the "Global Positioning System" (GPS). Perhaps you have heard of it?

7.4.4 The Global Positioning System

The Global Positioning System is a setup that allows anyone, with the aid of a small device, to tell exactly where they are on the earth's surface. It is made up of a number of satellites in precise, well-known orbits around the earth. Each of these satellites contains a very precise clock and a microwave transmitter. Each time the clock 'ticks' (millions of times every second!) it sends out a microwave pulse which is 'stamped' with the time and the ID of that particular satellite.

A hand-held GPS locator then receives these pulses. Because it is closer to some satellites than to others, the pulses it receives take less time to reach it from some satellites than from others. The result is that the pulses it receives at a given instant are not all stamped with the same time. The locator then uses the differences in these time-stamps to figure out which satellites it is closest to, and by how much. Since it knows the orbits of the satellites very precisely, this tells the device exactly where it itself it located. This technology allows the device to pinpoint its location on the earth's surface to within a one meter circle.

To achieve this accuracy, the clocks in the satellites must be very precise, and the time stamps must be very accurate. In particular, they must be much more accurate than one part in ten billion. If they were off by that much, then every second the time stamps would become off by 10^{-10} seconds. But, in this time, microwaves (or light) travel a distance $(3 \times 10^8 m/s)(10^{-10} sec) = 3 \times 10^{-2} m = 3cm$ and the GPS locator would think it was 'drifting away' at $3cm/sec$. While this is not very fast, it would add up over time. This drift rate is $72m/hr$, which would already spoil the accuracy of the GPS system. Over long times, the distance becomes even greater. The drift rate can also be expressed as $1.5km/day$ or $500km/year$. So, after one year, a GPS device in Syracuse, NY might think that it is in Philadelphia!

By the way, since the GPS requires this incredible precision, you might ask if it can measure the effects of regular speed-dependent special relativity time dilation as well (since the satellites are in orbit and are therefore 'moving.') The answer is that it can. In fact, for the particular satellites used in the GPS system, these speed-dependent effects turn out to be of a comparable size to the gravitational time dilation effect. Note that these effects actually go in opposite directions: the gravity effect makes the higher (satellite) clock run *fast* while the special relativity effect makes the faster (satellite) clock run *slow*.

The numbers are fun to work out[5]. Here, I will just report the results. Which effect is larger turns out to depend on the particular orbit. Low orbits (like that of the space shuttle) are higher speed, so in this case the special relativity effect dominates and the orbiting clocks run more slowly than on the earth's surface. High orbits (like that of the GPS satellites) are lower speed, so the gravity effect wins and their clocks run faster than clocks on the earth's surface. For the case of GPS clocks, the special relativity effect means that the amount of the actual time dilation is less than the purely gravitational effect by about a factor of two.

7.5 The moral of the story

OK, we have done a nice calculation and we were able to figure out how clocks run at different heights in a gravitational field. We have also seen how important this is for the running of things like GPS. But, what does all of this mean? And, why is this often considered a new subject (called 'General Relativity'), different from our old friend Special Relativity?

7.5.1 Local frames vs. Global frames

Let us briefly retrace our logic. While thinking about various frames of reference in a gravitational field, we discovered that freely falling reference frames are useful. In fact, they are really the most useful frames of reference, as they are similar to inertial frames. This fact is summarized by the equivalence principle which says "freely falling frames are *locally* equivalent to inertial frames."

The concept of these things being *locally* equivalent is a subtle one, so let me remind you what it means. The idea is that freely falling reference frames are indistinguishable from inertial reference frames so long as we are only allowed to perform experiments in a tiny region of spacetime. More technically, suppose that we make then mistake of pretending that a freely falling frame actually is an inertial frame of special relativity, but that we limit ourselves to measurements within a region of spacetime of size ϵ. When we then go and predict the results of experiments, we will make small errors in, say, the position of objects. However, these errors will be very small when ϵ is small; in fact, the percent error will go to zero like ϵ^2.

The same sort of thing happens in calculus. There, the corresponding statement is that a curved line is *locally* equivalent to a straight line.

Anyway, the important point is that we would make an error by pretending that freely falling frames are exactly the same as inertial frames. Physicists say that the two are locally equivalent, but are not "globally" equivalent. The term 'global' (from globe, whole, etc.) is the opposite of local and refers to the frame everywhere (as opposed to just in a small region).

So, if freely falling frames are not globally inertial frames, then where are the inertial frames? They cannot be the frames of reference that are attached to

[5] And we will learn more about how to do so in the next chapter.

the earth's surface. After all, if a frame is globally like an inertial frame then it must also be like an inertial frame locally. However, frames tied to the surface of the earth are locally like uniformly accelerated frames, not inertial frames.

But, there are really not any other frames left to consider. To match an inertial frame locally requires free fall, but that will not let us match globally. We are left with the conclusion that:

> In a generic gravitational field, there is no such thing as a global inertial frame.

What can I mean by this? Well, one can take various perspectives on this, but the bottom line is that we (following Einstein) merely assumed that the speed of light was constant in all (globally) inertial frames of reference. However, no such reference frame will exist in a generic gravitational field[6].

And what if we retreat to Newton's first law, asking about the behavior of objects on which no forces act? The trouble is that, as we have discussed, to identify an inertial frame in this way we would need to first identify an object on which no forces act. But, which object is this? Recall that any freely falling object seems to pass the 'no forces' tests as well (or better than!) an object sitting on the earth!

However, if freely falling objects are indeed free of force, then Newton's first law tells us that they do not accelerate relative to each other ..... in gross contradiction with experiment! (To see this, consider a freely falling rock dropped above the north pole and also one dropped above the south pole. These clearly accelerate toward one another.)

This strongly suggests that global inertial frames do not exist and that we should therefore abandon the concept and move on. In its place, we will now make use of *local* inertial frames, a.k.a. freely falling frames. It is just this change that marks the transition from 'special' to 'general' relativity. Special relativity is just the special case in which global inertial frames exist.

Actually, there is another reason why the study of gravity is known as "General Relativity." The point is that in special relativity (actually, even before) we noticed that the concept of velocity is intrinsically a relative one. That is to say, it does not make sense to talk about whether an object is moving or at rest, but only whether it is moving or at rest relative to some other object. However, we did have an absolute notion of acceleration: an object could be said to be accelerating without stating explicitly what frame was being used to make this statement. The result would be the same no matter what inertial frame was used.

However, now even the concept of acceleration becomes relative in a certain sense. Suppose that you are in a rocket in deep space and that you cannot look outside to see if the rockets are turned on. You drop an object and it falls. Are you accelerating, or are you in some monster gravitational field? There is no right answer to this question as the two are identical. In this sense, the concept

[6] Of course, the *locally measured* speed of light is always c in a freely falling frame.

of acceleration is now relative as well – it is equivalent to being in a gravitational field.

While this point is related to why the study of gravity historically acquired the name "General Relativity," it is not clear that this is an especially useful way to think about things. In particular, I want to stress that one can still measure one's proper acceleration as the acceleration relative to a nearby (i.e., local!) freely falling frame.

$\star\star$ Thus, there *is* an absolute distinction between freely falling and not freely falling. Whether you wish to identify these terms with non-accelerating and accelerating is just a question of semantics – though most modern relativists find it convenient to do so. As a result, I will use a language in which acceleration is *not* a relative concept but in which it implicitly means "acceleration measured locally with respect to freely falling frames."

7.5.2 And what about the speed of light?

There is a question that you probably wanted to ask a few paragraphs back, but then I went on to other things.... I said that in a general gravitational field there are no frames of reference in which light rays always travel in straight lines at constant speed. So, after all of our struggles, have we finally thrown out the constancy of the speed of light?

No, not completely. There is one very important statement left. Suppose that we measure the speed of light at some event (E) in a frame of reference that falls freely at event E. Then, near event E things in this frame work just like they do in inertial frames – so, light moves at speed c and in a straight line. Said in our new language:

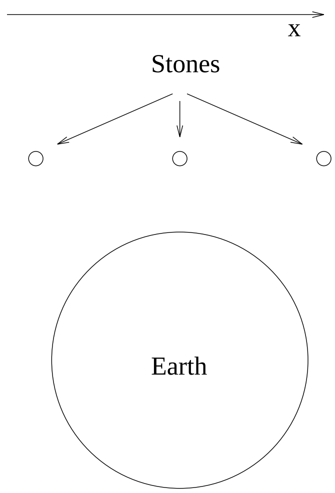
As measured locally in a freely falling frame, light always moves in straight lines at speed c.

7.6 Homework Problems

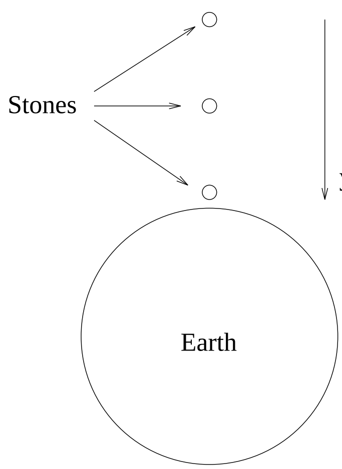
7-1. This first problem is an exercise in thinking about things from the perspective of a freely falling object. Remember that Newton's law of gravity tells us that (i) objects near the earth are attracted to the *center* of the earth and (ii) objects closer to the earth are attracted more strongly than are objects farther away. **You should include both of these effects as you work this problem.**

(a) Suppose that three small stones are released from high above the earth as shown below. Sketch their worldlines on a spacetime diagram drawn in the reference frame of the middle stone. Use the x direction (shown below) for the spatial direction of your diagram. (That is, sketch the motion in the x, t plane.) **[Hint: In the case shown below, the three stones are nearly the same distance**

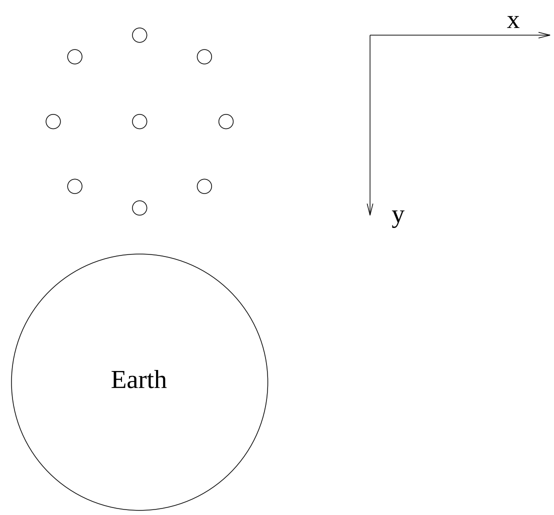
from the center of the earth. As a result, their accelerations have nearly the same magnitude. However, the direction of the acceleration varies from stone to stone.] How would the spacetime diagram be different if the stones have an initial velocity relative to the earth? Take this initial velocity to be the same for all stones.



(b) Suppose that three small stones are released from high above the earth as shown below. Sketch their worldlines on a spacetime diagram drawn in the reference frame of the middle stone. Use the y direction (shown below) for the space direction of your diagram. (That is, sketch the motion in the y, t plane.) How would the diagram be different if the stones have an initial velocity relative to the earth? **In all cases, assume that the stones pass smoothly through the earth's surface – don't worry about them hitting the earth.**



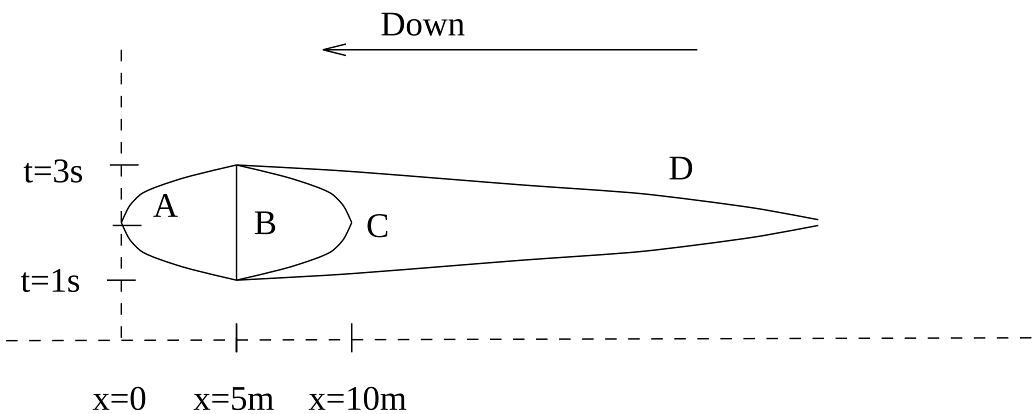
(c) Suppose that a set of small stones are released from high above the earth as shown below. Describe what happens to the configuration of stones as time passes, and describe the relative acceleration between the outer stones and the central stone. Are the stones close enough together that we can describe them by 'local measurements' (in the sense of the equivalence principle) with respect to the central stone? That is, are they close enough together that these freely falling objects are indistinguishable from inertial objects in a global inertial frame?



7-2. The earth falls freely in a gravitational field that is largely due to the moon[7]. Moreover, the earth is surrounded by a sphere of liquid called the ocean. Based on your answers to problem 1, do you expect the ocean to be perfectly round? Draw a picture of its shape.

7-3. The spacetime diagram below is drawn in the reference frame of a small lab sitting on the earth (which experiences a Newtonian gravitational field of $g = 10m/s^2$). Which of the worldlines shown below has the greatest proper time? Explain why the answer you chose is correct.

[7]For the purposes of this problem, the moon is more important than the Sun. Can you see why? Hint: Think back to problem 1.



7-4. Due the effects of General Relativity (and also due to the fact that the earth is not completely round) the effective Newtonian gravitational field of the earth is not exactly GM/r^2. Other terms contribute, and these must be considered by the designers of the Global Positioning System, as they effect the rate at which the clocks run on the GPS satellites. A more accurate model of the earth's gravitational field is

$$g = GM/r^2 + a/r^3 + b/r^4 \tag{7.23}$$

in terms of the distance r away from the center of the earth. Here, a and b are certain constants having to do with the exact shape of the earth.

Using this model, compute the ratio between the rate of ticking of a clock (A) at distance r_A from the center and a clock (B) at a distance r_B from the center. Express the answer in terms of the constants G, M, r_A, r_B, a and b.

Note: This problem is really just a calculus problem. In section 7.4 we discussed equation (7.16) which tells us the relationship between the rates at which clocks run at different places l in a gravitational field. When this gravitational field is produced by a round object like the earth, it is natural to use the radial distance r from the center of the earth as our coordinate l. So, the problem above just consists of performing the corresponding integral for the specified function $g(r)$. **Note: In (7.16) we used l as the distance variable. In this problem I have used r. Just replace l in (7.16) by r to use that formula here.**

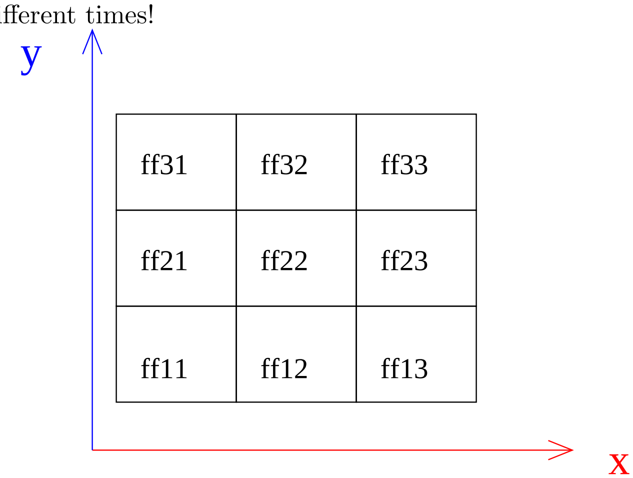
Chapter 8

General Relativity and Curved Spacetime

Read Einstein, ch. 23-29, Appendices 3-5

In chapter 7 we saw that we could use the equivalence principle to calculate the effects of a gravitational field over a finite distance by carefully patching together local inertial frames. If we are very, very careful, we can calculate the effects of any gravitational field in this way. However, this approach turns out to be a real mess.

Consider for example the case where the gravitational field changes with time. Then, it is not enough just to patch together local inertial frames at different positions. One must make a quilt of them at different places as well as at different times!



As you might guess, this process becomes even more complicated if we consider all 3+1 dimensions. One then finds that clocks at different locations in the gravitational field may not agree about simultaneity even if the gravitational field does not change with time.... but that is a story that we need not go into here[1].

What Einstein needed was a new way of looking at things – a new *language* in which to discuss gravity that would organize all of this into something relatively simple. Another way to say this is that he needed a better conception of what a gravitational field actually *is.* This next step was very hard for Albert. It took him several years to learn the appropriate mathematics and to make that mathematics into useful physics. Instead of going through all of the twists and turns in the development of the subject, I'll try to give you the rough, boiled down version, of how all it all works out.

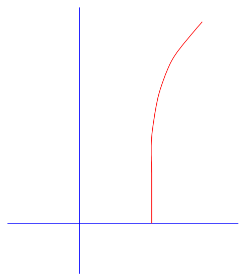
8.1 A return to geometry

You see, Einstein kept coming back to the idea that freely falling observers are like inertial observers – or at least as close as we can get. Recall that, in the presence of a general gravitational field, there really are no global inertial frames. When we talked about our 'error' in thinking of a freely falling frame as inertial, it is not the case that there is a better frame which is more inertial than is a freely falling frame. Instead, when gravity is present there are simply no frames of reference that act precisely in the way that global inertial frames act.

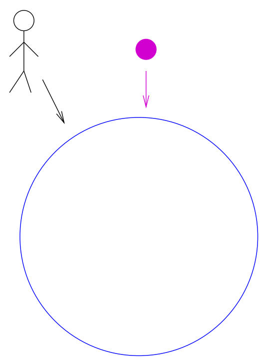
Anyway, Einstein focussed on the fact that freely falling frames are locally the same as inertial frames. However, he knew that things were tricky for measurements across a finite distance. Consider, for example, the reference frame of a freely falling person. Suppose that this person holds out a rock and releases it. The rock is then also a freely falling object, and the rock is initially at rest with respect to the person.

However, the rock need not remain exactly at rest with respect to the person. Suppose, for example, that the rock is released from slightly higher up in the gravitational field. Then, Newton would have said that the gravitational field was weaker higher up, so that the person should accelerate toward the earth faster than does the rock. This means that there is a relative acceleration between the person and the rock, and that the person finds the rock to accelerate away! A spacetime diagram in the person's reference frame looks like this:

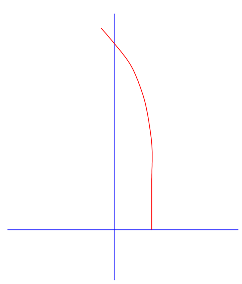
[1]If you're interested, you might look up the difference between 'stationary' and 'static' spacetimes in a more technical book on General Relativity.



Suppose, on the other hand, that the rock is released to the person's side. Then, Newton would say that both person and rock accelerate toward the center of the earth. However, this is not in quite the same direction for the person as for the rock:



So, again there is a relative acceleration. This time, however, the person finds the rock to accelerate toward her. So, she would draw a spacetime diagram for this experiment as follows:



The issue is that we would like to think of the freely falling worldlines as inertial worldlines. That is, we would like to think of them as being 'straight lines in spacetime.' However, we see that we are forced to draw them on a spacetime diagram as curved. Now, we can straighten out any one of them by using the reference frame of an observer moving along that worldline. However, this makes the other freely falling worldlines appear curved. How are we to understand this?

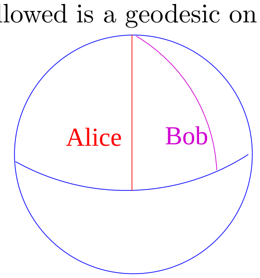
8.1.1 Straight Lines in Curved Space

Eventually Einstein found a useful analogy with something that at first sight appears quite different – a curved surface. The idea is captured by the question "What is a straight line on a curved surface?"

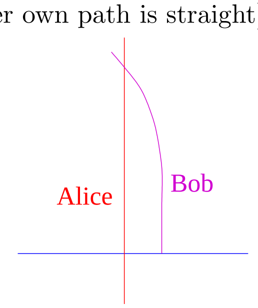
To avoid language games, mathematicians made up a new word for this idea: "geodesic." A geodesic can be thought of as the "straightest possible line on a curved surface." More precisely, we can *define* a geodesic as a line of minimal distance – the shortest line between two points[2]. The idea is that we can *define* a straight line to be the shortest line between two points.

Actually, there is another definition of geodesic that is even better, but requires more mathematical machinery to state precisely. Intuitively, it just captures the idea that the geodesic is 'straight.' It tells us that a geodesic is the path on a curved surface that would be traveled, for example, by an ant (or a person) walking on the surface who always walks straight ahead and does not turn to the right or left.

As an example, suppose you stand on the equator of the earth, face north, and then walk forward. Where do you go? If you walk far enough (over the ocean, etc.) you will eventually arrive at the north pole. The path that you have followed is a geodesic on the sphere.



Note that this is true no matter where you start on the equator. So, suppose there are in fact *two* people walking from the equator to the north pole, Alice and Bob. As you can see, Alice and Bob end up moving *toward* each other. So, if we drew a diagram of this process using Alice's frame of reference (so that her own path is straight), it would look like this:

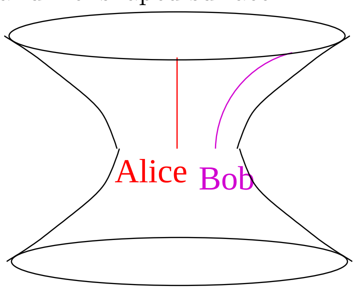


[2]Technically a geodesic is a line of locally minimal distance, meaning that the line is shorter than any nearby line.

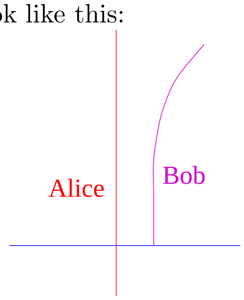
By the way, the above picture is not supposed to be a spacetime diagram. It is simply supposed to be a map of part of the (two dimensional) earth's surface, on which both paths have been drawn. This particular map is drawn in such a way that Alice's path appears as a straight line. As you probably know from looking at maps of the earth's surface, no flat map will be an accurate description *globally*, over the whole earth. There will always be some distortion somewhere. However, a flat map is perfectly fine *locally*, say in a region the size of the city of Syracuse (if we ignore the hills).

Now, does this look or sound at all familiar?

What if we think about a similar experiment involving Alice and Bob walking on a funnel-shaped surface:



In this case they begin to drift apart as they walk so that Alice's map would look like this:

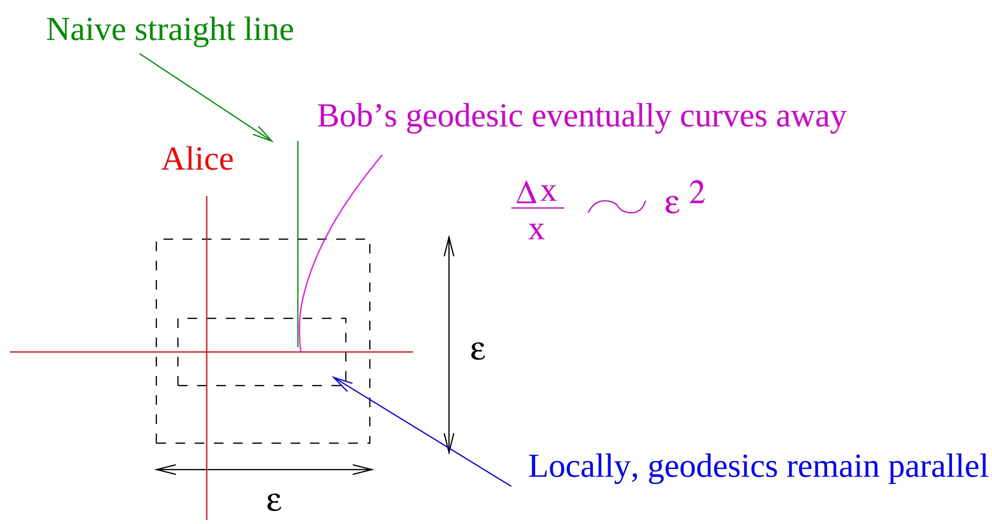


So, we see that straight lines (geodesics) on a curved surface act much like freely falling worldlines in a gravitational field. It is useful to think through this analogy at one more level: Consider two people standing on the surface of the earth. We know that these two people remain the same distance apart as time passes. Why do they do so? Because the earth itself holds them apart and prevents gravity from bringing them together. The earth exerts a force on each person, keeping them from falling freely.

Now, what is the analogy in terms of Alice and Bob's walk across the sphere or the funnel? Suppose that Alice and Bob do not simply walk independently, but that they are actually connected by a stiff bar. This bar will force them to always remain the same distance apart as they walk toward the north pole. The point is that, in doing so, Alice and Bob will be unable to follow their natural (geodesic) paths. As a result, Alice and Bob will each feel some push or pull from the bar that keeps them a constant distance apart. This is much like our two people standing on the earth who each feel the earth pushing on their feet to hold them in place.

8.1.2 Curved Surfaces are Locally Flat

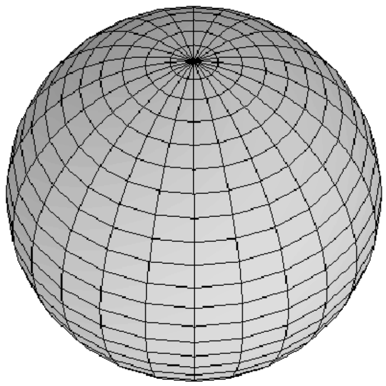
Note that straight lines (geodesics) on a curved surface act much like freely falling worldlines in a gravitational field. In particular, exactly the same problems arise in trying to draw a flat map of a curved surface as in trying to represent a freely falling frame as an inertial frame. A quick overview of the errors made in trying to draw a flat map of a curved surface are shown below:



We see that something like the equivalence principle holds for curved surfaces: flat maps are very accurate in small regions, but not over large ones.

In fact, we know that we can in fact build up a curved surface from a bunch of flat ones. One example of this happens in an atlas. An atlas of the earth contains many flat maps of small areas of the earth's surface (the size of states, say). Each map is quite accurate and together they describe the round earth, even though a single flat map could not possibly describe the earth accurately.

Computer graphics people do much the same thing all of the time. They draw little flat surfaces and stick them together to make a curved surface.



This is much like the usual calculus trick of building up a curved line from little pieces of straight lines. In the present context with more than one dimension, this process has the technical name of "differential geometry."

8.1.3 From curved space to curved spacetime

The point is that this process of building a curved surface from flat ones is just exactly what we want to do with gravity! We want to build up the gravitational field out of little pieces of "flat" inertial frames. Thus, we might say that gravity is the curvature of spacetime. This gives us the new language that Einstein was looking for:

1) (Global) Inertial Frames ⇔ Minkowskian Geometry ⇔ Flat Spacetime: We can draw it on our flat paper or chalk board and geodesics behave like straight lines.

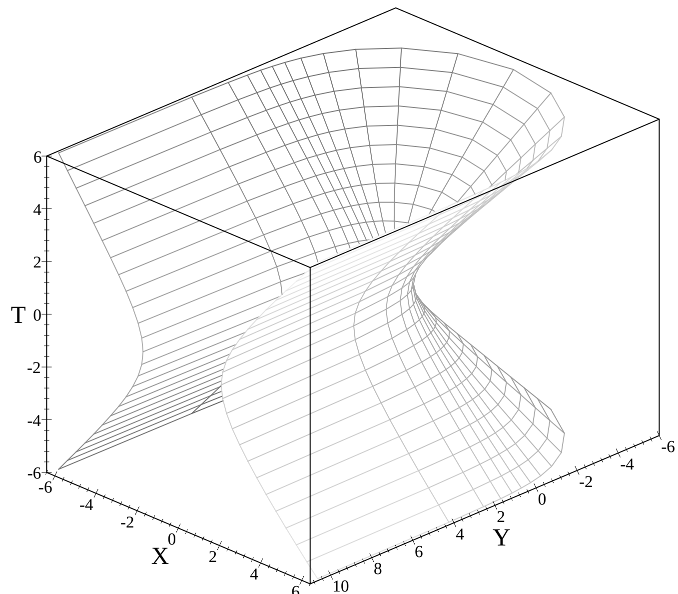
2) Worldlines of Freely Falling Observers ⇔ Straight lines in Spacetime

3) Gravity ⇔ The Curvature of Spacetime

Similarly, we might refer to the relation between a worldline and a line of simultaneity as the two lines being at a "right angle *in spacetime*[3]." It is often nice to use the more technical term "orthogonal" for this relationship.

By the way, the examples (spheres, funnels, etc.) that we have discussed so far are all curved *spaces*. A curved spacetime is much the same concept. However, we can't really put a curved spacetime in our 3-D Euclidean space. This is because the geometry of spacetime is fundamentally Minkowskian, and not Euclidean. Remember the minus sign in the interval? Anyway, what we can do is to once again think about a spacetime diagram for 2+1 Minkowski space – time will run straight up, and the two space directions (x and y) will run to the sides. Light rays will move at 45 degree angles to the (vertical) t-axis as usual. With this understanding, we can draw a (1+1) curved spacetime inside this 2+1 spacetime diagram. An example is shown below:

[3]As opposed to a right angle on a spacetime diagram drawn in a given frame.

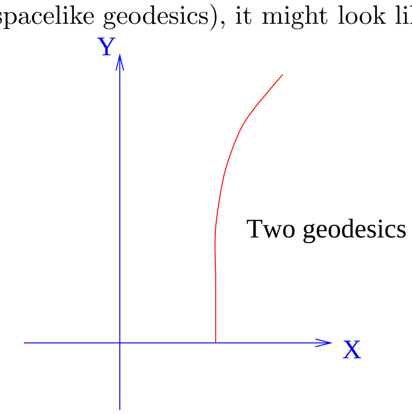


Note that one can move along the surface in either a timelike manner (going up the surface) or a spacelike manner (going across the surface), so that this surface does indeed represent a (1+1) space*time*. The picture above turns out to represent a particular kind of gravitational field that we will be discussing more in a few weeks. To see the similarity to the gravitational field around the earth, think about two freely falling worldlines (a.k.a. "geodesics," the straightest possible lines) that begin near the middle of the diagram and start out moving straight upward. Suppose for simplicity that one geodesic is on one side of the fold while the second is on the other side. You will see that the two worldlines separate, just as two freely falling objects do at different heights in the earth's gravitational field. Thus, if we drew a two-dimensional map of this curved spacetime using the reference frame of one of these observers, the results would be just like the spacetime diagram we drew for freely falling stones at different heights! This is a concrete picture of what it means to say that gravity is the curvature of spacetime.

Well, there is one more subtlety that we should mention...... it is important to realize that the extra dimension we used to draw the picture above was just a crutch that we needed because we think best in flat spaces. One can in fact talk about curved spacetimes without thinking about a "bigger space" that contains points "outside the spacetime." This minimalist view is generally a good idea, as we will discuss more in the sections below.

8.2 More on Curved Space

Let us remember that the spacetime in which we live is fundamentally four (=3+1) dimensional and ask if this will cause any new wrinkles in our story. It turns out to create only a few. The point is that curvature is fundamentally associated with two-dimensional surfaces. Roughly speaking, the curvature of a four-dimensional spacetime (labelled by x, y, z, t) can be described in terms of xt curvature, yt curvature, etc. associated with two-dimensional bits of the spacetime. However, this is relativity, in which space and time act pretty much the same. So, if there is xt, yt, and zt curvature, there should also be xy, yz, and xz curvature! This means that the curvature can show up even if we consider only straight lines in space (determined, for example, by stretching out a string) in addition to the effects on the motion of objects that we have already discussed. For example, if we draw a picture showing spacelike straight lines (spacelike geodesics), it might look like this:



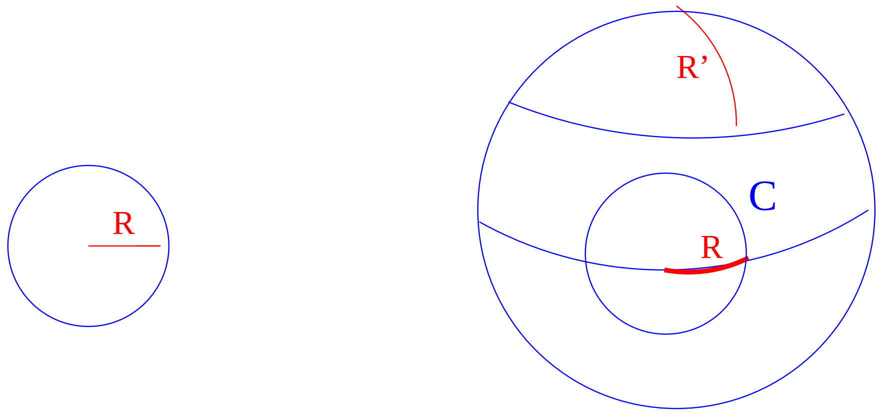
So, curved space is as much a part of gravity as is curved spacetime. This is nice, as curved spaces are easier to visualize. Let us now take a moment to explore these in more depth and build some intuition about curvature in general.

Curved spaces have a number of fun properties. Some of my favorites are:

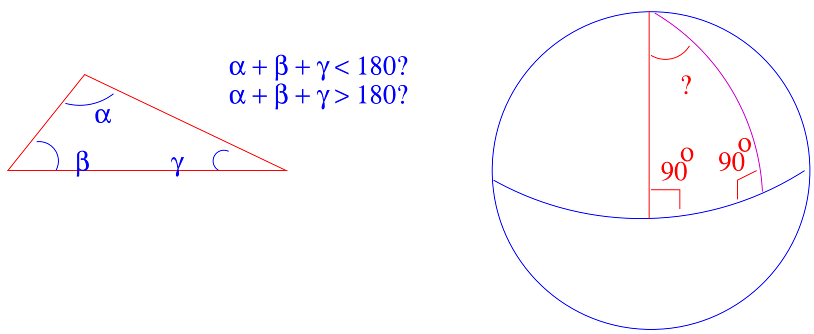
$C \neq 2\pi R$: The circumference of a circle is typically *not* 2π times its radius. Let us take an example: the equator is a circle on a sphere. What is it's center? We are only supposed to consider the two-dimensional surface of the sphere itself as the third dimension was just a crutch to let us visualize the curved two-dimensional surface. So this question is really 'what point on the sphere is equidistant from all points on the equator?' In fact, there are *two* answers: the north pole and the south pole. Either may be called the center of the sphere.

Now, how does the distance around the equator compare to the distance (measured along the sphere) from the north pole to the equator? The arc running from the north pole to the equator goes 1/4 of the way around the sphere. This is the radius of the equator in the relevant sense. Of course, the equator goes once around the sphere. Thus, its circumference is exactly four times its radius.

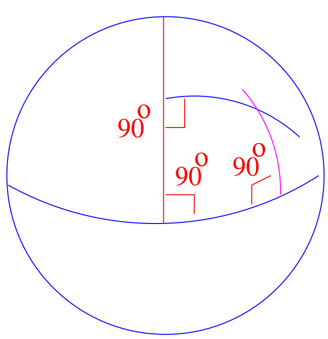
$A \neq \pi R^2$: The area of a circle is typically *not* π times the square of its radius. Again, the equator on the sphere makes a good example. With the radius defined as above, the area of this circle is much less than πR^2. I'll let you work out the math for yourself.



$\Sigma(angles) \neq 180^o$**:** The angles in a triangle do not in general add up to 180^o. An example on a sphere is shown below.



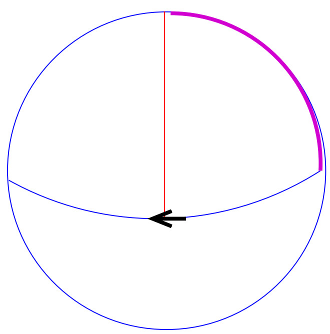
Squares do not close: A polygon with four sides of equal length and four right angles (a.k.a., a square) in general does not close.



Vectors (arrows) "parallel transported" around closed curves are rotated: This one is a bit more complicated to explain. Unfortunately, to describe this property as precisely as the ones above would require the introduction of more complicated mathematics. Nevertheless, the discussion below should provide you with both the flavor of the idea and an operational way to go about checking this property.

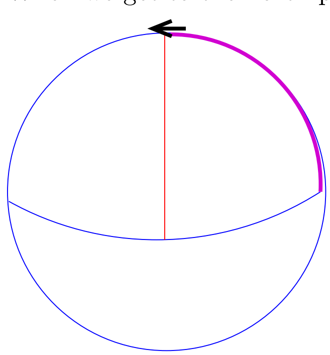
In a flat space (like the 3-D space that most people think we live in until they learn about relativity....), we know what it means to draw an arrow, and then to pick up this arrow and carry it around without turning it. The arrow can be carried around so that it always remains parallel to its original direction.

Now, on a curved surface, this is not possible. Suppose, for example, that we want to try to carry an arrow around a triangular path on the sphere much like the one that we discussed a few examples back. For concreteness, let's suppose that we start on the equator, with the arrow also pointing along the equator as shown below:

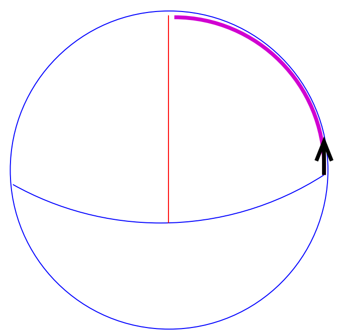


We now wish to carry this vector to the north pole, keeping it always pointing in the same direction as much as we can. Well, if we walk along the path shown, we are going in a straight line and never turning. So, since we start with the arrow pointing to our left, we should keep the arrow pointing to our left at all times. This is certainly what we would do in a flat space.

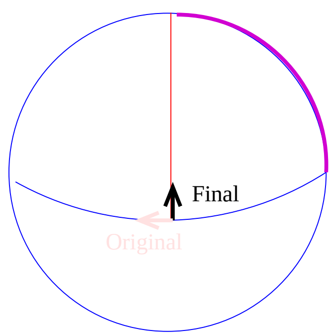
When we get to the north pole, the arrow looks like this:



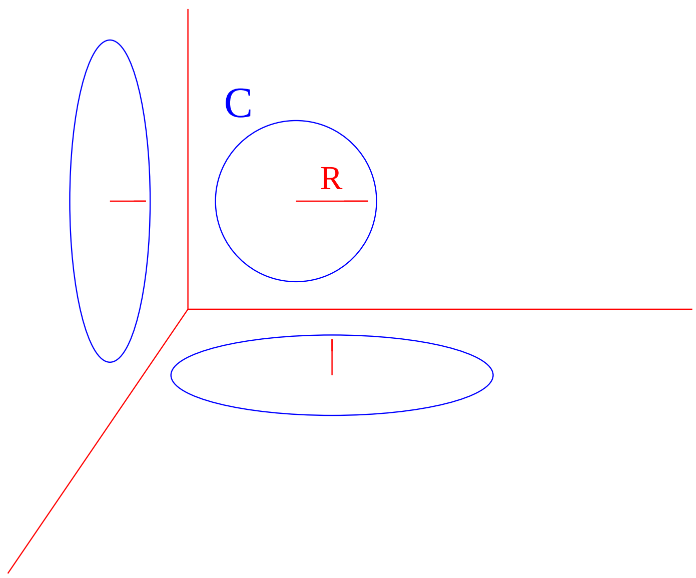
Now we want to turn and walk toward the equator along a different side of the triangle. We turn (say, to the right) , but we are trying to keep the arrow always pointing in the same direction. So the arrow should not turn with us. As a result, it points straight behind us. We carry it down to the equator so that it points straight behind us at every step:



Finally, we wish to bring the arrow back to where it started. We see that the arrow has rotated 90^o relative to the original direction:



All of these features will present in any space (say, a surface of simultaneity) in a curved spacetime. Now, since we identify the gravitational field with the curvature of spacetime then the above features must also be encoded in the gravitational field. But there is a lot of information in these features. In particular there are independent curvatures in the xy, yz, and xz planes that control, say, the ratio of circumference to radius of circles in these various planes.



But wait doesn't this seem to mean that the full spacetime curvature (gravitational field) contains a lot more information than just specifying an acceleration g at each point? After all, acceleration is related to how thing behave in time, but we have just realized that at least parts of the spacetime curvature are associated only with space. How are we to deal with this? For the answer, proceed on to the next section below.

8.3 Gravity and the Metric

Einstein: XXIII-XXVII

Let's recall where we are. A while back we discovered the equivalence principle: that locally a gravitational field is equivalent to an acceleration in special relativity. Another way of stating this is to say that, locally, a freely falling frame is equivalent to an inertial frame in special relativity. We noticed the parallel between this principle and the underlying ideas being calculus: that locally every curve is a straight line.

What we found in the current chapter is that this parallel with calculus is actually very direct. A global inertial frame describes a *flat* spacetime – one in which, for example, geodesics follow straight lines and do not accelerate relative to one another. A general spacetime with a gravitational field can be thought of as being curved. Just as a general curved line can be thought of as being made up of tiny bits of straight lines, a general curved spacetime can be thought of as being made of of tiny bits of flat spacetime – the local inertial frames of the equivalence principle.

This gives a powerful geometric picture of a gravitational field. It is nothing else than a curvature of spacetime itself

Now, there are several ways to discuss curvature. We are used to looking at curved spaces *inside* of some larger (flat) space. Einstein's idea was that the only relevant things are those that can be measured in terms of the curved surface itself and which have nothing to do with it (perhaps) being part of some larger flat space. As a result, one would gain nothing by assuming that there is such a larger flat space. In Einstein's theory, there is no reason to suppose that one exists.

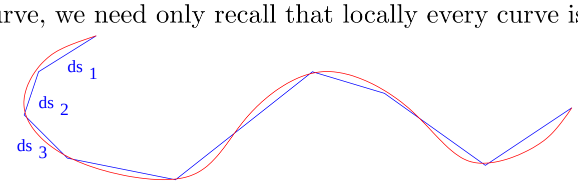
Our task for this section is to learn how to describe this in a useful way. For example, we noticed above that this new understanding of gravity means that the gravitational field contains more information than just giving an acceleration at various points in spacetime. The acceleration is related to curvature in spacetime associated with a time direction (say, in the xt plane), but there are also parts of the gravitational field associated with the (purely spatial) xy, xz, and yz planes.

Let's begin by thinking back to the flat spacetime case (special relativity). What was the object which encoded the flat Minkowskian geometry? It was the interval: $(interval)^2 = -c^2\Delta t^2 + \Delta x^2$. This is a special case of something known as a 'metric,' which we will explore further in the rest of this section.

8.3.1 Building Intuition in flat space

To understand fully what information is contained in the interval, it is perhaps even better to think first about flat *space*, for which the analogous quantity is the distance Δs between two points: $\Delta s^2 = \Delta x^2 + \Delta y^2$.

Much of the important information in geometry is not the distance between two points per se, but the closely related concept of *length*. For example, one of the properties of flat space is that the *length* of the circumference of a circle is equal to 2π times the *length* of its radius. Now, in flat space, distance is most directly related to length for straight lines: the distance between two points is the length of the straight line connecting them. To link this to the length of a curve, we need only recall that locally every curve is a straight line.



In particular, what we need to do is to approximate any curve by a set of tiny (infinitesimal) straight lines. Because we wish to consider the limit in which these straight lines are of zero size, let us denote the length of one such line by ds. The relation of Pythagoras then tells us that $ds^2 = dx^2 + dy^2$ for that straight line, where dx and dy are the infinitesimal changes in the x and y coordinates between the two ends of the infinitesimal line segment. To find the length of a curve, we need only add up these lengths over all of the straight line segments. In the language of calculus, we need only perform the integral:

$$Length = \int_{curve} ds = \int_{curve} \sqrt{dx^2 + dy^2}. \tag{8.1}$$

You may not be used to seeing integrals written in a form like the one above. Let me just pause for a moment to note that this can be written in a more familiar form by, say, taking out a factor of dx from the square root. We have

$$Length = \int_{curve} dx \sqrt{1 + \left(\frac{dy}{dx}\right)^2}. \tag{8.2}$$

So, if the curve is given as a function $y = y(x)$, the above formula does indeed allow you to calculate the length of the curve.

$\star\star$ Now, what does this all really mean? What is the 'take home' lesson from this discussion? The point is that the length of every curve is governed by the formula

$$ds^2 = dx^2 + dy^2. \tag{8.3}$$

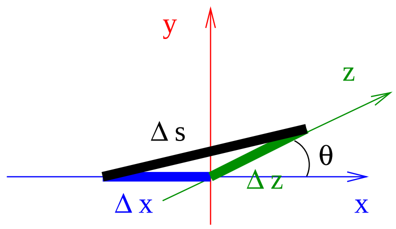
Thus, this formula encodes lots of geometric information, such that the fact that the circumference of a circle is 2π times its radius. As a result, 8.3 will be *false* on a curved surface like a sphere. A formula of the form $ds^2 = stuff$ is known as a *metric*, as it tells us how to measure things (in particular, it tells us how to measure lengths). What we are saying is that this formula will take a different form on a curved surface and will not match with (8.3).

8.3.2 On to Angles

What other geometric information is there aside from lengths? Here, you might consider the examples we talked about last time during class: that flat spaces are characterized by having 180^o in every triangle, and by squares behaving nicely. So, one would also like to know about angles. Now, the important question is: "Is information about angles also contained in the metric?"

It turns out that it is. You might suspect that this is true on the basis of trigonometry, which relates angles to (ratios of) distances. Of course, trigonometry is based on flat space, but recall that any space is locally flat, and notice that an angle is something that happens at a point (and so is intrinsically a local notion).

To see just how angular information is encoded in the metric, let's look at an example. The standard (Cartesian) metric on flat space $ds^2 = dx^2 + dy^2$ is based on an 'orthogonal' coordinate system – one in which the constant x lines intersect the constant y lines at right angles. What if we wish to express the metric in terms of x and, say, some other coordinate z which is not orthogonal to x?



In this case, the distances Δx, Δz, and Δs are related in a slightly more complicated way. If you have studied much vector mathematics, you will have seen the relation:

$$\Delta s^2 = \Delta x^2 + \Delta z^2 + 2\Delta x \Delta z \cos\theta. \tag{8.4}$$

In vector notation, this is just $|\vec{x} + \vec{z}|^2 = |\vec{x}|^2 + |\vec{z}|^2 + 2\vec{x} \cdot \vec{z}$.

Even if you have not seen this relation before, it should make some sense to you. Note, for example, that if $\theta = 0$ we get $\Delta s = 2\Delta x$ (since x and z are parallel and our 'triangle' is just a long straight line), while for $\theta = 180^o$ we get $\Delta s = 0$ (since x and z now point in opposite directions and, in walking along the two sides of our triangle, we cover the same path twice in opposite directions, returning to our starting point.).

For an infinitesimal triangle, we would write this as:

$$ds^2 = dx^2 + dz^2 - 2dxdz\cos\theta. \tag{8.5}$$

So, the angular information lies in the "cross term" with a $dxdz$. The coefficient of this term tells us the angle between the x and z directions.

8.3.3 Metrics on Curved space

This gives us an idea of what a metric on a general curved space should like like. It should have a part proportional to dx^2, a part proportional to dy^2, and a part proportional to $dxdy$. In general, we write this as:

$$ds^2 = g_{xx}dx^2 + 2g_{xy}dxdy + g_{yy}dy^2. \tag{8.6}$$

What makes this metric different from the ones above (and therefore not necessarily flat) is that g_{xx}, g_{xy}, and g_{yy} are in general functions of the coordinates x, y. In contrast, the functions were constants for the flat metrics above. Note that this fits with our idea that curved spaces are locally flat since, close to any particular point (x, y) the functions g_{xx}, g_{xy}, g_{yy} will not deviate too much from the values at that point. In other words, any smooth function is locally constant.

Now, why is there a 2 with the $dxdy$ term? Note that since $dxdy = dydx$, there is no need to have a separate g_{yx} term. The metric is always symmetric, with $g_{yx} = g_{xy}$. So, $g_{xy}dxdy + g_{yx}dydx = 2g_{xy}dxdy$.

If you are familiar with vectors, then I can tell you a bit more about how lengths and angles are encoded. Consider the 'unit' vectors $\hat{x}$ and $\hat{y}$. By 'unit' vectors, I mean the vectors that go from $x = 0$ to $x = 1$ and from $y = 0$ to $y = 1$. As a result, their length is one in terms of the coordinates. This may or may not be the *physical* length of the vectors. For example, I might have decided to use coordinates with a tiny spacing (so that $\hat{x}$ is very short) or coordinates with a huge spacing (so that $\hat{x}$ is large). What the metric tells us directly are the dot products of these vectors:

$$\begin{aligned} \hat{x} \cdot \hat{x} &= g_{xx}, \\ \hat{x} \cdot \hat{y} &= g_{xy}, \\ \hat{y} \cdot \hat{y} &= g_{yy}. \end{aligned} \tag{8.7}$$

Anyway, this object ($g_{\alpha\beta}$) is called the *metric* (or, the metric tensor) for the space. It tells us how to measure all lengths and angles. The corresponding object for a spacetime will tell us how to measure all proper lengths, proper times, angles, etc. It will be much the same except that it will have a time part with g_{tt} negative[4] instead of positive, as did the flat Minkowski space.

Rather than write out the entire expression (8.6) all of the time (especially when working in, say, four dimensions rather than just two) physicists use a condensed notation called the 'Einstein summation convention'. To see how this works, let us first relabel our coordinates. Instead of using x and y, let's use x^1, x^2 with $x^1 = x$ and $x^2 = y$. Then we have:

$$ds^2 = \sum_{\alpha=1}^{2} \sum_{\beta=1}^{2} g_{\alpha\beta} dx^\alpha dx^\beta = g_{\alpha\beta} dx^\alpha dx^\beta. \tag{8.8}$$

It is in the last equality that we have used the Einstein summation convention – instead of writing out the summation signs, the convention is that we implicitly sum over any repeated index.

8.3.4 A first example

To get a better feel for how the metric works, let's look at the metric for a flat plane in polar coordinates (r, θ). It is useful to think about this in terms of the unit vectors $\hat{r}, \hat{\theta}$.

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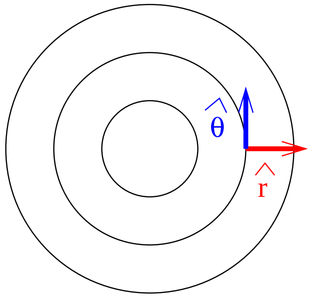
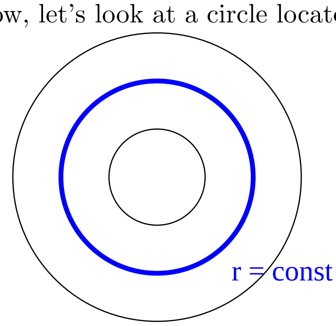
[4]More technically, g should have one negative and three positive eigenvalues at each point in spacetime.



From the picture above, we see that these two vectors are perpendicular: $\hat{r} \cdot \hat{\theta} = 0$. Normally, we measure the radius in terms of length, so that $\hat{r}$ has length one and $\hat{r} \cdot \hat{r} = 1$. The same is not true for θ: one radian of angle at large r corresponds to a much longer arc than does one radian of angle at small r. In fact, one radian of angle corresponds to an arc of length r. The result is that $\hat{\theta}$ has length r and $\hat{\theta} \cdot \hat{\theta} = r^2$. So, for theta measured in radians and running from 0 to 2π, the metric turns out to be:

$$ds^2 = dr^2 + r^2 d\theta^2. \tag{8.9}$$

Now, let's look at a circle located at some constant value of r.



To find the circumference of the circle, we need to compute the length of a curve *along* the circle. Now, along the circle, r does not change, so we have $dr = 0$. (Recall that dr is just the infinitesimal version of Δr.) So, we have $ds = r d\theta$. Thus, the length is:

$$C = \int_0^{2\pi} ds = \int_0^{2\pi} r d\theta = 2\pi r. \tag{8.10}$$

Let's check something that may seem trivial: *What is the radius of this circle?* The radius (R) is the length of the curve that runs from the origin out to the circle along a line of constant θ. Along this line, we have $d\theta = 0$. So, along this curve, we have $ds = dr$. The line runs from $r = 0$ to $r = r$, so we have

$$R = \int_0^r dr = r. \tag{8.11}$$

So, we do indeed have $C = 2\pi R$. Note that while the result $R = r$ may seem obvious it is true only because we used an r coordinate which was marked off in terms of radial distance. In general, this may not be the case. There are times when it is convenient to use a radial coordinate which directly measures something other than distance from the origin and, in such cases, it is very important to remember to calculate the actual 'Radius' (the distance from the origin to the circle) using the metric.

8.3.5 A second example

Now let's look at a less trivial example. Suppose I tell you that the metric of some surface is given by:

$$ds^2 = \frac{dr^2 + r^2 d\theta^2}{(1+r^2)}. \tag{8.12}$$

Is this space flat? Well, let's compare the circumference (C) of a circle at constant r to the radius (R) of that circle.

Again, the circumference is a line of constant r, so we have $dr = 0$ for this line and $ds = \frac{r}{\sqrt{1+r^2}} d\theta$. The circle as usual runs from $\theta = 0$ to $\theta = 2\pi$. So, we have

$$C = \int_0^{2\pi} \frac{r}{\sqrt{1+r^2}} d\theta = \frac{2\pi r}{\sqrt{1+r^2}}. \tag{8.13}$$

Now, how about the radius? Well, the radius R is the length of a line at, say, $\theta = 0$ that connects $r = 0$ with $r = r$. So, we have

$$R = \int_0^{r} \frac{dr}{\sqrt{1+r^2}} = \sinh^{-1} r. \tag{8.14}$$

(This is yet another neat use of hyperbolic trigonometry .... it allows us to explicitly evaluate certain integrals that would otherwise be a real mess.)

Clearly, $C \neq 2\pi R$. In fact, studying the large r limit of the circumference shows that the circumference becomes constant at large r. This is certainly not true of the radius: $R \to \infty$ as $r \to \infty$.

Thus, C is much less than $2\pi R$ for large R.

8.3.6 Some parting comments on metrics

This is perhaps the right place to make a point: We often think about curved spaces as being curved *inside* some larger space. For example, the two-dimensional surface of a globe can be thought of as a curved surface that sits inside some larger (flat) three-dimensional space. However, there is a notion of curvature (associated with the geometry of the surface – the measurements of circles, triangles, and rectangles drawn in that surface – and encoded by the metric) that does not refer in any way to anything outside the surface itself. So, in order for a four dimensional spacetime to be curved, there does not need to be any 'fifth dimension' for the universe to be 'curved into.' The point is that what physicists mean by saying that spacetime is curved is not that it is 'bent' in some new dimension, but rather they mean that the geometry on the spacetime is more complicated than that of Minkowski space. For example, they mean that not every circle has circumference $2\pi R$.

Another comment that should be made involves the relationship between the metric and the geometry. We have seen that the metric determines the geometry: it allows us to compute, for example, the ratio of the circumference of a circle to its radius. One might ask if the converse is true: *Does the geometry determine the metric?*

The answer is a resounding "no." We have, in fact, already seen *three* metrics for flat space: We had one metric in (orthogonal) Cartesian coordinates, one in 'tilted' Cartesian coordinates where the axes were at some arbitrary angle θ, and one in polar coordinates. Actually, we have seen infinitely many different metrics since the metric was different for each value of the tilt angle θ for the tilted Cartesian coordinates. So, the metric carries information not only about the geometry itself, but also about the coordinates you happen to be using to describe it.

The idea in general relativity is that the real physical effects depend only on the geometry and *not* upon the choice of coordinates[5]. After all, the circumference of a circle does not depend on whether you calculate it in polar or in Cartesian coordinates. As a result, one must be careful in using the metric to make physical predictions – some of the information in the metric is directly physical, but some is an artifact of the coordinate system and disentangling the two can sometimes be subtle.

The choice of coordinates is much like the choice of a reference frame. We saw this to some extent in special relativity. For a given observer (say, Alice) in a given reference frame, we would introduce a notion of position (x_{Alice}) as measured by Alice, and we would introduce a notion of time (t_{Alice}) as measured by Alice. In a different reference frame (say, Bob's) we would use different coordinates (x_{Bob} and t_{Bob}). Coordinates describing inertial reference frames were related in a relatively simple way, while coordinates describing an accelerated reference frame were related to inertial coordinates in a more complicated way.

However, whatever reference frame we used and whatever coordinate system we chose, the physical *events* are always the same. Either a given clock ticks 2 at the event where two light rays cross or it does not. Either a blue paintbrush leaves a mark on a meter stick or it does not. Either an observer writes "I saw the light!" on a piece of paper or she does not. As a result, the true physical predictions do not depend on the choice of reference frame or coordinate system at all.

So long as we understand how to deal with physics in funny (say, accelerating) coordinate systems, such coordinate systems will still lead to the correct physical results. The idea that physics should not depend on the choice of coordinates

[5] The idea is similar to the principle in special relativity that physical effects are the same no matter which reference frame you use to compute them.

is called *General Coordinate Invariance*. Invariance is a term that captures the idea that the physics itself does not change when we change coordinates. This turns out to be an important principle for the mathematical formulation of General Relativity as we will discuss further in section 8.4.

8.4 What is the metric of spacetime?

We have now come to understand that the gravitational field is encoded in the metric. Once a metric has been given to us, we have also learned how to use it to compute various objects of interest. In particular, we have learned how to test a space to see if it is flat by computing the ratio of circumference to radius for a circle. However, all of this still leaves open what is perhaps the most important question: just which metric is it that describes the spacetime in which we live?

First, let's again recall that there really is no one 'right' metric, since the metric will depend on the choice of coordinates and there is no one 'right' choice of coordinates. But there is a certain part of the metric that is in fact independent of the choice of coordinates. That part is called the 'geometry' of the spacetime. It is mathematically very complicated to write this part down by itself. So, in practice, physicists work with the metric and then make sure that the things they calculate do not depend on the choice of coordinates.

OK then, what determines the right geometry? Recall that the geometry is nothing other than the gravitational field. So, we expect that the geometry should in some way be tied to the matter in the universe: the mass, energy, and so on should control the geometry. Figuring out the exact form of this relationship is a difficult task, and Einstein worked on it for a long time. We will not reproduce his thoughts in any detail here. However, in the end he realized that there were actually not many possible choices for how the geometry and the mass, energy, etc. should be related.

8.4.1 The Einstein equations

It turns out that, if we make five assumptions, then there is really just one family of possible relationships. These assumptions are:

1) Gravity is spacetime curvature, and so can be encoded in a metric.

2) General Coordinate Invariance: Real physics is independent of the choice of coordinates used to describe it.

3) The basic equations of general relativity should give the dynamics of the metric, telling how the metric changes in time.

4) Energy (including the energy in the gravitational field) is conserved.

5) The (local) equivalence principle.

Making these five assumptions, one is led to a relation between an object $G_{\alpha\beta}$ (called the Einstein Tensor) which encodes *part* of the spacetime curvature and an object $T_{\alpha\beta}$ (called the Stress-Energy or Energy-Momentum Tensor) which encodes all of the energy, momentum, and stresses in everything else ("matter," electric and magnetic fields, etc.). Here, α, β run over the various coordinates (t, x, y, z). What is a stress? One example of a stress is pressure. It turns out that, in general relativity, pressure contributes to the gravitational field directly[6], as do mass, energy, and momentum. The relationship can be written:

$$G_{\alpha\beta} = \kappa T_{\alpha\beta} + \Lambda g_{\alpha\beta} \tag{8.15}$$

The $g_{\alpha\beta}$ in the equation above is just the metric itself. This relation is known as *The Einstein equations*. Note that (8.15) actually represents one equation for each value of α and β. This is 16 equations in all[7].

We see that κ controls just the overall strength of gravity. Making κ larger is the same thing as making $T_{\alpha\beta}$ bigger, which is the same as adding more mass and energy. On the other hand, Λ is something different. Note that it controls a term which relates just to the geometry and not to the energy and mass of the matter. However, this term is added to the side of the equation that contains the energy-momentum tensor. As a result, Λ can be said to control the amount of energy that is present in spacetime that has no matter in it at all. Partially for this reason, Λ is known as the *Cosmological constant*.

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[6] It turns out that pressure is intimately linked to energy and momentum even in special relativity. To understand this properly requires an argument that is more technical than the usual level of these notes. However, for the curious I will outline it here. See e.g. B. Shutz, *A first course in general relativity* (Cambridge, 1985), Chapter 4 for more details. The point is that we will need to discuss energy density, just as the Newtonian gravitational field is related to the mass *distribution* (and therefore the mass density) and as the electric field is related to the density of charge through $\nabla \cdot \vec{E} = \rho$. Now, this is a density in space (energy per 3-d volume) as opposed to a density in spacetime (energy per 4-d volume). This involves some split of spacetime into space and time, such as occurs on a surface of simultaneity. As a result, the energy density can best be thought of a the *flux* of energy through some spacelike surface which may think of as being 'purely spatial.' Similarly, the momentum density is the flux of momentum through a spacelike surface which we might consider 'purely spatial.' However, if we change frames then our purely spatial surface will point a bit in the time direction. As a result we could consider it to be a piece of a spacelike surface plus a bit of a timelike surface. This means that the flux of energy and momentum through timelike surfaces will also be important. It turns out that the flux of energy through a timelike surface (the transfer of energy from one place to another) is just another way to look at the momentum density that we have already discussed. But what about the flux of momentum across a timelike surface? Momentum is transferred from object to object and from place to place by forces. Thus, a momentum flux across a timelike surface is a force density, otherwise known as a pressure or a tension.

[7] However, there is some redundancy. Just as $g_{xy} = g_{yx}$, so $G_{\alpha\beta}$ and $T_{\alpha\beta}$ are similarly 'symmetric'. This means that 6 of the Einstein equations merely repeat other equations and that there are really only 10 equations to be solved.

8.4.2 The Newtonian Approximation

As we said (but did not explicitly derive), equation (8.15) can be deduced from the five above assumptions on purely mathematical grounds. It is not necessary to use Isaac Newton's theory of gravity here as even partial input. So, what is the connection to Newton's ideas about gravity?

Recall that we argued before that Newton's law of gravity can only be correct when the objects are slowly moving – otherwise special relativity would be relevant and all sorts of things would go wrong. There is in fact another restriction on when Newtonian gravity is valid. The point is that, in Newtonian gravity, mass creates a gravitational field. But, we know now that energy and mass are very closely related. So, all energy should create some kind of gravity. However, we have also seen that a field (like the gravitational field itself) can carry energy. As a result, *the gravitational field must also act as a source of further gravity*. That is, once relativity is taken into account, gravitational fields should be stronger than Newton would have expected. For a very weak field (where the field itself would store little energy), this effect should be small. But, for a strong gravitational field, this effect should be large. So, Newton's law of gravity should only be correct for slowly moving objects in fairly weak gravitational fields.

If one does study the Einstein equations for the case of slowly moving objects and weak gravitational fields, one indeed obtains the Newtonian law of gravity for the case $\Lambda = 0$, $\kappa = 8\pi G$, where $G = 6.67 \times 10^{-11} Nm^2/kg^2$ is Newton's gravitational constant. So, to the extent that these numbers are determined by experimental data, they must be the correct values.

In summary, given a lot of thought, Einstein came up with the above five assumptions about the nature of gravity. Then, by mathematics alone he was able to show that these assumptions lead to equation (8.15). For weak gravitational fields and slowly moving objects something like Newton's law of gravity also follows, but with two arbitrary parameters κ and Λ. One of these (κ) is just 8π times Newton's own arbitrary parameter G. As a result, except for one constant (Λ) Newton's law of gravity has also followed from the five assumptions using only mathematics. Finally, by making use of experimental data (the same data that Newton used originally!) Einstein was able to determine the values of κ and Λ. The Einstein equations then take on the pleasing form:

$$G_{\alpha\beta} = 8\pi G T_{\alpha\beta}. \tag{8.16}$$

8.4.3 The Schwarzschild Metric

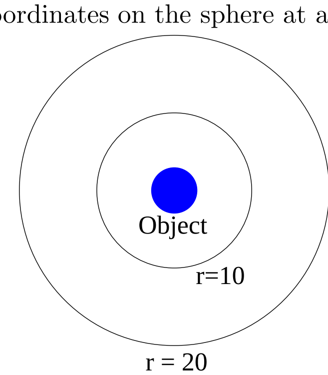
Of course, the natural (and interesting!) question to ask is "What happens when the gravitational field is strong and Newton's law of gravity does *NOT* hold?" We're not actually going to solve the Einstein equations ourselves – they're pretty complicated even for the simplest of cases. However, I will tell you the answer for the interesting special case of round objects.

When an object is perfectly round (spherical), the high *symmetry* of the situation simplifies the mathematics. The point is that if the object is round, and if the object completely determines the gravitational field, then the gravitational field must be round as well. So, the first simplification we will perform is to assume that our gravitational field (i.e., our spacetime) is *spherically symmetric*.

The second simplification we will impose is to assume that there is no matter (just empty spacetime), at least in the region of spacetime that we are studying. In particular, the energy, momentum, etc., of matter are equal to zero in this region. As a result, we will be describing the gravitational field of an object (the earth, a star, etc.) only in the region *outside* of the object. This would describe the gravitational field well above the earth's surface, but not down in the interior.

For this case, the Einstein equations were solved by a young German mathematician named Schwarzschild. There is an interesting story here, as Schwarzschild solved these equations during his spare time while he was in the trenches fighting (on the German side) in World War I. I believe the story is the Schwarzschild got his calculations published but, by the time this happened, he had been killed in the war.

Because of the spherical symmetry, it was simplest for Schwarzschild to use what are called spherical coordinates (r, θ, ϕ) as opposed to Cartesian Coordinates (x, y, z). Here, r tells us how far out we are, and θ, ϕ are latitude and longitude coordinates on the sphere at any value of r.

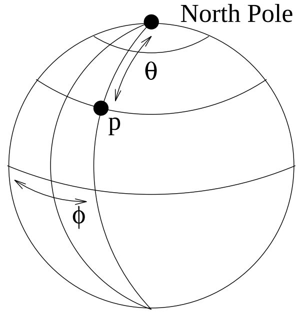


Schwarzschild found that, for any spherically symmetric spacetime and outside of the matter, the metric takes the form:

$$ds^2 = -\left(1 - \frac{R_s}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{R_s}{r}} + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{8.17}$$

Here, the parameter R_s depends on the total mass of the matter inside. In particular, it turns out that $R_s = 2MG/c^2$.

The last part of the metric, $r^2(d\theta^2 + \sin^2\theta d\phi^2)$, is just the metric on a standard sphere of radius r. This part follows just from the spherical symmetry itself. Recall that θ is a latitude coordinate and ϕ is a longitude coordinate. The factor of $\sin^2\theta$ encodes the fact that circles at constant θ (i.e., with $d\theta = 0$) are smaller near the poles ($\theta = 0, \pi$) than at the equator ($\theta = \pi/2$).



8.5 The experimental verification of General Relativity

Now that the Schwarzschild metric (8.17) is in hand, we know what is the spacetime geometry around any round object. Now, what can we do with it? Well, in principle, one can do just about anything. The metric encodes all of the information about the geometry, and thus all of the information about geodesics. Recall that any freely falling worldline (like, say, that of an orbiting planet) is a geodesic. So, one thing that can be done is to compute the orbits of the planets. Another would be to compute various gravitational time dilation effects.

Having arrived at the Schwarzschild solution, we are finally at the point where Einstein's ideas have a lot of power. They now *predict* the curvature around any massive object (the sun, the earth, the moon, etc.). So, Einstein started looking for predictions that could be directly tested by experiment to check that he was actually right. This makes an interesting contrast with special relativity, in which quite a bit of experimental data was already available *before* Einstein constructed the theory. In the case of GR, Einstein was guided for a long time by a lot of intuition (i.e., guesswork) and, for the most part, the experiments would only be done later, after he had constructed the theory. Recall that although we have mentioned a few pieces of experimental evidence already (such as the Pound-Rebke and GPS experiments) these occurred only in 1959 and in the 1990's! Einstein finished developing General Relativity in 1916 and certainly wanted to find an experiment that could be done soon after.

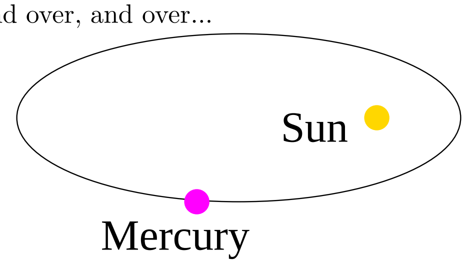
8.5.1 The planet Mercury

We have seen that Einstein's theory of gravity agrees with Newton's when the gravitational fields are weak (i.e., far away from any massive object). But, the discrepancy increases as the field gets stronger. So, the best place (around here) to look for new effects is close to the sun. One might therefore start by considering the orbit of Mercury.

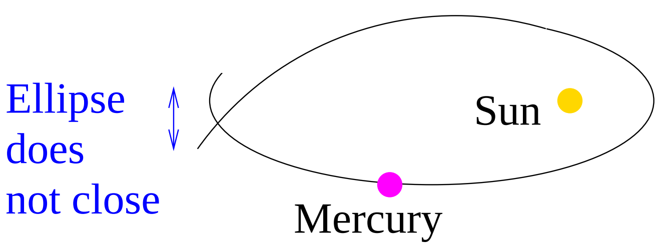
Actually, there is an interesting story about Mercury and its orbit. Astronomers had been tracking the motion of the planets for hundreds of years. Ever since Newton, they had been comparing these motions to what Newton's law of gravity predicted.

The agreement was incredible. In the early 1800's, they had found small discrepancies (30 seconds of arc in 10 years) in the motion of Uranus. For awhile people thought that Newton's law of gravity might not be exactly right. However, someone then had the idea that maybe there were *other* objects out there whose gravity affected Uranus. They used Newton's law of gravity to predict the existence of new planets: Neptune, and later Pluto. They could even tell astronomers where to look for Neptune within about a degree of angle on the sky.

However, there was one discrepancy with Newton's laws that the astronomers could not explain. This was the 'precession' of Mercury's orbit. The point is that, if there were nothing else around, Newton's law of gravity would say that Mercury would move in a perfect ellipse around the sun, retracing its path over, and over, and over...



Of course, there are small tugs on Mercury by the other planets that modify this behavior. However, the astronomers knew how to account for these effects. Their results seemed to say that, even if the other planets and such were not around, Mercury would do a sort of spiral dance around the sun, following a path that looks more like this:



Here, I have drawn the ellipse itself as rotating (a.k.a. 'precessing') about the sun. After all known effects had been taken into account, astronomers found that Mercury's orbit precessed by an extra *43 seconds of arc per century.* This is certainly not very much, but the astronomers already understood all of the other planets to a much higher accuracy. So, what was going wrong with Mercury? Most astronomers thought that it must be due to some sort of gas or dust surrounding the Sun (a big 'solar atmosphere') that was somehow affecting Mercury's orbit.

However, Einstein knew that his new theory of gravity would predict a precession of Mercury's orbit for two reasons. First, he predicted a slightly stronger gravitational field (since the energy in the gravitational field itself acts as a source of gravity). Second, in Einstein's theory, space itself is curved and this effect will also make the ellipse precess (though, since the velocity of mercury is small, this effect turns out to be much smaller than the one due to the stronger gravitational field).

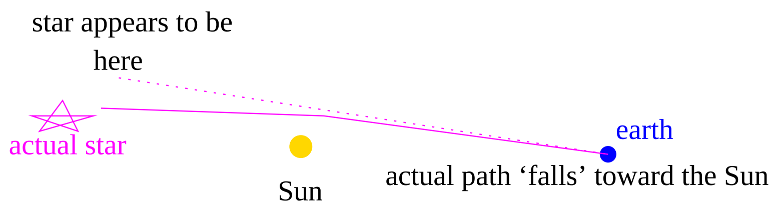
The number that Einstein calculated from his theory was 43 seconds of arc per century. That is, his prediction agreed with the experimental data to better than 1%! Clearly, Einstein was thrilled.

This was big news. However, it would have been even bigger news if Einstein had predicted this result *before* it had been measured. Physicists are always skeptical of just explaining known effects. After all, maybe the scientist (intentionally or not) fudged the numbers or the theory to get the desired result? So, physicists tend not to *really* believe a theory until it predicts something *new* that is then verified by experiments. This is the same sort of idea as in double blind medical trials, where even the researchers don't know what effect they want a given pill to have on a patient!

8.5.2 The Bending of Starlight

Luckily, Einstein had an idea for such an effect and now had enough confidence in his theory to push it through. The point is that, as we have discussed, light will fall in a gravitational field. For example, a laser beam fired horizontally across the classroom will be closer to the ground on the side where it hits the far wall it was when it left the laser.

Similarly, a ray of light that goes skimming past a massive object (like the sun) will fall a bit toward the sun. The net effect is that this light ray is bent. Suppose that the ray of light comes from a star. What this means in the end is that, when the Sun is close to the line connecting us with the star, the star appears to be in a slightly different place than when the Sun is not close to that light ray. For a light ray that just skims the surface of the Sun, the effect is about .875 seconds of arc.



However, this is not the end of the story. It turns out that there is also another effect which causes the ray to bend. This is due to the effect of the curvature of *space* on the light ray. This effect turns out to be exactly the same size as the first effect, and with the same sign. As a result, Einstein predicted a total bending angle of 1.75 seconds of arc – twice what would come just from the observation that light falls in a gravitational field.

This is a tricky experiment to perform, because the Sun is bright enough that any star that close to the sun is very hard to see. One solution is to wait for a solar eclipse (when the moon pretty much blocks out the light from the sun itself) and then one can look at the stars nearby. Just such an observation was performed by the British physicist Sir Arthur Eddington in 1919. The result indicated a bending angle of right around 2 seconds of arc. More recently, much more accurate versions of this experiment have been performed which verify Einstein's theory to high precision. See *Theory and experiment in gravitational physics* by Clifford M. Will, (Cambridge University Press, New York, 1993) QC178.W47 1993 for a modern discussion of these issues.

8.5.3 Other experiments: Radar Time Delay

The bending of starlight was the really big victory for Einstein's theory. However, there are two other classic experimental tests of general relativity that should be mentioned. One of these is just the effect of gravity on the frequency of light that we have already discussed. As we said before, this had to wait quite a long time (until 1959) before technology progressed to the stage where it could be performed.

The last major class of experiments is called 'Radar Time delay.' These turn out to be the most accurate tests of Einstein's theory, but they had to wait until even more modern times. The point is that the gravitational field effects not only the path through space taken by a light ray, but that it also effects the time that the light ray takes to trace out that path. As we have discussed once or twice before, time measurements can be made extremely accurately. So, these experiments can be done to very high precision.

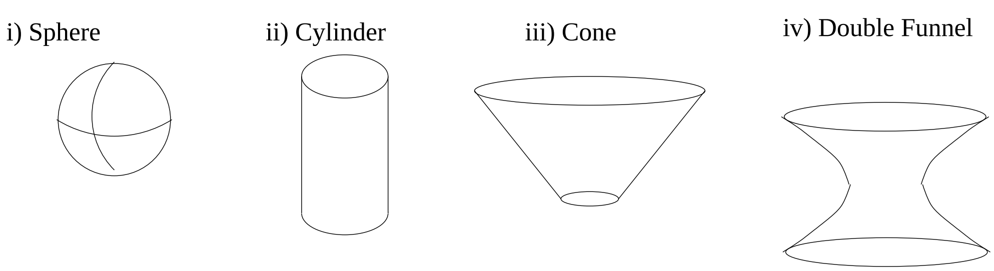
The idea behind these experiments is that you then send a microwave (a.k.a. radar) signal (which is basically a long wavelength light wave) over to the other side of the sun and back. You can either bounce it off a planet (say, Venus) or a space probe that you have sent over there for just this purpose. If you measure the time it takes for the signal to go over and then return, this time is always *longer* than it would have been in flat spacetime. In this way, you can carefully test Einstein's theory. For more details, see fig. 7.3 of *Theory and experiment in gravitational physics* by Clifford M. Will, (Cambridge University Press, New York, 1993) QC178.W47 1993. There is a copy in the Physics Library. A less technical version of this book is *Was Einstein Right? putting General Relativity to the test* by Clifford M. Will (Basic Books, New York, 1993), also available in the Physics Library. You can also order a copy of this book from Amazon.com for about $15.

8.6 Homework Problems

The following problems provides practice in working with curved *spaces* (i.e., not space-times) and the corresponding metrics.

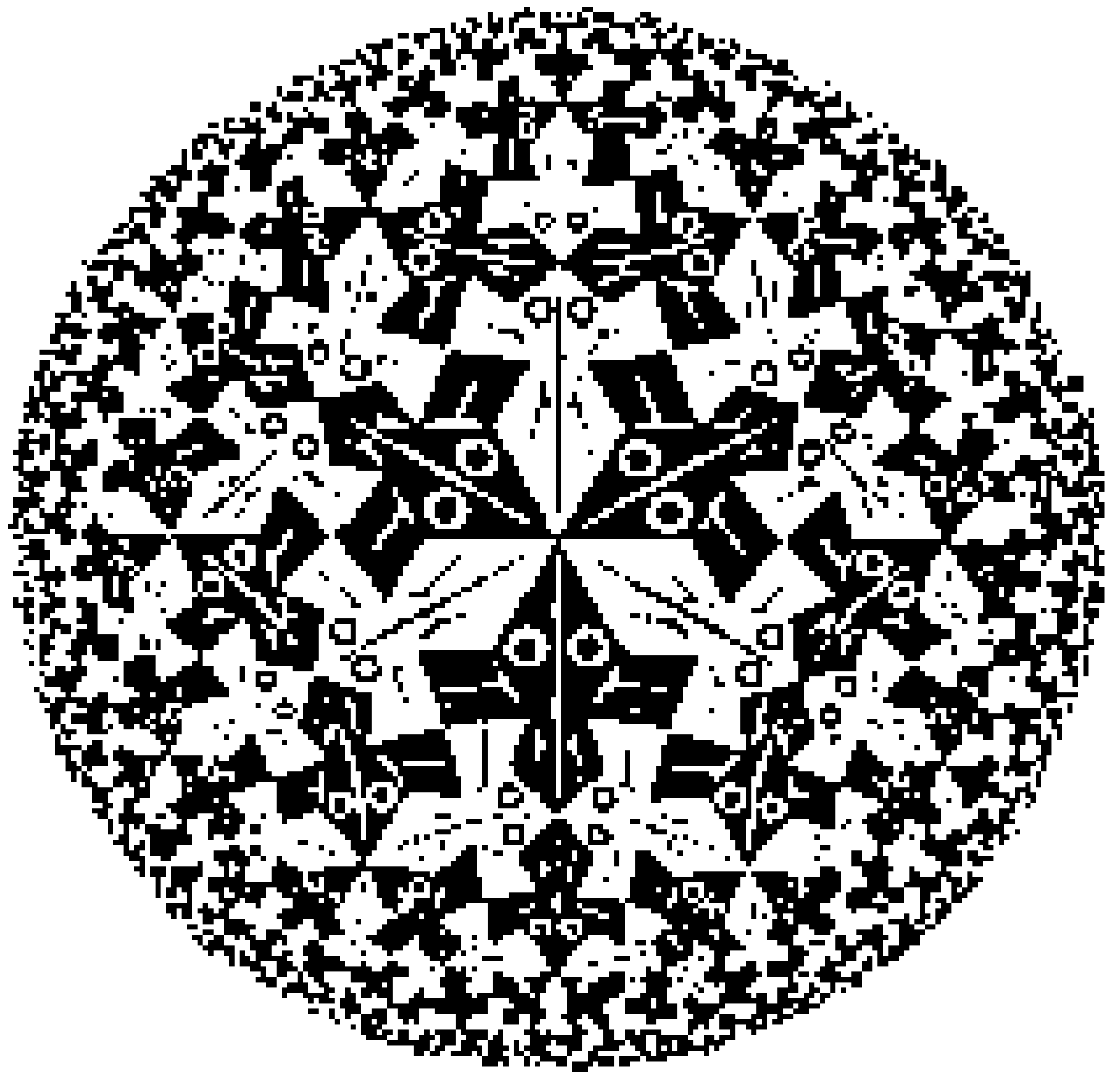
8-1. In this problem, you will explore four effects associated with curvature on each of 5 different spaces. The effects are subtle, so think about them thoroughly, and be sure to read the instructions carefully at each stage. The best way to work these problems is to actually build paper models (say, for the cone and cylinder) and to try to hold a sphere and a funnel shape in your hand while you work on those parts of this problem. I also suggest that you review the discussion in section 8.2 before beginning this problem.

The five spaces are:



and

v) This last one (see below) is a drawing by M.C. Escher. It is intended to represent a mathematical space first constructed by Lobachevski. The idea is that this is a 'map' of the space, and that all of the white fish below (and, similarly, all of the black fish) are really the same size and shape. They only look different because the geometry of the space being described does not match the (flat) geometry of the paper. This is just the phenomenon that we see all the time when we try to draw a flat map of the round earth. Inevitably, the continents get distorted – Greenland for example looks enormous on most maps of North America even though it is actually rather small.



Escher's depiction of Lobachevskian space
(Think of all the black fish as congruent
and all the white fish as congruent.)

A careful study of the map below shows that the Lobachevskian space is *homogeneous*, meaning that every small region of this space is the same as every other small region of this space even even if it looks different on the map. This is just the statement that all of the fish are really the same size and shape even if some of them look shrunken and twisted. The sphere is also a homogeneous space, although the Lobachevskian space is *not* a sphere.

In particular, note the following properties:

I) As you can see in the middle, each fish is bordered by a set of straight (geodesic) line segments. The 'spine' of each fish is also a geodesic. You can find longer geodesics in the Lobachevskian space by tracing through a series of these short geodesics (spines and edges). Remember though not to insert any corners when you connect one short geodesic to the next.

II) Since the fish are all the same (real) size, you can use them as rulers to measure lengths. Give all lengths below in terms of 'fish.' As an example, you might work with a circle of radius three fish.

III) Since, for example, the nose of every black fish is the same shape, it represents the same (actual) angle everywhere you see a black fish nose on the diagram. Use the center part of the diagram (where there is no distortion to figure out how much this angle actually is). Then you can use the fish as protractors to measure angles in other parts of the diagram.

For each of the spaces above, answer the following questions:

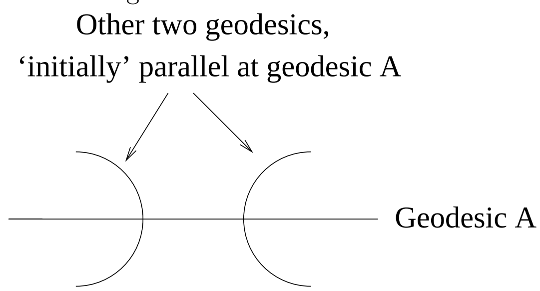
a) *Do circles satisfy $C > 2\pi R$, $C < 2\pi R$, or $C = 2\pi R$?* Important note: What I mean here by a circle (at least for i- iv) is the following: suppose you cut a piece of string of length R, tying one end to a pencil and another end to a thumbtack. A circle is what you get when you use the tack to attach the string to the surface and then move the pencil around *in such a way that the string always remains in full contact with the surface.* The radius R of this circle is the length of the string (again, assuming that the string always remains in full contact with the surface), and the circumference C is the length of the curve traced out by the pencil. You may want to use pieces of string to measure the radius and circumference.

$\star\star$ Note that circles drawn in this way *cannot* not go "all the way around" the hole in the cone, cylinder, or double funnel.

For the Lobachevskian space (v), the idea is the same. A circle consists of a set of point for whom the *real* distance to the center is the same. So, in this context, it is the set of points that are, say, 2 fish lengths away from some center point.

b) *Do the (interior) angles of triangles add up to more than* 180 *degrees, less than* 180 *degrees, or exactly* 180 *degrees?* Important note: Let us define a *triangle* to be a closed figure with three sides, each of which is a *geodesic*. For this part, consider only triangles which can be shrunk down to zero size without taking the triangle out of the surface. (For example, the triangle should not loop all the way around the hole in the cylinder, cone, or double funnel below.)

c) *Do initially parallel 'geodesics' bend toward each other, bend away from each other, or remain parallel?* Important note: The phrase *initially parallel* is important here. Certainly, two geodesics that start out pointing toward each other will move toward each other, and similarly for geodesics that initially point apart. What exactly do I mean by initially parallel, you may ask? Suppose that two geodesics are both orthogonal to a third geodesic A as shown below:



Then, we will say that, at A, the two geodesics are parallel. For example, if geodesics on our surface looked like the ones above, we would say that initially parallel geodesics diverge from each other. Note that this is true whether we follow the geodesics up the digram, or down. Note that the geodesics are parallel only at A so that (since we wish to consider only 'initially parallel' geodesics) we may talk about them curving toward or away from each other only relative to what they are doing at A.

d) *Suppose that you 'parallel transport' an arrow (vector) around a triangle in the surface, does the arrow rotate? If you trace the loop clockwise on the surface, does the arrow rotate clockwise or counter-clockwise?* The full discussion of parallel transport was in section 8.2. Briefly, though, let us recall that to parallel transport a vector along a geodesic (such as along one side of a triangle) means to carry the vector with you without rotating it (relative to you). This is because a geodesic is a straight line. Thus, if your vector starts off pointing straight ahead, it will always point straight ahead. Similarly, if the arrow starts off pointing 30^o to your right, it will always point 30^o to your right. Note, however, that the angle between your arrow and the path you are following *will* change when you go around a corner since the path is not straight there: the path turns although the arrow does not. For the Lobachevskian space, you will need to use the fish as protractors to keep track of the angle between the arrow being carried and the path being followed.

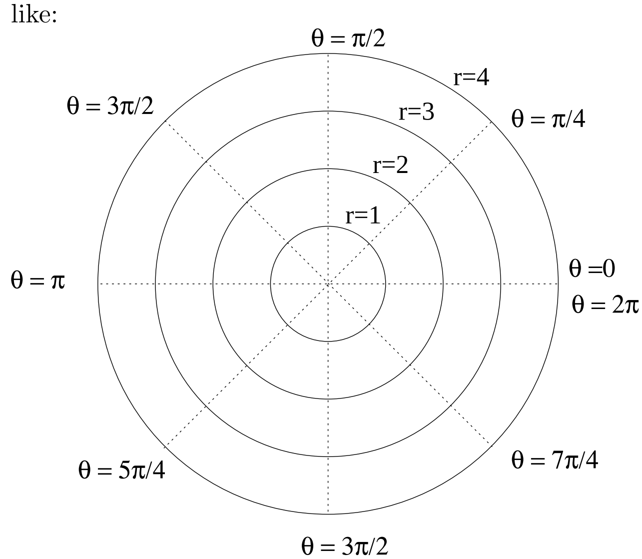
8-2. This problem will give you some practice with the mathematical description of curved surfaces. Remember that if r is constant along a curve, then the change dr in r is zero: $dr = 0$ for that curve. Similarly, if θ is constant along a curve, then we have $d\theta = 0$. The length of a curve is given by $L = \int ds$.

What this means is that, for each curve below where you want to find the length, you should first find ds in terms of dr or $d\theta$ (whichever one is not zero) and then do the appropriate (definite) integral. You may also need to think carefully about the limits of integration.

a) Consider the following two-dimensional geometries:

   i) $ds^2 = dr^2 + a^2 \sin^2(r/a) d\theta^2$
   and
   ii) $ds^2 = dr^2 + a^2 \sinh^2(r/a) d\theta^2$.

   Here, a is some fixed constant; i.e., a is some given number for each geometry. The above expressions refer to 'polar' coordinate systems in which the 'radius' r starts at zero and the 'angle' θ runs from 0 to 2π. That is to say, we could draw a flat picture of each geometry that looks like:



   However we should remember that, as with the Escher picture for problem (1), we will not be able to measure distances and angles off of this picture with a ruler and protractor. This picture is only a very qualitative one and does in any way represent the actual distances in the surface.

   The Question: For each geometry above, what is the circumference of a circle of radius R around the origin? **Note: I would like an expression $C(R)$ valid for any R, not just for the case $r = 4$ shown above.** For which geometry is the relation between C and R

more like a sphere, and for which is it more like the Lobachevskian geometry?

b) By using a different set of coordinates (ρ, θ), instead of (r, θ), I can write these geometries in a different way. The angle θ still goes from 0 to 2π when you go around the circle once.

The two metrics below represent the *same* two geometries as the two metrics in part A. Which is which? Hint: Compute the radius R of a circle with $\rho = \textit{constant}$ around the origin. Remember that the radius is the actual *distance* of the circle from the center, not just the value of the radial coordinate. This means that it is given by the actual *length* of an appropriate line. If the circumference is C, what is the radius R?

i) $ds^2 = a^2 \frac{d\rho^2}{a^2 - \rho^2} + \rho^2 d\theta^2$.

ii) $ds^2 = a^2 \frac{d\rho^2}{a^2 + \rho^2} + \rho^2 d\theta^2$.

c) Show that the geometries in (2b) are the same as the geometries in (2a), but just expressed in terms of different coordinates. To do so, rewrite metrics (2(b)i) and (2(b)ii)) in terms of the **distance** R from the origin to the circle at constant ρ and then compare them with (2(a)i) and (2(a)ii). Remember that $d\rho = \left(\frac{d\rho}{dR}\right) dR = \left(\frac{dR}{d\rho}\right)^{-1} dR$.

Chapter 9

Black Holes

Einstein: XXVIII and XXIX

In the last two chapters we have come to understand many features of General Relativity. We have grown used to gravitational time dilation and curved spacetime and we have seen how these are described through the spacetime metric. We have also learned something about the dynamics of the gravitational field and how spacetime curvature is tied to mass and energy. We have even begun to study the so-called 'Schwarzschild metric' which gives the exact description of curved spacetime outside of a round object.

In this chapter we will take the time to explore the Schwarzschild metric in more detail. We will discover an intriguing feature of this metric known as the horizon and, in this way, we will be led into a study of black holes. Although black holes are one of the most surprising objects in general relativity, it turns out that all of the features of the simplest round black holes can be extracted directly from the Schwarzschild metric. We will follow this approach for most of the chapter, with a few extra comments at the end on some more complicated sorts of black holes.

9.1 Investigating the Schwarzschild Metric

Let's take a look at the Schwarzschild metric in more detail. We computed the gravitational effect on time dilation back in section 7.4.2. However, in this computation we needed to know the gravitational acceleration $g(l)$. We could of course use Newton's prediction for $g(l)$, which experiments tell us is approximately correct near the earth. However, in general we expect this to be the correct answer only for weak gravitational fields. On the other hand, we know that the Schwarzschild metric describes the gravitational field around a spherical object even when the field is strong. So, what we will do is to first use the Schwarzschild metric to compute the gravitational time dilation effect directly. We will then be able to use the relation between this time dilation and the gravitational acceleration to compute the corrections to Newton's law of gravity.

9.1.1 Gravitational Time Dilation from the Metric

Suppose we want to calculate how clocks run in this gravitational field. This has to do with proper time $d\tau$, so we should remember that $d\tau^2 = -ds^2$. For the Schwarzschild metric we have:

$$d\tau^2 = -ds^2 = \left(1 - \frac{R_s}{r}\right) dt^2 - \frac{dr^2}{1 - \frac{R_s}{r}} - r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{9.1}$$

The Schwarzschild metric describes any spherically symmetric gravitational field in the region outside of all the matter. So, for example, it gives the gravitational field outside of the earth. In using the Schwarzschild metric, remember that $R_s = 2MG/c^2$.

Let's think about a clock that just sits in one place above the earth. It does not move toward or away from the earth, and it does not go around the earth. It just 'hovers.' Perhaps it sits in a tower, or is in some rocket ship whose engine is tuned in just the right way to keep it from going either up or down. Such a clock is called a *static* clock since, from it's point of view, the gravitational field does not change with time.

Consider the worldline of this clock through spacetime. Along this worldline, what is dr? How about $d\theta$ and $d\phi$?

Since r, θ, and ϕ do not change, we have $dr = d\theta = d\phi = 0$. So, on our clock's worldline we have just: $d\tau^2 = \left(1 - \frac{R_s}{r}\right) dt^2$. That is,

$$d\tau = \sqrt{1 - \frac{R_s}{r}} dt. \tag{9.2}$$

Note that if the clock is at $r = \infty$ then the square root factor is equal to 1. So, we might write $d\tau_\infty = dt$. In other words, $d\tau = \sqrt{1 - \frac{R_s}{r}} d\tau_\infty$, or,

$$\frac{\Delta\tau}{\Delta\tau_\infty} = \sqrt{1 - \frac{R_s}{r}}. \tag{9.3}$$

As saw before, clocks higher up run faster. Now, however, the answer seems to take a somewhat simpler form than it did back in section 7.4.2, when we were using only the Newtonian approximation.

9.1.2 Corrections to Newton's Law

Note that the Schwarzschild geometry is a time independent gravitational field. This means that we can use our results from section 7.4.2 to relate the rate at which various clocks run to the acceleration of freely falling observers. In other words, we can use this to compute the corrections to Newton's law of gravity.

Recall the relation (equation 7.16) is

$$\frac{\Delta\tau_b}{\Delta\tau_a} = \exp\left(\int_a^b \frac{\alpha(s)}{c^2} ds\right). \tag{9.4}$$

Here, $\alpha(s)$ is the acceleration of a static clock relative to a freely falling clock at s, and s measures distance. To compare this with our formula above, we want to take $a = s$ and $b = \infty$. Taking the ln of both sides gives us

$$\ln\left(\frac{\tau(s)}{\tau_\infty}\right) = \int_\infty^s \frac{\alpha(s)}{c^2} ds. \tag{9.5}$$

Now, taking a derivative with respect to s we find:

$$\frac{\alpha(s)}{c^2} = -\frac{d}{ds}\ln\left(\frac{\tau(s)}{\tau_\infty}\right). \tag{9.6}$$

Now, it is important to know what exactly s measures in this formula. Recall that when we derived this result we were interested in the actual physical height of a tower. As a result, this s describes *proper distance*, say, above the surface of the earth.

On the other hand, equation (9.3) is given in terms of r which, it turns out, does *not* describe proper distance. To see this, let's think about the proper distance ds along a radial line with $dt = d\theta = d\phi = 0$. In this case, we have $ds^2 = \frac{dr^2}{1-R_s/r}$, or $ds = \frac{dr}{\sqrt{1-R_s/r}}$, and

$$\frac{dr}{ds} = \sqrt{1 - R_s/r}. \tag{9.7}$$

However we can deal with this by using the chain rule:

$$\alpha = c^2 \frac{d}{ds}\ln\left(\frac{\tau(s)}{\tau_\infty}\right) = c^2\left(\frac{dr}{ds}\right)\frac{d}{dr}\ln\left(\frac{\tau(r)}{\tau_\infty}\right). \tag{9.8}$$

Going through the calculation yields:

$$\begin{aligned}
\alpha &= c^2\sqrt{1-R_s/r}\frac{d}{dr}\ln\sqrt{1-R_s/r} \\
&= c^2\sqrt{1-R_s/r}\frac{1}{2}\frac{d}{dr}\ln(1-R_s/r) \\
&= \frac{c^2}{2}\sqrt{1-R_s/r}\frac{1}{1-R_s/r}\frac{+R_s}{r^2} \\
&= \frac{c^2}{2\sqrt{1-R_s/r}}\frac{R_s}{r^2}.
\end{aligned} \tag{9.9}$$

Note that for $r \gg R_s$, we have $\alpha \sim \frac{c^2}{2}\frac{R_s}{r^2} = \frac{MG}{r^2}$. This is exactly Newton's result.

However, for small r, α is much bigger. In particular, look at what happens when $r = R_S$. There we have $\alpha(R_s) = \infty$! So, at $r = R_s$, it takes an infinite proper acceleration for a clock to remain static. A static person at $r = R_s$ would therefore feel *infinitely* heavy. This is clearly a various special value of the radius coordinate, r. This value is known as the Schwarzschild radius.

Now, let's remember that the Schwarzschild metric only gives the right answer outside of all of the matter. Suppose then that the actual physical radius of the matter is bigger than the associated Schwarzschild radius (as is the case for the earth and the Sun). In this case, you will not see the effect described above since the place where it would have occurred ($r = R_s$) in inside the earth where the matter is non-zero and the Schwarzschild metric does not apply.

But what if the matter source is very small so that its physical radius is less than R_s? Then the Schwarzschild radius R_s will lie outside the matter at a place you could actually visit. In this case, we call the object a "black hole." You will see why in a moment.

9.2 On Black Holes

Objects that are smaller than their Schwarzschild radius (i.e., black holes) are one of the most intriguing features of general relativity. We now proceed to explore them in some detail, discussing both the formation of such objects and a number of their interesting properties. Although black holes may seem very strange at first, we will soon find that many of their properties are in quite similar to features that we encountered in our development of special relativity some time ago.

9.2.1 Forming a black hole

A question that often arises when discussing black holes is whether such objects actually exist or even whether they could be formed in principle. After all, to get $R_s = 2MG/c^2$ to be bigger than the actual radius of the matter, you've got to put a lot of matter in a very small space, right? So, maybe matter just can't be compactified that much. In fact, it turns out that making black holes (at least big ones) is actually very easy. In order to stress the importance of understanding black holes and the Schwarzschild radius in detail, we'll first talk about just why making a black hole is so easy before going on to investigate the properties of black holes in more detail in section 9.2.2.

Suppose we want to make a black hole out of, say, normal rock. What would be the associated Schwarzschild radius? We know that $R_s = 2MG/c^2$. Suppose we have a big ball of rock or radius r. How much mass in in that ball? Well, our experience is that rock does not curve spacetime so much, so let's use the flat space formula for the volume of a sphere: $V = \frac{4}{3}\pi r^3$. The mass of the ball of rock is determined by its density, ρ, which is just some number[1]. The mass of the ball of rock is therefore $M = \rho V = \frac{4\pi}{3}\rho r^3$. The associated Schwarzschild radius is then $R_s = \frac{8\pi G}{3c^2}\rho r^3$.

Now, for large enough r, any cubic function is bigger than r. In particular, we get $r = R_s$ at $r = \left(\frac{3c^2}{8\pi G\rho}\right)^{1/2}$ and there is a solution no matter what the value of ρ! So, a black hole can be made out of rock, without even working hard to compress it more than normal, *so long as we just have enough rock*. Similarly, a black hole could be made out of people, so long as we had enough of them – just insert the average density of a person in the formula above. A black hole could even be made out of very diffuse air or gas, so long so as we had enough of it. For air at normal density[2], we would need a ball of air 10^{13} meters across. For comparison, the Sun is 10^9 meters across, so we would need a ball of gas $10,000$ times larger than the Sun (in terms of radius).

Black holes in nature seem to come come in two basic kinds. The first kind consists of small black holes whose mass is a few times the mass of the Sun. These form in extremely violent processes like supernova explosions of stars, when the interior of the star is compressed to enormous densities. The second kind consists of *huge* black holes, whose mass is 10^6 (a million) to 10^{10} (ten billion) times the mass of the sun. Some black holes may be even larger.

Astronomers tell us that there seems to be a large black hole at the center of every galaxy, or almost every galaxy – we'll talk about this more in section 9.5.1. As we have seen, these large black holes are much easier to form than are small ones and do not require especially violent processes. To pack the mass of 10^6 suns within the corresponding Schwarzschild radius does not require a density much higher than that of the Sun itself (which is comparable to the density of water or rock). One can imagine such a black hole forming in the center of a galaxy, where the stars are densely packed, just by having a few million stars wander in very close together. The larger black holes are even easier to make: to pack a mass of ten billion suns within the corresponding Schwarzschild radius requires a density of only 10^{-5} times the density of air! It could form from just a very large cloud of very thin gas.

9.2.2 Matter within the Schwarzschild radius

Since black holes exist (or at least could easily be made) we're going to have to think more about what is going on at the Schwarzschild radius. At first, the Schwarzschild radius seems like a very strange place. There, a rocket would require an *infinite* proper acceleration to keep from falling in. So what about

[1]More precisely, the density is a function of the pressure that the rock is under, and this pressure does increase as we pile rocks on top of each other. Nevertheless, for this discussion let us assume that we have a very tough rock that does not compress under pressure. Using compressible rock would only make it easier to form a black hole since this would put more rock inside a smaller radius.

[2]Say, at STP.

the matter that first formed the black hole itself? Where is that matter and what is it doing?

Let's go through this step by step. Let us first ask if there can be matter sitting at the Schwarzschild radius (as part of a static star or ball or gas). Clearly not, since the star or ball of gas cannot produce the infinite force that would be needed to keep its atoms from falling inward. The star or ball of gas must contract. Even more than this, the star will be already be contracting when it reaches the Schwarzschild radius and, since gravitation produces accelerations, it must cause this rate of contraction to increase.

Now, what happens when the star becomes smaller than its Schwarzschild radius? The infinite acceleration of static observers at the Schwarzschild radius suggests that the Schwarzschild metric may not be valid inside R_s. As a result we cannot yet say for sure what happens to objects that have contracted within R_s. However, we would certainly find it odd if the effects of gravity became weaker when the object was compressed. Thus, since the object has no choice but to contract (faster and faster) when it is of size R_s, one would expect smaller objects also to have no choice but accelerated contraction!

It now seems that in a finite amount of time the star must shrink to an object of *zero* size, a mathematical point. This most 'singular' occurrence (to quote Sherlock Holmes) is called a 'singularity.' But, once it reaches zero size, what happens then? This is an excellent question, but we are getting ahead of ourselves. For the moment, let's go back out to the Schwarzschild radius and find out what is really going on there.

9.2.3 The Schwarzschild radius and the Horizon

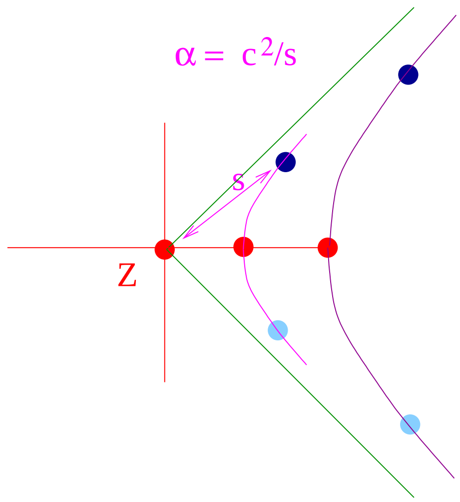
Not only does a clock require an infinite acceleration to remain static at the Schwarzschild radius, but something else interesting happens there as well. Let's look back at the formula we had for the time measured by a static clock:

$$\frac{\Delta\tau(r)}{\Delta\tau_\infty} = \sqrt{1 - R_s/r}. \tag{9.10}$$

Notice what happens at at the Schwarzschild radius. Since $r = R_s$, we have $\Delta\tau = 0$.. Our clock *stops*, and no time passes at all.

Now, this is certainly very weird, but perhaps it rings a few bells? It should sound vaguely familiar.... clocks running infinitely slow at a place where the acceleration required to keep from falling becomes infinite.... You may recall that the same thing occurred for the acceleration horizons back in special relativity.

This gives us a natural guess for what is going on near the Schwarzschild radius. In fact, let us recall that any curved spacetime is locally flat. So, if our framework holds together at the Schwarzschild radius we should be able to match the region near $r = R_s$ to some part of Minkowski space. Perhaps we should match it to the part of Minkowski space near an acceleration horizon? Let us guess that this is correct and then proceed to check our answer.



We will check our answer using the equivalence principle. The point is that an accelerating coordinate system in flat spacetime contains an apparent gravitational field. There is some nontrivial proper acceleration α that is required to remain static at each position. Furthermore, this proper acceleration is not the same at all locations, but instead becomes infinitely large as one approaches the horizon. What we want to do is to compare this apparent gravitational field (the proper acceleration $\alpha(s)$, where s is the proper distance from the horizon) near the acceleration horizon with the corresponding proper acceleration $\alpha(s)$ required to remain static a small proper distance s away from the Schwarzschild radius.

If the two turn out to be the same then this will mean that static observers have identical experiences in both cases. But, the experiences of static observers are related to the experiences of freely falling observers. Thus, if we then consider freely falling observers in both cases, they will also describe both situations in the same way. It will then follow that physics near the event horizon is identical to physics near an acceleration horizon – something that we understand well from special relativity.

Recall that in flat spacetime the proper acceleration required to maintain a constant proper distance s from the acceleration horizon (e.g., from event Z) is given (see section 5.1.3) by

$$\alpha = c^2/s. \tag{9.11}$$

Now, so far this does not look much like our result (9.9) for the black hole. However, we should again recall that r and s represent different quantities. That is, r does not measure proper distance. Instead, we have

$$ds = \frac{dr}{\sqrt{1 - R_s/r}} = \sqrt{\frac{r}{r - R_s}}dr. \tag{9.12}$$

I have rewritten this formula in this way because we only want to study what happens near the Schwarzschild radius. In other words, we are interested in the behavior when $r - R_s$ is small. To examine this, it is useful to introduce the quantity $\Delta = r - R_s$. We can then write the above formula as: $ds = \sqrt{\frac{r}{\Delta}} d\Delta$. Integrating, we get

$$s = \int_0^\Delta \sqrt{\frac{r}{\Delta}} d\Delta. \tag{9.13}$$

This integral is hard to perform exactly since $r = R_s + \Delta$ is a function of Δ. However, since we are only interested in small Δ (for our local comparison), r doesn't differ much from R_s. So, we can simplify our work and still maintain sufficient accuracy by replacing r in the above integral by R_s. The result is:

$$s \approx \sqrt{R_s} \int_0^\Delta \frac{d\Delta}{\sqrt{\Delta}} = 2\sqrt{R_s \Delta}. \tag{9.14}$$

Let us use this to write α for the black hole (let's call this α_{BH}) in terms of the proper distance s. From above, we have

$$\begin{aligned} \alpha_{BH} &= \frac{c^2}{2\sqrt{1 - R_s/r}} \frac{R_s}{r^2} \\ &= \frac{c^2}{2} \frac{\sqrt{r}}{r - R_s} \frac{R_s}{r_2} \\ &= \frac{c^2}{2} \frac{1}{\sqrt{\Delta}} \frac{r}{\sqrt{r}} \frac{R_s}{r^2} \\ &\approx \frac{c^2}{2\sqrt{\Delta R_s}} = \frac{c^2}{s}. \end{aligned} \tag{9.15}$$

Note that this is identical to the expression for α near an acceleration horizon. It worked! Thus we can conclude:

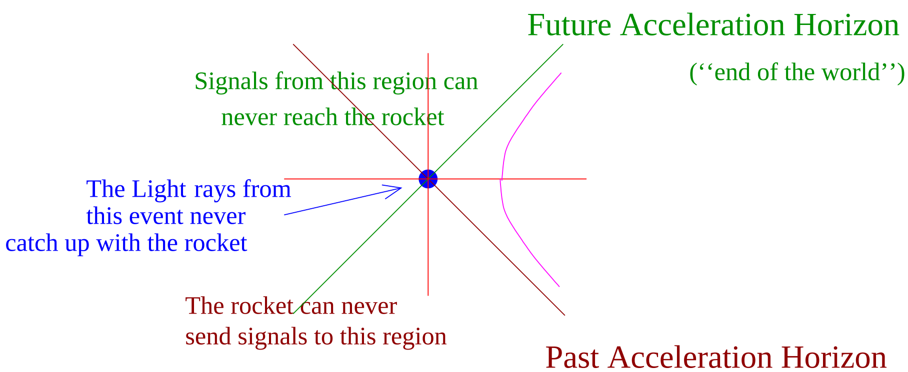
> Near the Schwarzschild radius, the black hole spacetime is just the same as flat spacetime near an acceleration horizon.

The part of the black hole spacetime at the Schwarzschild radius is known as the horizon of the black hole.

9.2.4 Going Beyond the Horizon

We are of course interested in what happens when we go below the horizon of a black hole. However, the connection with acceleration horizons tells us that we will need to be careful in investigating this question. In particular, so far we have made extensive use of static observers – measuring the acceleration of freely falling frames relative to them. Static observers were also of interest when discussing acceleration horizons – so long as they were outside of the acceleration horizons. Recall that the past and future acceleration horizons divided Minkowski space into four regions: static worldlines did not enter two of these at all, and in another region static worldlines would necessarily move 'backwards in time.' The fourth region was the normal 'outside' region, and we concluded that true static observers could only exist there.

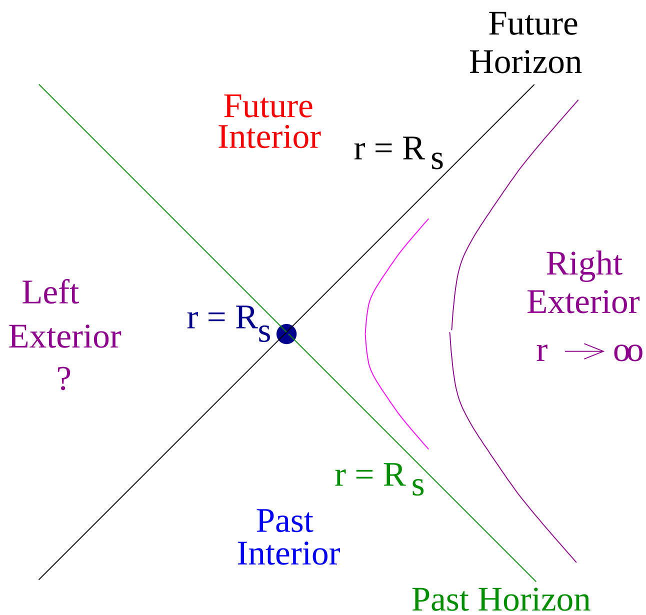
We have seen that the spacetime near the black hole horizon is just like that near an acceleration horizon. As a result, there will again be no static observers below the horizon. We suspected this earlier based on the idea that it takes infinite acceleration to remain static *at* the horizon and we expected the gravitational effects to be even stronger deeper inside. Based on our experience with acceleration horizons, we now begin to see how this may in fact be possible. It has become clear that we will need to abandon static observers in order to describe the region below the black hole horizon.



Suppose then that we think about freely falling observers instead. As we know, freely falling observers typically have the simplest description of spacetime. Using the connection with acceleration horizons, we see immediately how to draw a (freely falling) spacetime diagram describing physics near the Schwarzschild radius. It must look just like our diagram above for flat spacetime viewed from an inertial frame near an acceleration horizon! Note that $r = R_s$ for the black hole is like $s = 0$ for the acceleration horizon since $\alpha \to \infty$ in both cases.

The important part of this is that $s = 0$ is not only the event Z, but is in fact the entire horizon! This is because events separated by a light ray are separated by *zero* proper distance. It also follows from continuity since, arbitrarily close to the light rays shown below we clearly have a curve of constant r for r arbitrarily close to R_s. So, $r = R_s$ is also the path of a light ray, and forms a horizon in the black hole spacetime. In the black hole context, the horizon is often referred to as the 'event horizon[3] of the black hole.'

[3]Technically, relativists distinguish various types of horizons. However, all of these horizons coincide for the static round black hole we are discussing. The term 'event horizon' is common in popular treatments of black holes, so I wanted to be sure to make this connection here.



9.2.5 A summary of where we are

Let us review our discussion so far. We realized that, so long as we were outside the matter that is causing the gravitational field, any spherically symmetric (a.k.a. 'round') gravitational field is described by the Schwarzschild metric. This metric has a special place, at $r = R_s$, the 'Schwarzschild Radius.' *Any* object which is smaller than its Schwarzschild Radius will be surrounded by an event horizon, and we call such an object a black hole.

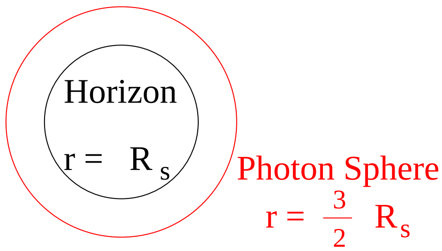
If we look far away from the black hole, at $r \gg R_s$, then the gravitational field is much like what Newton would have predicted for an object of that mass. There is of course a little gravitational time dilation, and a little curvature[4], but not much. Indeed, the Schwarzschild metric describes the gravitational field not only of a black hole, but of the earth, the Sun, the moon, and any other round object. However, for those more familiar objects, the surface of the object is at $r \gg R_s$. For example, on the surface of the Sun $r/R_s \sim 5 \times 10^5$.

So, far from a black hole, objects can orbit just like planets orbit the Sun. By the way, remember that orbiting objects are freely falling – they do not require rocket engines or other forces to keep them in orbit. However, suppose that we look closer in to the horizon. What happens then?

In your recent homework, you saw that something interesting happens to orbiting objects when they orbit at $r = 3R_s/2$. There, an orbiting object experiences no proper time: $\Delta\tau = 0$. *This means that the orbit at this radius is a light-like path.* In other words, a ray of light will orbit the black hole in a circle at

[4] e.g., dC/dR is a bit less than 2π.

$r = 3R_s/2$. For this reason, this region is known as the 'photon sphere.' This makes for some very interesting visual effects if you would imagine traveling to the photon sphere. A few years ago NASA funded a guy to make some nice computer generated movies showing how this would look. He hasn't included is the effect of gravity on the color of light, but otherwise what he has done gives a very good impression of what you would see. You can find his movies[5] at (http://antwrp.gsfc.nasa.gov/htmltest/rjn_bht.html).



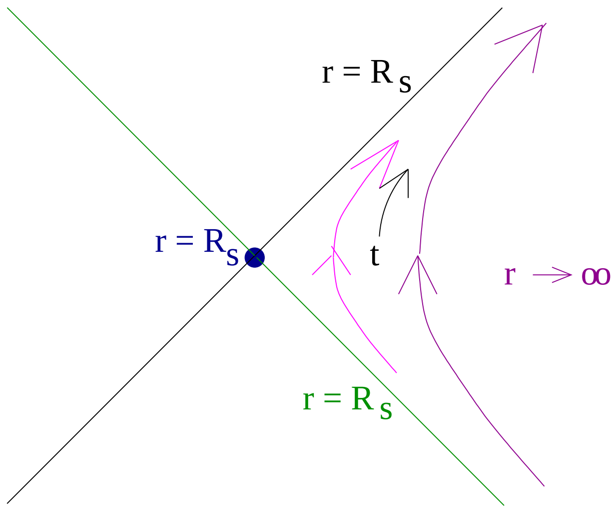
This is not to say that light cannot escape from the photon sphere. The point is that, if the light is moving straight *sideways* (around the black hole) then the black hole's gravity is strong enough to keep the light from moving farther away. However, if the light were directed straight outward at the photon sphere, it would indeed move outward, and would eventually escape.

And what about closer in, at $r < 3R_s/2$? Any circular orbit closer in is *spacelike*, and represents an object moving faster than the speed of light. So, given our usual assumptions about physics, nothing can orbit the black hole closer than $r = 3R_2/2$. Any freely falling object that moves inward past the photon sphere will continue to move to smaller and smaller values of r. However, if it ceases to be freely falling (by colliding with something or turning on a rocket engine) then it can still return to larger values of r.

Now, suppose that we examine even smaller r, and still have not run into the surface of an object that is generating the gravitational field. If we make it all the way to $r = R_s$ without hitting the surface of the object, we find a horizon and we call the object a black hole.

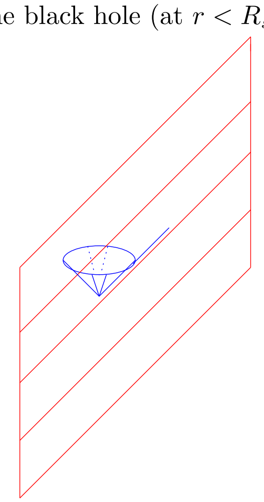
Recall that *even though it is at a constant value of* r*, the horizon contains the worldlines of outward directed light rays.* To see what this means, imagine an expanding sphere of light (like one of the ones produced by a firecracker) at the horizon. Although it is moving outward at the speed of light (which is infinite boost parameter.), the sphere does not get any bigger. The curvature of spacetime is such that the area of the spheres of light do not increase. A spacetime diagram looks like this:

[5] The movies are in MPEG format, so if you don't already have an MPEG player then you may need to install one on your computer to see them.



Here, I have used arrows to indicate the direction in which the time coordinate t increases on this diagram.

Not only do light rays directed along the horizon remain at $r = R_s$, any light ray at the horizon which is directed a little bit sideways (and not perfectly straight outward) cannot even stay at $r = R_s$, but must move to smaller r. The diagram below illustrates this by showing the horizon as a surface made up of light rays. If we look at a light cone emitted from a point on this surface, only the light ray that is moving in the same direction as the rays on the horizon can stay in the surface. The other light rays all fall behind the surface and end up inside the black hole (at $r < R_s$).



Similarly, any object of nonzero mass requires an infinite acceleration (directed straight outward) to remain at the horizon. With any finite acceleration, the object falls to smaller values of r. At any value of r less than R_s no object can ever escape from the black hole. This is clear from the above spacetime diagram, since to move from the future interior to, say, the right exterior the object would have to cross the light ray at $r = R_s$, which is not possible.

9.3 Beyond the Horizon

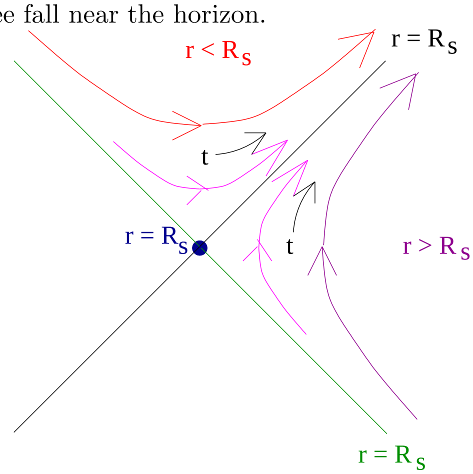
Of course, the question that everyone would like to answer is "What the heck is going on *inside* the black hole?" To understand this, we will turn again to the Schwarzschild metric. In this section we will explore the issue in quite a bit of detail and obtain several useful perspectives.

9.3.1 The interior diagram

To make things simple, let's suppose that all motion takes place in the r, t plane. This means that $d\theta = d\phi = 0$, and we can ignore those parts of the metric. The relevant pieces are just

$$ds^2 = -(1 - R_s/r)dt^2 + \frac{dr^2}{1 - R_s/r}. \tag{9.16}$$

Let's think for a moment about a line of constant r (with $dr = 0$). For such a line, $ds^2 = -(1 - R_s/r)dt^2$. The interesting thing is that, for $r < R_s$, this is *positive*. Thus, for $r < R_s$, a line of constant r is *spacelike*. You will therefore not be surprised to find that, near the horizon, the lines of constant r are just like the hyperbolae that are a constant proper time from where the two horizons meet. Below, I have drawn a spacetime diagram in a reference frame that is in free fall near the horizon.



The coordinate t increases along these lines, in the direction indicated by the arrows. This means that the t-direction is actually spacelike inside the black hole. The point here is not that something screwy is going on with time inside a black hole. Instead, it is merely that using the Schwarzschild metric in the way that we have written it we have done something ‘silly’ and labelled a space direction t. The problem is in our notation, not the spacetime geometry.

Let us fix this by changing notation when we are in this upper region. We introduce $t' = r$ and $r' = t$. The metric then takes the form

$$ds^2 = -(1 - R_s/t')dt'^2 + \frac{dr'^2}{1 - R_s/t'}. \tag{9.17}$$

You might wonder if the Schwarzschild metric is still valid in a region where the t direction is spacelike. It turns out that it is. Unfortunately, we were not able to discuss the Einstein equations in detail. If we had done so, however, then we could check this by directly plugging the Schwarzschild metric into equation (8.15) just as we would to check that the Schwarzschild metric is a solution outside the horizon.

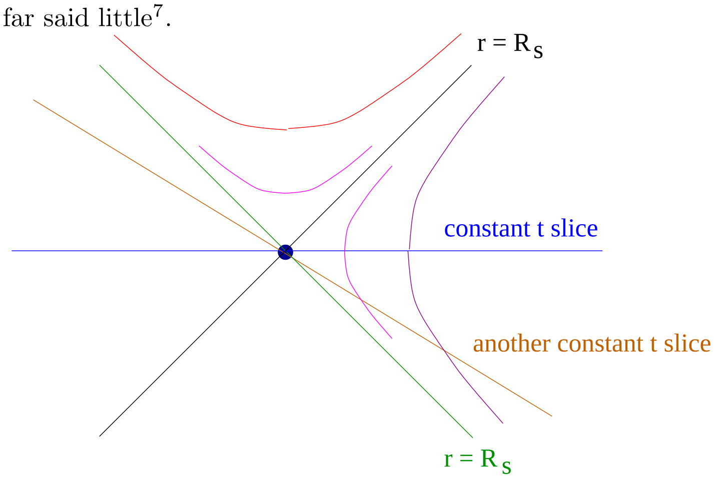
Finally, notice that the lines above look just the like lines we drew to describe the boost symmetry of Minkowski space associated with the change of reference frames. In the same way, these lines represent a symmetry of the black hole spacetime. After all, the lines represent the direction of increasing $t = r'$. But, the Schwarzschild metric is completely independent of $t = r'$ – it depends only on $r = t'$! So, sliding events along these lines and increasing their value of $t = r'$ does not change the spacetime in any way. Outside of the horizon, this operation moves events in time. As a result, the fact that it is a symmetry says that the black hole's gravitational field is not changing in time. However, inside the horizon, the operation moves events in a *spacelike* direction. Roughly speaking, we can interpret the fact that this is a symmetry as saying that the black hole spacetime is the same at every *place* inside.

However, the metric does depend on $r = t'$, so *the interior is dynamical.*

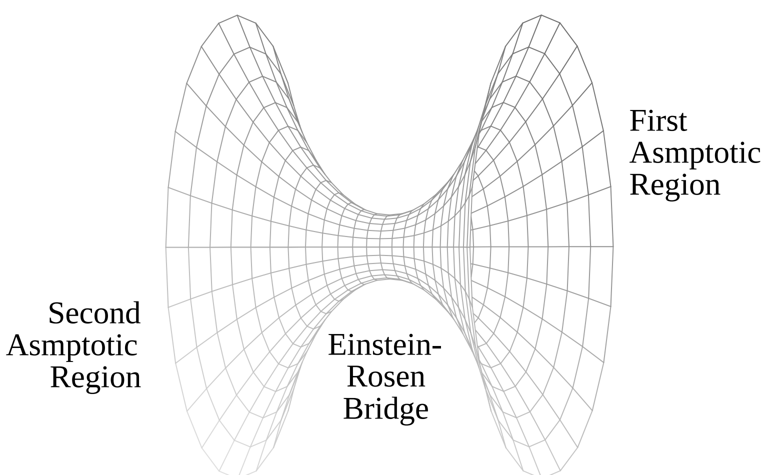
We have discovered a very important point: although the black hole spacetime is independent of time on the outside, it *does* in fact change with time on the inside. On the inside the only symmetry is one that relates different points in *space*, it says nothing about the relationship between events at different times.

Now, you might ask just how the spacetime changes in time. Recall that on one of the hyperbolae drawn above (in the future interior region) there is a symmetry that relates all of the points in space. Also recall that the full spacetime is 3+1 dimensional and that for every point on the diagram above the real spacetime contains an entire sphere of points. Even inside the horizon, the spacetime is spherically symmetric Now, the fact the points on our hyperbola are related by a symmetry means that the spheres are the same size (r) at each of these points! What changes as we move from one hyperbola to another (‘as time passes’) is that the size of the spheres (r) decreases. This is ‘why’ everything must move to smaller r inside the black hole – the whole spacetime is shrinking!

To visualize what this means, it is useful to draw a picture of the curved *space* of a black hole at some time. You began this process on a recent homework assignment when you considered a surface of constant t ($dt = 0$) and looked at circumference (C) vs. radius (R) for circles in this space[6]. You found that the space was not flat, but that the size of the circles changed more slowly with radius than in flat space. One can work out the details for any constant t slice in the exterior (since the symmetry means that they are all the same!). Two such slices are shown below. Note that they extend into both the 'right exterior' with which we are familiar and the 'left exterior', a region about which we have so far said little[7].



Ignoring, say, the θ direction and drawing a picture of r and ϕ (at the equator, $\theta = \pi/2$), any constant t slice looks like this:



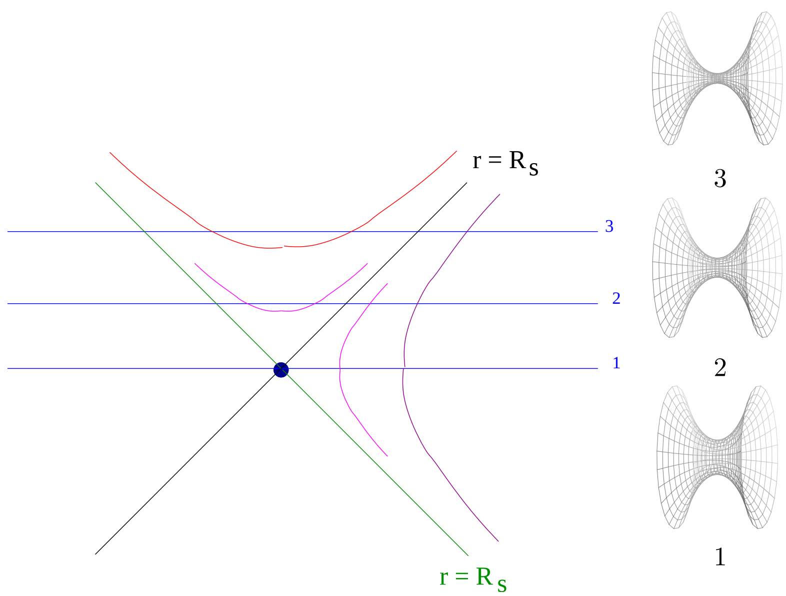
This is the origin of the famous idea that black holes can connect our universe

[6]Note that here we consider a different notion of a 'moment of time' than we used above when used the surface of constant r *inside* the black hole. In general relativity (since there are no global inertial frames) the notion of a 'moment of time' can be very general. Any spacelike surface will do.

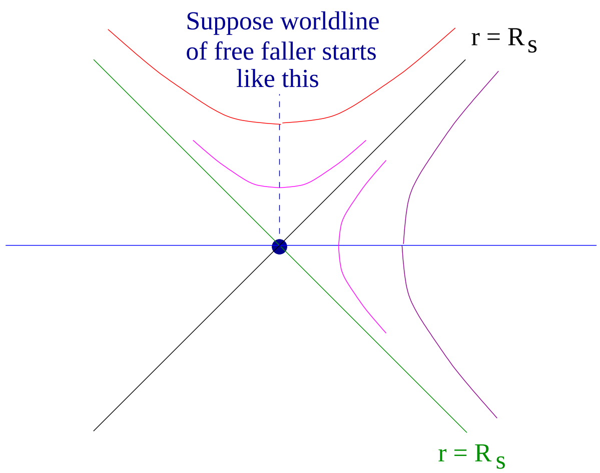
[7]The symmetry of the metric under $r' \to -r'$ (i.e., $t \to -t$) tells us that the left exterior region is a mirror image copy of the right exterior region.

(right exterior) to other universes (left exterior), or perhaps to some distant region of our own universe. If this idea bothers you, don't worry too much: as we will discuss later, the other end of the tunnel is not really present for the black holes commonly found in nature. Note that the left exterior looks just like the right exterior and represents another region 'outside' the black hole, connected to the first by a tunnel. This tunnel is called a 'wormhole,' or 'Einstein-Rosen bridge.'

So, what are these spheres inside the black hole that I said shrink with time? They are the 'throat' of the wormhole. Gravity makes the throat shrink, and begin to pinch off. That is, if we draw the shape of space on each of the slices numbered 0,1,2 below, they would look much like the Einstein-Rosen bridge above, but with narrower and narrower necks as we move up the diagram.



Does the throat ever pinch off completely? That is, does it collapse to $r = 0$ in a finite proper time? We can find out from the metric. Let's see what happens to a freely falling observer who falls from where the horizons cross (at $r = R_s$) to $r = 0$ (where the spheres are of zero size and the throat has collapsed). Our question is whether the proper time measured along such a worldline is finite. Consider an observer that starts moving straight up the diagram, as indicated by the dashed line in the figure below. We first need to figure out what the full worldline of the freely falling observer will be.



Will the freely falling worldline curve to the left or to the right? Recall that, since t is the space direction inside the black hole, this is just the question of whether it will move to larger t or smaller t. What do you think will happen?

Well, our diagram is exactly the same on the right as on the left, so there seems to be a *symmetry*. In fact, you can check that the Schwarzschild metric is unchanged if we replace t by $-t$. So, both directions must behave identically. If any calculation found that the worldline bends to the left, then there would be an equally valid calculation showing that the worldline bends to the right. As a result, the freely falling worldline will not bend in either direction and will remain at a constant value of t.

Now, how long does it take to reach $r = 0$? We can compute the proper time by using the freely falling worldline with $dt = 0$. For such a worldline the metric yields:

$$d\tau^2 = -ds^2 = \frac{dr^2}{R_s/r - 1} = \frac{r}{R_s - r} dr^2. \tag{9.18}$$

Integrating, we have:

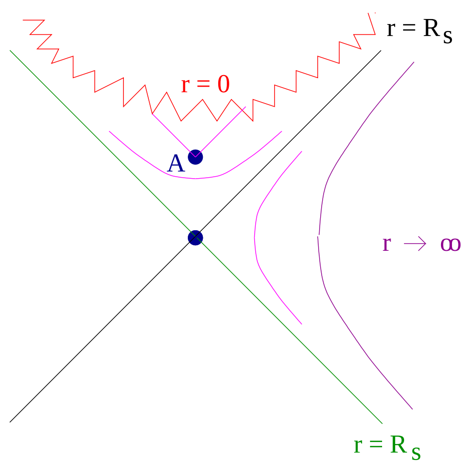
$$\tau = \int_{R_s}^{0} dr \sqrt{\frac{r}{R_s - r}}. \tag{9.19}$$

It is not important to compute this answer exactly. What is important is to notice that the answer is *finite*. We can see this from the fact that, near $r \approx R_s$ the integral is much like $\frac{dx}{\sqrt{x}}$ near $x = 0$. This latter integral integrates to $\sqrt{x}$ and is finite at $x = 0$. Also, near $r = 0$ the integral is much like $\frac{x}{R_s} dx$, which clearly gives a finite result. Thus, our observer measures a finite proper time between $r = R_s$ and $r = 0$ and the throat does collapse to zero size in finite time.

9.3.2 The Singularity

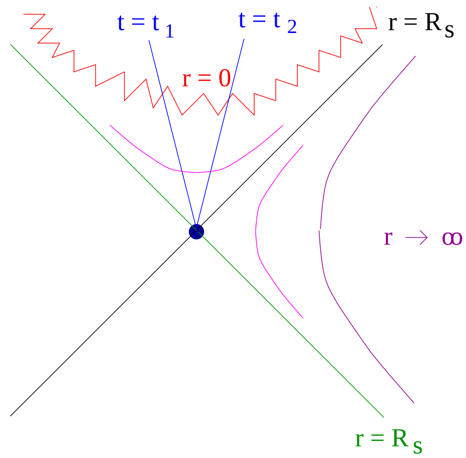
This means that we should draw the line $r = 0$ as one of the hyperbolae on our digram. It is clearly going to be a 'rather singular line' (to paraphrase Sherlock Holmes again), and we will mark it as special by using a jagged line. As you can see, this line is spacelike and so represents a certain *time*. We call this line the singularity. Note that this means that the singularity of a black hole is not a *place* at all!

The singularity is most properly thought of as being a very special *time*, at which the entire interior of the black hole squashes itself (and everything in it) to zero size. Note that, since it cuts all of the way across the future light cone of any events in the interior (such as event A below), there is no way for any object in the interior to avoid the singularity.



By the way, this is a good place to comment on what would happen to you if you tried to go from the right exterior to the left exterior through the wormhole. Note that, once you leave the right exterior, you are in the future interior region. From here, there is no way to get to the left exterior without moving faster than light. Instead, you will encounter the singularity. What this means is that the wormhole pinches off so quickly that even a light ray cannot pass through it from one side to the other. It turns out that this behavior is typical of wormholes.

Let's get a little bit more information about the singularity by studying the motion of two freely falling objects. As we have seen, some particularly simple geodesics inside the black hole are given by lines of constant t. I have drawn two of these (at t_1 and t_2) on the diagram below.



One question that we can answer quickly is how far apart these lines are at each r (say, measured along the line $r = const$). That is, "What is the proper length of the curve at constant r from $t = t_1$ to $t = t_2$?" Along such a curve, $dr = 0$ and we have $ds^2 = (R_s/r - 1)dt^2$. So, $s = (t_1 - t_2)\sqrt{R_s/r - 1}$. As $r \to 0$, the separation becomes infinite. Since a freely falling object reaches $r = 0$ in finite proper time, this means that any two such geodesics move *infinitely* far apart in a finite proper time. It follows that the relative acceleration (a.k.a. the gravitational tidal force) diverges at the singularity. (This means that the spacetime curvature also becomes infinite.) Said differently, it would take an infinite proper acceleration acting on the objects to make them follow (non-geodesic) paths that remain a finite distance apart. Physically, this means that it requires an infinite force to keep any object from being ripped to shreds near the black hole singularity.

9.3.3 Beyond the Singularity?

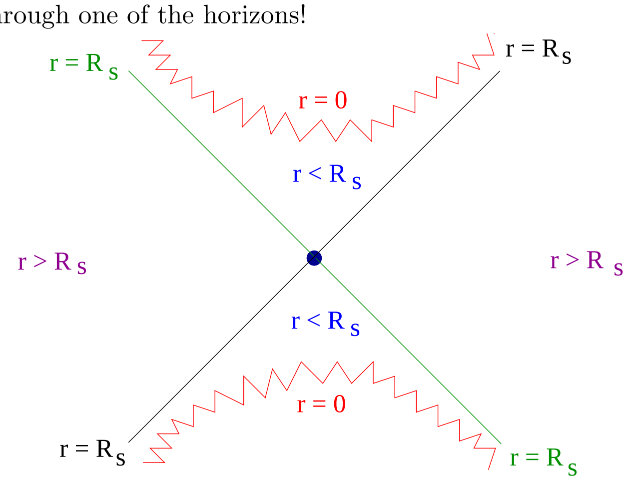
Another favorite question is "what happens beyond (after!) the singularity?" The answer is not at all clear. The point is that just as Newtonian physics is not valid at large velocities and as special relativity is valid only for very weak spacetime curvatures, we similarly expect General Relativity to be an incomplete description of physics in the realm where curvatures become truly enormous. This means that all we can really say is that a region of spacetime forms where the theory we are using (General Relativity) can no longer be counted on to correctly predict what happens.

The main reason to expect that General Relativity is incomplete comes from another part of physics called quantum mechanics. Quantum mechanical effects should become important when the spacetime becomes very highly curved. Roughly speaking, you can see this from the fact that when the curvature is strong local inertial frames are valid only over very tiny regions and from the fact the quantum mechanics is always important in understanding how very small things work. Unfortunately, no one yet understands just how quantum mechanics and gravity work together. We say that we are searching for a theory of "quantum gravity." It is a very active area of research that has led to a number of ideas, but as yet has no definitive answers. This is in fact the area of my own research.

Just to give an idea of the range of possible answers to what happens at a black hole singularity, it may be that the idea of spacetime simply ceases to be meaningful there. As a result, the concept of time itself may also cease to be meaningful, and there may simply be no way to properly ask a question like "What happens after the black hole singularity?" Many apparently paradoxical questions in physics are in fact disposed of in just this way (as in the question 'which is *really* longer, the train or the tunnel?'). In any case, one expects that the region near a black hole singularity will be a very strange place where the laws of physics act in entirely unfamiliar ways.

9.3.4 The rest of the diagram and dynamical holes

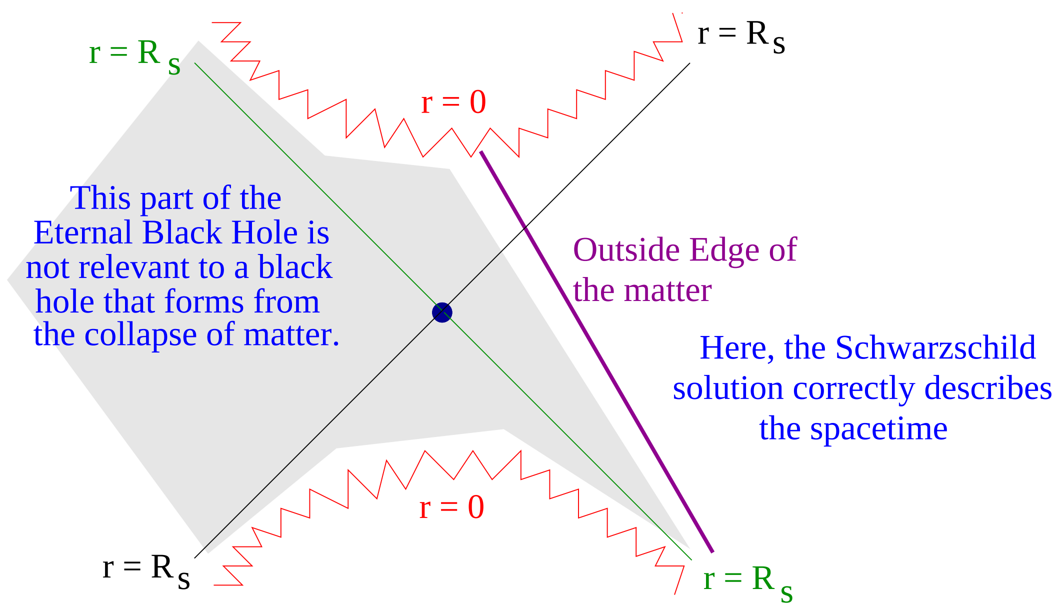
There still remains one region of the diagram (the 'past interior') about which we have said little. Recall that the Schwarzschild metric is time symmetric (under $t \to -t$). As a result, the diagram should have a top/bottom symmetry, and the past interior should be much like the future interior. This part of the spacetime is often called a 'white hole' as there is no way that any object can remain inside: everything must pass outward into one of the exterior regions through one of the horizons!



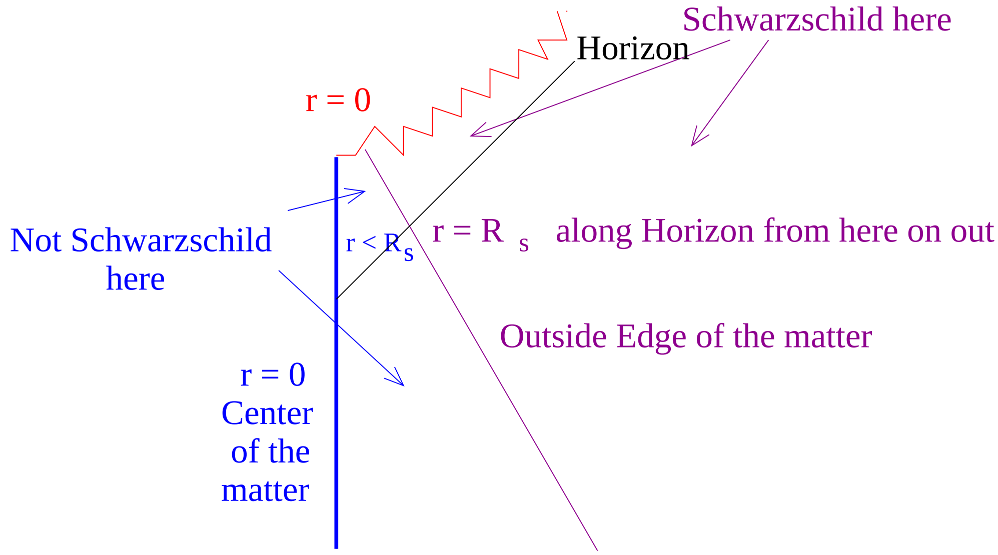
As we mentioned briefly with regard to the second exterior, the past interior does not really exist for the common black holes found in nature. Let's talk about how this works. So far, we have been studying the pure Schwarzschild solution. As we have discussed, it is only a valid solution in the region in which no matter is present. Of course, a little bit of matter will not change the picture much. However, if the matter is an important part of the story (for example, if it is matter that causes the black hole to form in the first place), then the modifications will be more important.

Let us notice that in fact the 'hole' (whether white or black) in the above spacetime diagram has existed since infinitely far in the past. If the Schwarzschild solution is to be used exactly, the hole (including the wormhole) must have been created at the beginning of the universe. We expect that most black holes were not created with the beginning of the universe, but instead formed later when too much matter came too close together. Recall that a black hole must form when, for example, too much thin gas gets clumped together.

Once the gas gets into a small enough region smaller than its Schwarzschild radius, we have seen that a horizon forms and the gas must shrink to a smaller size. No finite force (and, in some sense, not even infinite force) can prevent the gas from shrinking. Now, *outside* of the gas, the Schwarzschild solution should be valid. So, let me draw a worldline on our Schwarzschild spacetime diagram that represents the outside edge of the ball of gas. This breaks the diagram into two pieces: an outside that correctly describes physics outside the gas, and an inside that has no direct physical relevance and must be replaced by something that depends on the details of the matter:



We see that the 'second exterior' and the 'past interior' are in the part of the diagram with no direct relevance to relevance to black holes that form from collapsing matter. A careful study of the Einstein equations shows that, inside the matter, the spacetime looks pretty normal. A complete spacetime diagram including both then region inside the matter and the region outside would look like this:



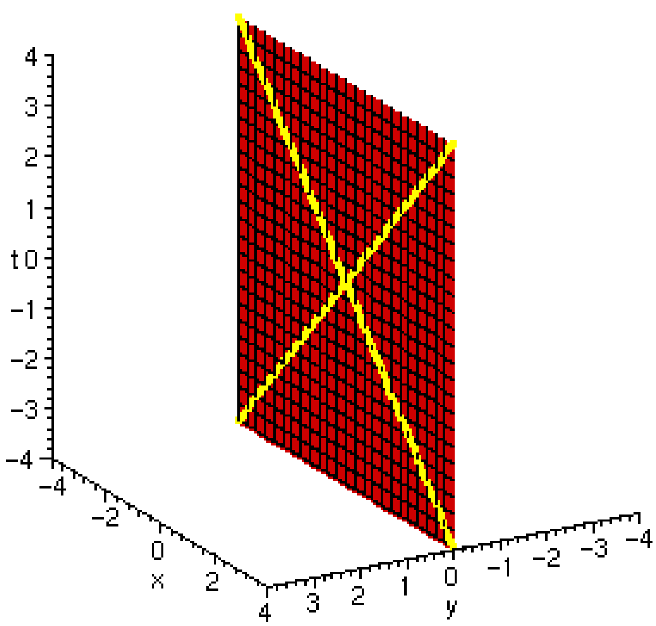
9.3.5 Visualizing black hole spacetimes

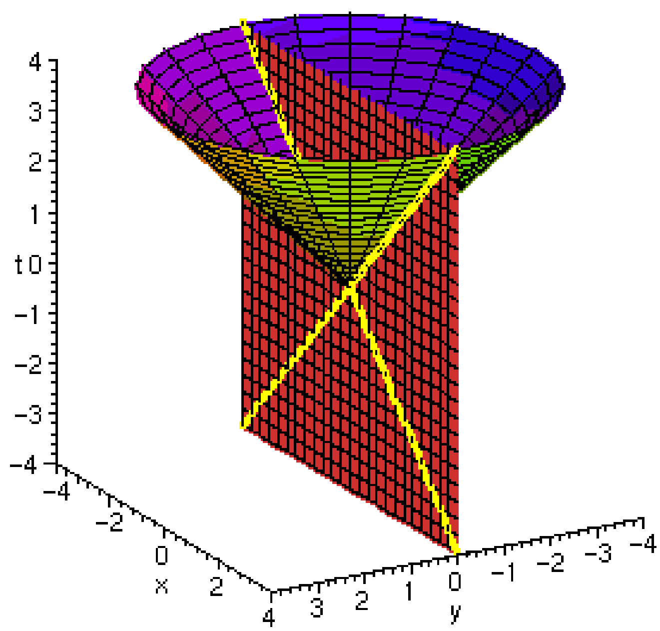
We have now had a fairly thorough discussion about Schwarzschild black holes include the outside, the horizon, the inside, and the "extra regions" (second exterior and past interior). One of the things that we emphasized was that the spacetime at the horizon of a black hole is locally flat, just like everywhere else in the spacetime. Also, the curvature at the horizon depends on the mass of the black hole. The result is that, if the black hole is large enough, the spacetime at the horizon is less curved than it is here on the surface of the earth, and a person could happily fall through the horizon without any discomfort (yet).

It is useful to provide another perspective on the various issues that we have discussed. The idea is to draw a few pictures that I hope will be illustrative. The point is that the black hole horizon is an effect caused by the curvature of spacetime, and the way that our brains are most used to thinking about curved spaces is to visualize them *inside* of a larger flat space. For example, we typically draw a curved (two-dimensional) sphere) as sitting inside a flat three-dimensional space.

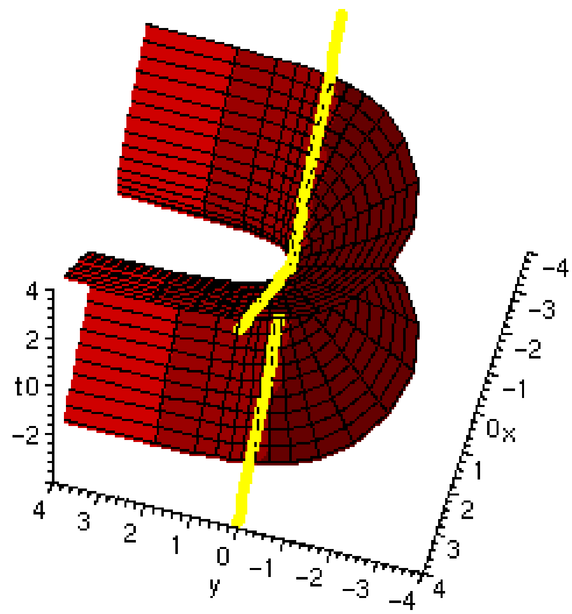
Now, the r, t plane of the black hole that we have been discussing and drawing on our spacetime diagrams forms a curved two-dimensional space*time*. It turns out that this two-dimensional spacetime can also be drawn as a curved surface inside of a flat *three*-dimensional space*time*. These are the pictures that we will draw and explore in this section.

To get an idea of how this works, let me first do something very simple: I will draw a *flat* two-dimensional spacetime inside of a flat three-dimensional spacetime. As usual, time runs up the diagram, and we use units such that light rays move along lines at 45^o angles to the vertical. Note that any worldline of a light ray in the 3-D spacetime that happens to lie entirely in the 2-D spacetime will also be the worldline of a light ray in the 2-D spacetime, since it is clearly a curve of zero proper time. A pair of such crossed light rays are shown below where the light cone of the 3-D spacetime intersects the 2-D spacetime.

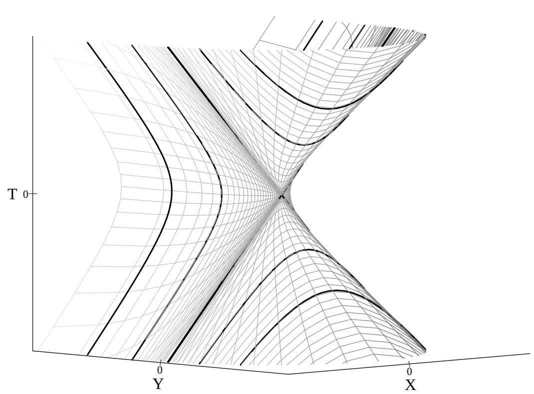


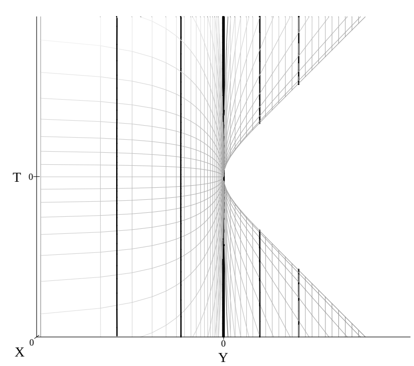


Now that we've got the idea, I'll show you a picture that represents the (2-D) r, t plane of our black hole, drawn as a curved surface inside a 3-D flat spacetime. It looks like this:

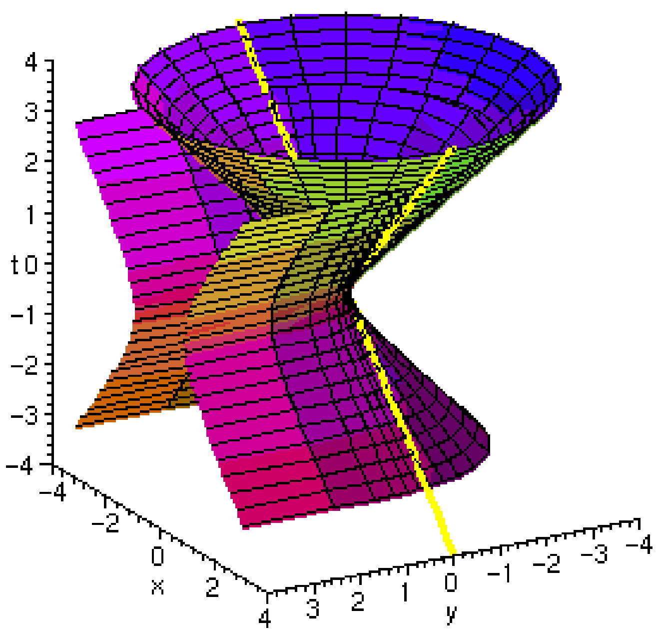


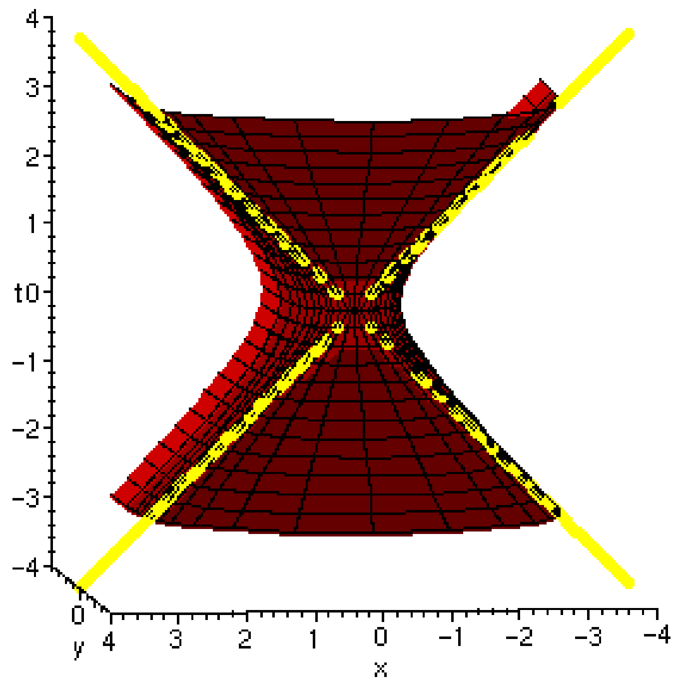
Below, I have drawn in the curves of constant r so that you can visualize them more easily. Note that larger r is farther from the center of the diagram, and in particular farther out along the 'flanges.' One flange represents the left exterior, and one represents the right exterior.





The most important thing to notice is that we can once again spot two lines that 1) are the worldlines of light rays in the 3D flat space and 2) lie entirely within the curved 2D surface. As a result, they again represent worldlines of light rays in the black hole spacetime. They are marked with yellow lines on the first picture I showed you (above) of the black hole spacetime and also on the diagrams below. *Note that they do not move at all outward toward larger values of r.* These are the horizons of the black hole.

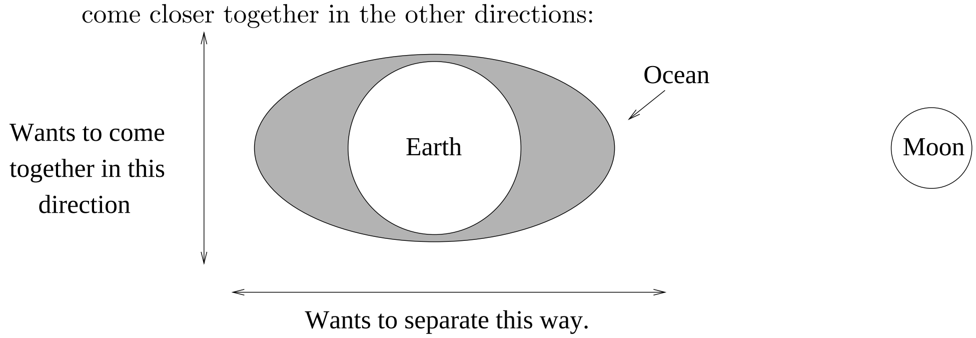




Another thing we can see from these diagrams is the symmetry we discussed. The symmetry of the 2-D black hole spacetime is the same as the boost symmetry of the larger 3D Minkowski space. Inside the black hole, this symmetry moves events in a spacelike direction. We can also see from this picture that, inside the black hole, the spacetime *does* change with time.

9.4 Stretching and Squishing: Tidal effects in General Relativity

We have now seen several manifestations of what are called 'tidal effects' in general relativity, where gravity by itself causes the stretching or squashing of an object. We discussed these a lot in homework problems 1 and 2, but even earlier our most basic observation in general relativity was that gravity causes freely falling observers to accelerate *relative to each other*. That is to say that, on a spacetime diagram, freely falling worldlines may bend toward each other or bend away. In problem 1 and 2 I asked how you thought this effects the ocean around the earth as the earth falls freely around the moon. The answer was that it *stretches* the ocean in the direction pointing toward (and away from) the moon, while it *squishes* (or compresses) the ocean in the perpendicular directions. This is because different parts of the ocean would like to separate from each other along the direction toward the moon, while they would like to come closer together in the other directions:



As stated in the homework solutions, this effect is responsible for the tides in the earth's oceans. (You know: if you stand at the beach for 24 hours, the ocean level rises, falls, then rises and falls again.) Whenever gravity causes freely falling observers (who start with no relative velocity) to come together or to separate, we call this a *tidal effect.* As we have seen, tidal effects are the fundamental signature of spacetime curvature, and in fact tidal effects are a direct measure of spacetime curvature.

Of course any other object (a person, rocket ship, star, etc.) would feel a similar stretching or squishing in a gravitational field. Depending on how you are lined up, your head might be trying to follow a geodesic which would cause it to separate from your feet, or perhaps to move closer to your feet. If this effect were large, it would be quite uncomfortable, and could even rip you into shreds (or squash you flat).

However, as we have seen, this effect is generally small for geodesics that are close together: if the stones in problem 1 were only as far apart as your head and feet are, the effect would have been completely negligible! So, unless the spacetime curvature is really big, it usually does not harm small objects like people – only large objects like planets and stars.

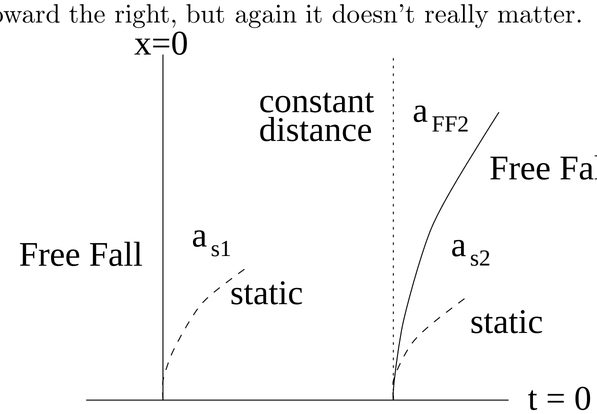
On the other hand, we argued that this tidal effect will become *infinitely* large at the singularity of a black hole. There the effect certainly will be strong enough to rip apart even tiny objects like humans, or cells, or atoms, or even subatomic particles. It is therefore of interest to learn how to compute how strong this effect actually is. We know that it is small far away from a black hole and that it is large at the singularity, but how big is it at the horizon? This last question is the key to understanding what you would *feel* as you fell through the horizon of a Schwarzschild black hole.

These are exactly the questions we're going to ask (and answer) in the current section. We will investigate tidal effects in more detail and discover how to compute them directly from the metric. Before we begin, I should warn you that this will be one of the more technical sections of these notes as the derivations are quite involved. It is sufficiently technical that I will almost certainly not go through this in class. You will find, however, that the results of this section are essential for doing certain homework problems. You should read through this section but, if you find yourself becoming lost in the details of the argument, you may want to skip ahead to subsections 9.4.4 and 9.4.5 which will give you an overview of the results.

9.4.1 The setup

So, let's suppose that somebody tells us what the spacetime metric is (for example, it might be the Schwarzschild metric). For convenience, let's suppose that it is independent of time and spherically symmetric. In this case, we discussed in class how to find the acceleration of static observers relative to freely falling observers who are at the same event in spacetime. What we are going to do now is to use this result to compute the relative acceleration of two neighboring freely falling observers.

To start with, let's draw a spacetime diagram in the reference frame of one of the freely falling observers. What this means is that lines drawn straight up (like the dotted one below) represent curves that remain a constant distance away from our first free faller. If you followed Einstein's discussion, this is what he would call a 'Gaussian' coordinate system. We want our two free fallers to start off with the same velocity – this is analogous to using 'initially parallel geodesics'. For the sake of argument, let's suppose that the geodesics separate as time passes, though the discussion is exactly the same if they come together. The freely falling observers are the solid lines, and the static observers are the dashed lines. To be concrete, I have chosen the static observers to be accelerating toward the right, but again it doesn't really matter.



The coordinate x measures the distance from the first freely falling observer.

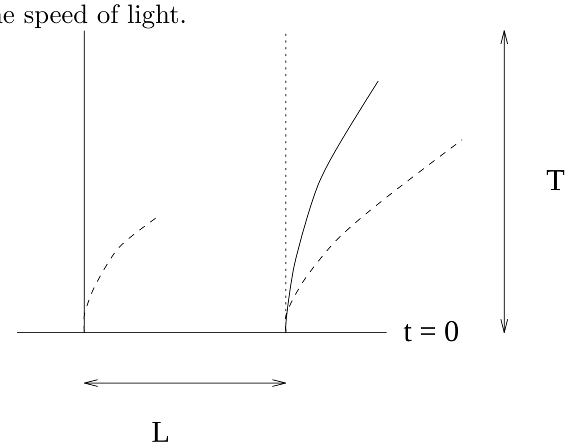
What we would like to know is how fast the second geodesic is accelerating away from the first. Let us call this acceleration a_{FF2}, the acceleration of the second free faller. Since we are working in the reference frame of the first free faller, the corresponding acceleration a_{FF1} is identically zero.

Now, what we already know is the acceleration of the two static observers *relative to* the corresponding free faller. In other words, we know the acceleration a_{s1}

of the first static observer relative to the first free faller, and we know the acceleration a_{s2} of the second static observer relative to the second free faller. Note that the *total* acceleration of the second static observer in our coordinate system is $a_{FF2} + a_{s2}$ – her acceleration relative to the second free faller plus the acceleration of the second free faller in our coordinate system. This is represented pictorially on the diagram above.

Actually, there is something else that we know: since the two static observers are, well, *static*, the proper distance between them (as measured by *them*) can never change. We will use this result to figure out what a_{FF2} is.

The way we will proceed is to use the standard Physics/Calculus trick of looking at *small* changes over *small* regions. Note that there are two parameters (T and L, as shown below) that tell us how big our region is. L is the distance between the two free fallers, and T is how long we need to watch the system. We will assume that both L and T are very small, so that the accelerations a_{s1} and a_{s2} are not too different, and so that the speeds involved are all much slower than the speed of light.

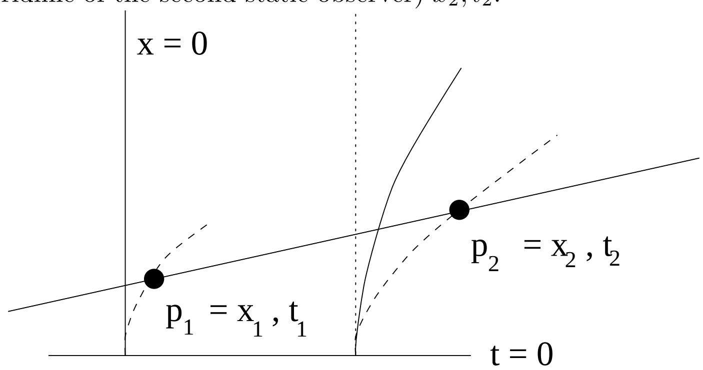


Now, pick a point (p_1) on the worldline of the first static observer. Call the coordinates of that point x_1, t_1. (We assume $t_1 < T$.) Since the velocity is still small at that point, we can ignore the difference between acceleration and proper acceleration and the Newtonian formula:

$$x_1 = \frac{1}{2} a_{s1} t_1^2 + O(T^4) \tag{9.20}$$

is a good approximation. The notation $O(T^4)$ is read "terms of order T^4." This represents the error we make by using only the Newtonian formula. It means that the errors are proportional to T^4 (or possibly even smaller), and so become much smaller than the term that we keep (t_1^2) as $T \to 0$. Note that since this is just a rough description of the errors, we can use T instead of t_1.

Recall that the two static observers will remain a constant distance apart *as determined by their own measurements.* To write this down mathematically, we need to understand how these observers measure distance. Recall that any observer will measure distance along a line of simultaneity. I have sketched in this line of simultaneity below, and called the point p_2 (where it intersects the worldline of the second static observer) x_2, t_2.



Now, since spacetime is curved, this line of simultaneity need not be perfectly straight on our diagram. However, we also know that, in a very small region near the first Free Faller (around whom we drew our diagram), space is approximately flat. This means that the curvature of the line of simultaneity has to vanish near the line $x = 0$. Technically, the curvature of this line (the second derivative of t with respect to x) must itself be 'of order $(x_2 - x_1)$. This means that p_1 and p_2 are related by an equation that looks like:

$$\begin{aligned}\frac{t_2 - t_1}{x_2 - x_1} &= \text{slope at } \mathrm{p_1} \; + \; [\text{curvature at } \mathrm{p_1}] \; (\mathrm{x_2 - x_1}) + \mathrm{O}([\mathrm{x_2 - x_1}]^2) \\ &= \text{slope at } \mathrm{p_1} \; + \mathrm{O}([\mathrm{x_2 - x_1}]^2) + \mathrm{O}(\mathrm{T}^2[\mathrm{x_2 - x_1}]). \qquad (9.21)\end{aligned}$$

Again, we need only a rough accounting of the errors. As a result, we can just call the errors $O(L^2)$ instead of $O([x_2 - x_1]^2)$.

Remember that, in flat space, the slope of this line of simultaneity would be v_{s1}/c^2, where v_{s1} is the velocity of the first static observer. Very close to $x = 0$, the spacetime can be considered to be flat. Also, as long as t_1 is small, the point p_1 *is* very close to $x = 0$. So, the slope at p_1 is essentially v_{s1}/c^2. Also, for small t_1 we have $v_{s1} = a_{s1}t$. Substituting this into the above equation and including the error terms yields

$$\begin{aligned} t_2 &= t_1 + (a_{s1}t_1/c^2)(x_2 - x_1) + O(L^3) \\ &= t_1 \left(1 + \frac{a_{s1}}{c^2}(x_2 - x_1)\right) + O(L^3) + O(T^2 L), \qquad (9.22)\end{aligned}$$

OK, we're making progress here. We've already got two useful equations written down (9.20 and 9.23), and we know that a third will be the condition that the proper distance between p_1 and p_2 will be the same as the initial separation L between the two free fallers:

$$L^2 = (x_2 - x_1)^2 - c^2(t_2 - t_1)^2. \qquad (9.23)$$

In addition, there is clearly an analogue of equation (9.20) for the second static observer (remembering that the second one does not start at $x = 0$, but instead starts at $x = L$):

$$x_2 = L + \frac{1}{2}(a_{s2} + a_{FF2})t_2^2 + O(T^4). \tag{9.24}$$

9.4.2 The solution

So, let's try using these equations to solve the problem at hand. The way the I will proceed is to substitute equation (9.22) for t_2 in equation (9.24). That way we express both positions in terms of just t_1. The result is

$$x_2 = L + \frac{1}{2}(a_{s2} + a_{FF2})\left(1 + \frac{a_{s1}}{c^2}(x_2 - x_1)\right)^2 t_1^2 + O(T^4) + O(L^3T^2). \tag{9.25}$$

OK, now we will want to use the condition (9.23) that the proper distance between the static observers does not change. This equation involves the difference $x_2 - x_1$. Subtracting equation (9.20) from equation (9.25), we get:

$$\begin{aligned} x_2 - x_1 &= L + \frac{1}{2}(a_{s2} + a_{FF2} - a_{s1})t_1^2 + (a_{s2} + a_{FF2})\frac{a_{s1}}{c^2}(x_2 - x_1)t_1^2 \\ &+ \frac{1}{2}(a_{s2} + a_{FF2})\frac{a_{s1}^2}{c^4}(x_2 - x_1)^2 t_1^2 + O(T^4) + O(L^3T^2). \end{aligned} \tag{9.26}$$

And, actually, we won't need to keep the $(x_2 - x_1)^2$ term, so we can write this as:

$$x_2 - x_1 = L + \frac{1}{2}(a_{s2} + a_{FF2} - a_{s1})t_1^2 + (a_{s2} + a_{FF2})\frac{a_{s1}}{c^2}(x_2 - x_1)t_1^2 + O(T^4) + O(L^2T^2). \tag{9.27}$$

Now, this equation involves $x_2 - x_1$ on both the left and right sides, so let's solve it for $x_2 - x_1$. As you can check, the result is:

$$x_2 - x_1 = \left(L + \frac{1}{2}(a_{s2} + a_{FF2} - a_{s1})t_1^2\right)\left(1 - (a_{s2} + a_{FF2})\frac{a_{s1}}{c^2}t_1^2\right)^{-1} + O(T^4) + O(T^2L^2). \tag{9.28}$$

But there is a standard 'expansion' $(1 - x)^{-1} = 1 + x + O(x^2)$ that we can use to simplify this. We find:

$$x_2 - x_1 = L + \frac{1}{2}(a_{s2} + a_{FF2} - a_{s1})t_1^2 + L(a_{s2} + a_{FF2})\frac{a_{s1}}{c^2}t_1^2 + O(T^4) + O(T^2L^2). \tag{9.29}$$

Believe it or not, we are almost done!!!! All we have to do now is to substitute this (and also equation 9.22 for the times) into the requirement that $\Delta x^2 - c^2\Delta t^2 = L^2$. Below, we will only keep terms up through T^2 and L^2. Note that:

$$(x_2-x_1)^2 = L^2+L(a_{s2}+a_{FF2}-a_{s1})t_1^2+2L^2(a_{s2}+a_{FF2})\frac{a_{s1}}{c^2}t_1^2+O(T^4)+O(T^2L^3) \tag{9.30}$$

while

$$(t_2-t_1)^2 = t_1^2 a_{s1}^2 L^2/c^2 + O(L^3T^2). \tag{9.31}$$

So, since the proper distance between p_1 and p_2 must be L^2,

$$\begin{aligned} L^2 &= \Delta x^2 - c^2\Delta t^2 \\ &= L^2 + L(a_{s2}+a_{FF2}-a_{s1})t_1^2 + L^2(2a_{s2}+2a_{FF2}-a_{s1})\frac{a_{s1}}{c^2}t_1^2 \\ &+ O(T^4)+O(T^2L^3). \end{aligned} \tag{9.32}$$

Canceling the L^2 terms on both sides leaves only terms proportional to $t_1^2 L$. So, after subtracting the L^2, let's also divide by $t_1^2 L$. This will leave:

$$0 = (a_{s2}+a_{FF2}-a_{s1}) + L(2a_{s2}+2a_{FF2}-a_{s1})\frac{a_{s1}}{c^2} + O(T^2/L) + O(L^2). \tag{9.33}$$

Reminder: What we want to do is to solve for a_{FF2}, the acceleration of the second free faller. In preparation for this, let's regroup the terms above to collect things with a_{FF2} in them:

$$0 = a_{FF2}(1+2La_{s1}/c^2) + (a_{s2}-a_{s1}) + L(2a_{s2}-a_{s1})\frac{a_{s1}}{c^2} + O(T^2/L) + O(L^2). \tag{9.34}$$

Now, before we solve for a_{FF2}, I want to make one more simplification. Remember that we started off by assuming that the region was very small. If it is small enough, then in fact a_{s1} and a_{s2} are not very different. In fact, we will have $a_{s1} - a_{s2} = O(L)$. This simplifies the last term a lot since $L(2a_{s2}-a_{s1}) = La_{s1} + O(L^2)$. Using this fact, and solving the above equation for a_{FF2} we get:

$$\begin{aligned} a_{FF2} &= -\frac{a_{s2}-a_{s1}+La_{s1}^2/c^2}{1+La_{s1}/c^2} + O(T^2/L) + O(L^2) \\ &= -(a_{s2}-a_{s1}) - La_{s1}^2/c^2 + O(T^2/L) + O(L^2). \end{aligned} \tag{9.35}$$

9.4.3 The Differential equation

Whew!!!!!!!!!!! Well, that's the answer. But now we want to convert it into a more useful form which will apply without worrying about whether our region is small. What we're going to do is to take the limit as T and L go to zero and turn this into a differential equation. Technically, we will take T to zero faster that L so that $T^2/L^2 \to 0$. Note that we are really interested in how things change with position at $t = 0$, so that is is natural to take T to zero before taking L to zero.

OK, so imagine not just two free fallers, but a whole set of them at every value of x. Each of these starts out with zero velocity, and each of them has an accompanying static observer. The free faller at x will have some acceleration $a_{FF}(x)$, and the static observer at x will have some acceleration $a_s(x)$ relative to the corresponding free faller. If L is very small above, notice that $a_{s2} - a_{s1} = L\frac{da_s}{dx} + O(L^2)$ and that (since $a_{FF1} = 0$), $a_{FF2} = L\frac{da_{FF}}{dx} + O(L^2)$.

So, we can rewrite equation (9.35) as:

$$L\frac{da_{FF}}{dx} = -L\frac{da_s}{dx} - La_s^2/c^2 + O(T^2/L) + O(L^2). \tag{9.36}$$

We can now divide by L and take the limit as T/L and L go to zero. The result is a lovely differential equation:

$$\frac{da_{FF}}{dx} = -\frac{da_s}{dx} - a_s^2/c^2 \tag{9.37}$$

By the way, the important point to remember about the above expression is that the coordinate x represents *proper distance.* (Sound familiar?)

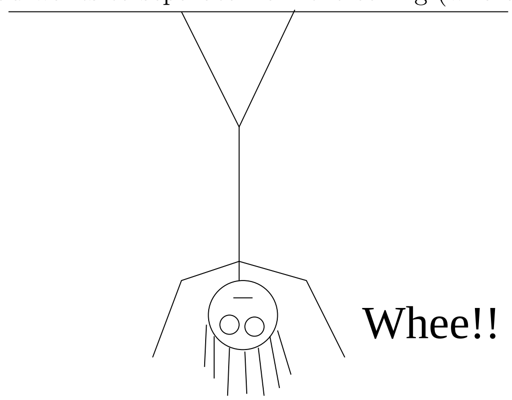
9.4.4 What does it all mean?

One of the best ways to use this equation is to undo part of the last step. Say that you have two free falling observers close together that have no initial velocity. Then, if their separation L is small enough, their relative acceleration is $L\frac{da_{FF}}{dx}$ or

$$\text{relative acceleration } = -L\left(\frac{da_s}{dx} + a_s^2/c^2\right). \tag{9.38}$$

Let's take a simple example of this. Suppose that you are near a black hole and that your head and your feel are both freely falling objects. Then, this formula tells you at what acceleration your head would separate from (or, perhaps, accelerate toward) your feet. Of course, your head and feet are not, in reality, separate freely falling objects. The rest of your body will pull and push on them to keep your head and feet roughly the same distance apart at all times. However, your head and feet will *want* to separate or come together, so depending on how big the relative acceleration is, keeping your head and feet in the proper places will cause a lot of stress on your body.

For example, suppose that the relative acceleration is $10m/s^2$ (1g) away from each other. In that case, the experience would feel much like what you feel if you tie your legs to the ceiling and hang upside down. In that case also, your head wants to separate from the ceiling (where your feet are) at $10m/s^2$.



However, if the relative acceleration we a lot bigger, it would be *extremely* uncomfortable. In fact, a good analogy with the experience would be being on a Medieval rack – an old torture device where they pulled your arms one way and your feet in the opposite direction. If the relative acceleration were great enough, I'm afraid that your head would in fact separate from your feet.

9.4.5 Black Holes and the Schwarzschild Metric

OK, so how big is this near a black hole? Well, remember that the acceleration of static observers (relative to freely falling observers) in the Schwarzschild metric is given by:

$$a_s = \frac{c^2}{2}\left(\frac{R_s}{r^2}\right)(1 - R_s/r)^{-1/2}. \tag{9.39}$$

We would like to take the derivative of this with respect to the proper distance S in the radial direction. That is, we will work along a line of constant t, ϕ, and θ. In this case, as we have seen before,

$$\frac{dr}{dS} = \sqrt{1 - R_s/r}. \tag{9.40}$$

So,

$$\frac{da_s}{dS} = (\sqrt{1 - R_s/r})\frac{da_s}{dr}. \tag{9.41}$$

A bit of computation yields

$$\frac{da_s}{dS} = -c^2\left(\frac{R_s}{r^3}\right) - \frac{c^2}{4}\left(\frac{R_s}{r^2}\right)^2(1 - R_s/r)^{-1}. \tag{9.42}$$

On the other hand, we have:

$$a_s^2/c^2 = \frac{c^2}{4}\left(\frac{R_s}{r^2}\right)^2 (1 - R_s/r)^{-1}. \tag{9.43}$$

To evaluate the relative acceleration, equation (9.38) tells us to add these two results together. Clearly, there is a major cancellation and all that we have left is:

$$\text{relative acceleration} = \mathrm{c}^2\left(\frac{\mathrm{R_s}}{\mathrm{r}^3}\right)\mathrm{L}. \tag{9.44}$$

This gives the relative acceleration of two freely falling observers who, at that moment, are at rest with respect to the static observers. (The free fallers are also located at radius r and are separated by a radial distance L, which is much smaller than r.) The formula holds anywhere that the Schwarzschild metric applies. In particular, anywhere outside a black hole.

Now for the question you have all been waiting for .... what happens at the horizon (or, perhaps just barely outside)? Well, this is just $r = R_s$. In this case, equation (9.44) reads

$$\text{relative acceleration} = \mathrm{c}^2\left(\frac{\mathrm{L}}{\mathrm{R_s^2}}\right). \tag{9.45}$$

The most important thing to notice about this formula is that the answer is *finite*. Despite the fact that a static observer at the horizon would need an infinite acceleration relative to the free fallers, any two free fallers have only a finite acceleration relative to each other.

The second thing to notice is that, for a *big* black hole (large R_s), this relative acceleration is even *small*. (However, for a small black hole, it can be rather large.) I'll let you plug in numbers on your own and see how the results come out. Have fun!

9.5 Black Hole Astrophysics and Observations

We have now come to understand basic round (Schwarzschild) black holes fairly well. We have obtained several perspectives on black hole exteriors and interiors and we have also learned about black hole singularities. However, there are several issues associated with black holes that we have yet to discuss. Not least of these is the observational evidence that indicates that black holes actually exist! This section will be devoted to this evidence and to the physics that surrounds it.

9.5.1 The observational evidence for black holes

We argued back in section 9.2.1 that big black holes should not be too hard to make. So, the question arises, *are* there really such things out there in the universe? If so, how do we find them? Black holes are *dark* after all, they themselves do not shine brightly like stars do.

Well, admittedly most of the evidence is indirect. Nevertheless, it is quite strong. Let's begin by reviewing the evidence for a black hole at the center of our own galaxy.

What is quite clear is that there is something massive, small, and dark at the center of our galaxy. Modern techniques allow us to make high resolution photographs of stars orbiting near the galactic center. In class, I will give you a handout contains copies of some of these photographs so that you can see the quality for yourselves. Note that one can track the motion of the objects directly through the photographs. One can also measure the velocities using the Doppler shift. The result is that we know a lot about the orbits of these objects, so that we can tell a lot about the mass of whatever object lies at the very center (marked with an * on one diagram in the handout) that they are orbiting around. Note: Along with other material on this subject, the photographs and diagrams in the handout come from a lovely talk about the status of black hole observations by Andrew Fabian (http://online.itp.ucsb.edu/online/bhole_c99/fabian/). Go ahead and look at this talk if you want to see the diagrams and photos before I hand them out.

What the data shows quite clearly is that there is a mass of 2.61×10^6 solar masses (M_O is the mass of the sun) contained in a region of size .02 parsecs (pc). Now, a parsec is around 3×10^16m. So, this object has a radius of less than $6 \times 10^{14}m$. In contrast, the Schwarzschild radius for a 2.61×10^6 solar mass object is around $10^{10}m$. So, what we get from direct observations of the orbits of stars is that this object is smaller than $10,000R_s$.

That may not sound like a small bound (since 10,000 is a pretty big number), but an important point is that an object of that mass at $r = 10,000R_s$ could not be very dense. If we simply divide mass by volume, we would find an average density of 10^{-9} that of water! We know an awful lot about how matter behaves at that density and the long and short of it is that the gravitational field of this object should make such a diffuse gas of stuff contract ...... You might then ask what happens when it becomes dense enough to form a solid. This brings us to another interesting observation....

It turns out that, at the very position at the center of our galaxy where the massive object (black hole?) should be located, a strong radio signal is being emitted. The source of this signal has been named "Sagittarius A^*" (Sgr A^*). It therefore natural to assume that this signal is coming from the massive object that we have been discussing. As we will see below, it is natural for radio signals to be emitted not from black holes themselves, but from things falling into black holes. Precision radio measurements using what is called "very-long baseline interferometry" (VLBI) tell us that the radio signal is coming from a small region. In terms of R_s for the mass we have been discussing, the region's size is about $30R_s$.

It therefore appears that the object itself is within $30R_s$. If the mass were spread uniformly over a volume of $30R_s$, it would have a density about three times greater than that of air. However, the proper acceleration (of static observers relative to freely falling ones) would be about 100g's. Again, we know a lot about how matter behaves under such conditions. In particular, we know that matter at that density behaves like a gas. However, the 100g acceleration means that the pressure in the gas must be quite high in order to keep the gas from collapsing. In particular, the pressure would reach one atmosphere about 1km inside the object. One hundred thousand km inside, the pressure would reach one hundred thousand atmospheres! Since we are thinking of an object of size $30R_s = 3 \times 10^{11}m$ (which is 300 million km), one hundred thousand km is less than .1% of the way to the center. So, the vast majority of the object is under much more than one hundred thousand (10^5) atmospheres of pressure. At 10^5 atmospheres of pressure, all forms of matter will have roughly the density of a solid. The matter supports this pressure by the electrons shells of the atoms bumping up against one another.

So, using what we know about matter, the object must surely be even smaller: small enough that have at least the density of water. Such an object (for this mass) would have a size of less than $3R_s$. So, we are getting very close. At the surface of such an object, the relative accelerations of freely falling and static observers would be around $10,000$g's. At a depth of $10,000km$ (again .1% of the way to the center), the pressure would be $10^{14}N/m^2$, or roughly one billion atmospheres. At this pressure, any kind of matter will compress to more than 30 times the density of water. So, again, we should redo the calculation, but now at 30 times the density of water.....

At this density, the object would be within its Schwarzschild radius. It would be a black hole. We conclude that we the experimental bounds and what we know about physics the object at the center of our galaxy either is a black hole already or is rapidly collapsing to become one. Oh, the time such an object would it take to collapse from $30R_s$ is about 15 minutes. Astronomers have been monitoring this thing for awhile, so I guess it's a black hole by now.

9.5.2 Finding other black holes

So, while the astronomical measurements do not directly tell us that Sagittarius A^* is a black hole, when combined with what we know about (more or less) ordinary matter, the conclusion that the object is a black hole is hard to escape. Much the same story is true for other "black hole candidates" as the astronomers call them. The word candidate is added to be intentionally conservative, but at this point I don't know of anyone who actually doubts that they are in fact black holes. Black hole candidates at the center of other galaxies are identified in much the same way that Sagittarius A^* was found. Astronomers study how stars orbit around those galactic centers to conclude that there is "massive compact object" near the center. Typically, such objects are also associated with strong emissions of radio waves.

Similar techniques are used for finding smaller black holes as well. The small black holes that we think we have located are in so-called 'binary systems.' The way that these black holes were found was that astronomers found certain stars which seemed to be emitting a lot of high energy x-rays. This is an unusual thing for a star to do, but it is not so odd for a black hole (as we will discuss shortly). On closer inspection of the star, it was found that the star appeared to "wobble" back and forth. This is just what the star would seem to do if it was in fact orbiting close to a small massive dark object that could not be seen directly. This is why they are called binary systems, since there seem to be two objects in the system. These massive dark objects have masses between 5 and 10 solar masses. Actually, there are also cases where the dark companion has a mass of less then 2 solar masses, but those are known to be neutron stars (see below).

We had a discussion in the previous section about how our knowledge of normal matter led to the conclusion that Sagittarius A^* is a black hole. Well, we also have a pretty good idea of how star-like objects work in the solar mass range. In actual stars, what happens is that the objects become dense enough that nuclear fusion occurs. This generates large amounts of heat that increases the pressure in the matter (remember the ideal gas law?) far above what it would be otherwise. It is this pressure that keeps the object from collapsing to higher density. Thus, the reason that a star has a relatively low density (the average density of the sun is a few times that of water) is that it is very hot! This of course is also the reason that stars shine.

Now, the dark companions in the binary systems do not shine. It follows that they are *not* hot. As a result, they must be much smaller and much more dense. Our understanding of physics tells us that massive cold objects will collapse under their own weight. In particular, a cold object greater than 1.4 times the mass of the sun will not be a star at all. It will be so dense that the electrons will be crushed into the atomic nuclei, with the result that they will be absorbed into the protons and electron + proton will turn into a neutron. Thus the object ceases to be normal matter (with electrons, protons, and neutrons) at all, but becomes just a big bunch of neutrons. This number of 1.4 solar masses is called the Chandrasekhar limit after the physicist who discovered it. In practice, when we look at the vast numbers of stars in the universe, we have never found a cold star of more than 1.4 solar masses though we have found some that are close.

So, any cold object of more than 1.4 solar masses must be at least as strange as a big bunch of neutrons. Well, neutrons can be packed very tightly without resistance, so that in fact such 'neutron stars' naturally have the density of an atomic nucleus. What this means is that one can think of a neutron star as being essentially one incredibly massive atomic nucleus (but will all neutrons and no protons).

The density of an atomic nucleus is a huge $10^{18} kg/m^3$. (This is 10^{15} times that of normal matter.) Let us ask: suppose we had a round ball of nuclear matter at this density. How massive would this ball need to be for the associated Schwarzschild radius to be larger than the ball itself? The answer is about 4 times the mass of the sun. So, working with a very simple model in which the density is constant (and always equal to the density of normal nuclei, which are under significantly less pressure) inside the object, we find that any cold object with a mass greater than four solar masses will be a black hole! It turns out that *any* model where the density increases with depth and pressure yields an even stronger bound. As a result, modern calculations predict that any cold object with a mass of greater than 2.1 solar masses will be a black hole.

As one can see from the observational data in the handout, the dark companions in the binary systems all have masses significantly greater than 2 solar masses. By the way, it is reassuring to note that every neutron star that has been found has been in the range between 1.4 and 2.1 solar masses.

9.5.3 A few words on Accretion and Energy

Even with the above arguments, one might ask what direct measurements could be made of the size of the dark companions. Can we show directly that their size is comparable to the Schwarzschild radius? To do so one needs to use the energy being released from matter falling into a black hole. This leads us to a brief discussion of what are called accretion disks.

The idea is shown on one of the pictures that I have handed out. In general, matter tends to flow into black holes. This addition of matter to an object is called "accretion." Black holes (and neutron stars) are very small, so that a piece of matter from far away that becomes caught in the gravitational field is not likely to be directed straight at the black hole or neutron star, but instead is likely to go into some kind of orbit around it. The matter piles up in such orbits and then, due to various interactions between the bits of matter, some bits slowly loose angular momentum and move closer and closer to the center. Eventually, they either fall through the horizon of the black hole or hit the surface of the neutron star.

In cases where the compact object is in a binary system, the matter flowing in comes mostly from the shining star. This process makes the accreting matter into a disk, as shown in the picture[8] below. This is why astronomers often talk about 'accretion disks' around black holes and neutron stars.

[8]This picture was created by Jillian Bornak, a past PHY312 student.

Now, an important point is that a lot of energy is released when matter falls toward a black hole. Why does this happen? Well, as an object falls, its speed relative to static observers becomes very large. When many such of matter bump into each other at high these speeds, the result is a lot of very hot matter! This is where those x-rays come from that I mentioned awhile back. The matter is hot enough that x-rays are emitted as thermal radiation.

By the way, it is worth talking a little bit about just how we can calculate the extra 'kinetic energy' produced when objects fall toward black holes (or neutron stars). To do so, we will run in reverse a discussion we had long ago about light falling in a gravitational field.

Do you recall how we first argued that there must be something like a gravitational time dilation effect? It was from the observation that a photon going upward through a gravitational field must loose energy and therefore decrease in frequency. Well, let's now think about a photon that falls *down* into a gravitational field from far away to a radius r. Recall that clocks at r run slower than clocks far away by a factor of $\sqrt{1 - R_s/r}$. Since the lower clocks run more slowly, from the viewpoint of these clocks the electric field of the photon seems to be oscillating very *quickly*. So, this must mean that the frequency of the photon (measured by a static clock at r) is higher by a factor of $1/\sqrt{1 - R_s/r}$ than when the frequency is measured by a clock far away. Since the energy of a photon is proportional to its frequency, the energy of the photon has increased by $1/\sqrt{1 - R_s/r}$.

Now, in our earlier discussion of the effects of gravity on light we noted that the energy in light could be turned into any other kind of energy and could then be turned back into light. We used this to argue that the effects of gravity on light must be the same as on any other kind of energy. So, consider an object of mass m which begins at rest far away from the black hole. It contains an energy $E = mc^2$. So, by the time the object falls to a radius r, its energy (measured locally) must have increased by the same factor was would the energy of a photon; to $E = mc^2 1/\sqrt{1 - R_s/r}$. What this means is that if the object gets anywhere even close to the Schwarzschild radius, it's energy will have increased by an amount comparable to its rest mass energy. Roughly speaking, this means that objects which fall toward a black hole or neutron star and collide with each other release energy on the same scale as a star or a thermonuclear bomb. This is the source of those x-rays and the other hard radiation that we detect from the accretion disk.

Actually, there is one step left in our accounting of the energy. After all, we don't sit in close to the black hole and measure the energy of the x-rays. Instead, we are far away. So, we also need to think about the energy that the x-rays loose as they climb back out of the black hole's gravitational field. To this end, suppose our object begins far away from the black hole and falls to r. As we said above, its energy is now $E = mc^2/\sqrt{1 - R_s/r}$. Suppose that the object now comes to rest at r. The object will then have an energy $E = mc^2$ as measured at r. So, stopping this object will have *released* an energy of

$$\Delta E = mc^2 \left(\frac{1}{\sqrt{1 - R_s/r}} - 1 \right). \tag{9.46}$$

as measured at r. This is how much energy can be put into x-ray photons and sent back out. But, on it's way back out, such photons will decrease in energy by a factor of $\sqrt{1 - R_s/r} - 1$. So, the final energy that gets out of the gravitational field is:

$$\begin{aligned} \Delta E_\infty &= mc^2 \sqrt{1 - R_s/r} \left(\frac{1}{\sqrt{1 - R_s/r}} - 1 \right) \\ &= mc^2 (1 - \sqrt{1 - R_s/r}). \end{aligned} \tag{9.47}$$

In other words, the total energy released to infinity is a certain fraction of the energy in the rest mass that fell toward the black hole. This fraction goes to 1 if the mass fell all the way down to the black hole horizon. Again, so long as r was within a factor of 100 or so of the Schwarzschild radius, this gives an efficiency comparable to thermonuclear reactions.

I can now say something about the question that we asked at the beginning of this section. Using direct observations, how strongly can we bound the size of a black hole candidate? It turns out that one can study the detailed properties of the spectrum of radiation produced by an accretion disk, and that one can match this to what one expects from an accretion disk living in the Schwarzschild geometry. Current measurements focus on a particular (x-ray) spectral line associated with iron. In the best case, the results show that the region emitting radiation is within $25 R_s$.

9.5.4 So, where does all of this energy go, anyway?

This turns out to be a very interesting question. There is a lot of energy being produced by matter falling into a black hole or a neutron star. People are working very hard with computer models to figure out just how much matter falls into black holes, and therefore just how much energy is produced. Unfortunately, things are sufficiently complicated that one cannot yet state results with certainty. Nonetheless, some very nice work has been done in the last few years by Ramesh Narayan and his collaborators showing that in certain cases there appears to be much less energy coming out than there is going in. Where is this energy going? It is not going into heating up the object or the accretion disk, as such effects would increase the energy that we see coming out (causing the object to shine more brightly). If their models are correct, one is forced to conclude that the energy is truly disappearing from the part of the spacetime that can communicate with us. In other words, the energy is falling behind the horizon of a black hole. As the models and calculations are refined over the next five years or so, it is likely that this missing energy will be the first 'direct detection' of the horizon of a black hole.

9.6 Black Hole Odds and Ends

Black holes are an enormous subject area and there are many parts of the story that we have not yet discussed. For most of these there is simply not enough time to address them in a one semester course. However, a few topics merit special mention either because of their common appearance in the popular media[9], because they will be useful in our discussion of cosmology in chapter 10, or because of their intrinsic interest. We will therefore be addressing Hawking radiation, Penrose diagrams, and more complicated types of black holes (more complicated than Schwarzschild black holes) in turn.

9.6.1 A very few words about Hawking Radiation

Strictly speaking, Hawking Radiation is not a part of this course because it does not fall within the framework of general relativity. However, since someone always asks about it, I feel the need to make a few very brief comments on the subject. Believe me, this is only the very tip of the iceberg!

Here's the story: do you recall that, when we discussed the black hole singularity, we said that what really happens there will not be described by general relativity? We mentioned that physicists expect a new and even more fundamental understanding of physics to be important there, and that the subject is called "quantum gravity." We also mentioned that very little is understood about quantum gravity at the present time.

Well, there is one thing that we think we do understand about quantum effects in gravity. This is something that happens *outside* the black hole and therefore *far* from the singularity. In this setting, the effects of quantum mechanics in the gravitational field itself are extremely small. So small that we believe that we can do calculations by simply splicing together our understanding of quantum mechanics (which governs the behavior of photons, electrons, and such things) and our understanding of gravity. In effect, use quantum mechanics together with the equivalence principle to do calculations.

Stephen Hawking did such a calculation back in the early 1970's. What he found came as a real surprise. Consider a black hole by itself, without an accretion disk or any other sort of obvious matter nearby. It turns out that the region around the black hole is not completely dark! Instead, it glows like a hot object, albeit at a very low temperature. The resulting thermal radiation is called Hawking radiation.

Now, I first want to say that this is an incredibly tiny effect. For a solar mass black hole the associated temperature is only 10^{-5} Kelvin, that is, 10^{-5} degrees above absolute zero. Large black holes are even colder, as the temperature is proportional to M^{-2}, where M is the black hole mass. So black holes are very, very cold. In particular, empty space has a temperature of about 3K due to

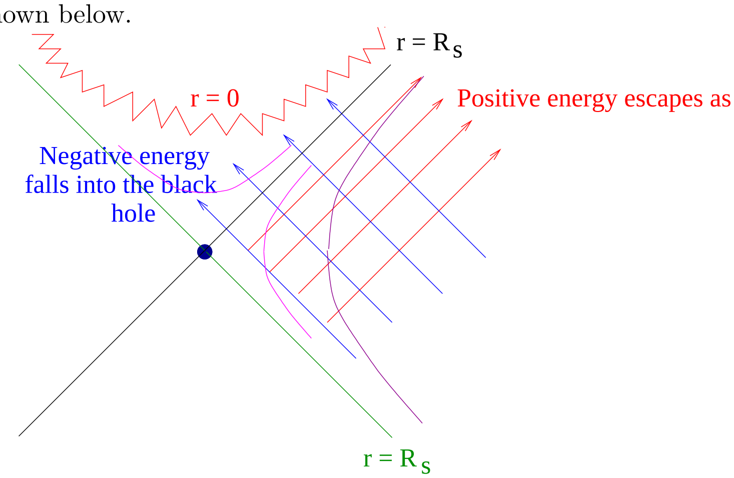
[9] So that you have likely heard of such things already or will encounter them in the future. I would like you to have at least some exposure to the real story about these things.

what is called the 'cosmic microwave background' (we will be talking about this soon), so a black hole is much colder than empty space. However, if one could make or find a very tiny black hole, that black hole would be very hot.

Second, let me add that the radiation does not come directly from the black hole itself, but from the space around the black hole. This is a common misconception about Hawking radiation: the radiation does not by itself contradict our statement that nothing can escape from within the horizon.

But, you may ask, how can radiation be emitted from the space around the black hole? How can there be energy created from nothing? The answer is that, in 'quantum field theory[10],' one can have negative energies as well as positive energies. However, these negative energies should always be very small and should survive only for a short time. What happens is that the space around the black hole produces a net zero energy, but it sends a positive energy flux of Hawking radiation outward away from the black hole while sending a negative energy flux inward across the horizon of the black hole. The negative energy is visible only for a short time between when it is created and when it disappears behind the horizon of the black hole.

The net effect is that the black hole looses mass and shrinks, while positive energy is radiated to infinity. A diagram illustrating the fluxes of energy is shown below.



9.6.2 Penrose Diagrams, or "How to put infinity in a box"

There are a few comments left to make about black holes, and this will require one further technical tool. The tool is yet another kind of spacetime diagram (called a 'Penrose diagram') and it will be useful both for discussing more complicated kinds of black holes *and* for discussing cosmology in chapter 10. Actually, it is not all that technical.

[10]Quantum field theory is what you get when you combine quantum mechanics and the idea that things of interest are fields, like the electric and magnetic fields. Quantum field theory is what is used to understand all of subatomic and particle physics, for example.

The point is that, as we have seen, it is often useful to compare what an observer very far from the black hole sees to what one sees close to the black hole. We say that an observer very far from the black hole is "at infinity." Comparing infinity with finite positions is even more important for more complicated sorts of black holes that we have not yet discussed. However, it is difficult to draw infinity on our diagrams since infinity is after all infinitely far away.

How can we draw a diagram of an infinite spacetime on a finite piece of paper? Think back to the Escher picture of the Lobachevskian space. By 'squishing' the space, Escher managed to draw the infinitely large Lobachevskian space inside a finite circle. If you go back and try to count the number of fish that appear along on a geodesic crossing the entire space, it turns out to be infinite. It's just that most of the fish are drawn incredibly small. Escher achieved this trick by letting the *scale* vary across his map of the space. In particular, at the edge an infinite amount of Lobachevskian space is crammed into a very tiny amount of Escher's map. In some sense this means that his picture becomes infinitely bad at the edge, but nevertheless we were able to obtain useful information from it.

We want to do much the same thing for our spacetimes. However, for our case there is one catch: *As usual, we will want all light rays to travel along lines at 45 degrees to the vertical.* This will allow us to continue to read useful information from the diagram. This idea was first put forward by (Sir) Roger Penrose[11], so that the resulting pictures are often called "Penrose Diagrams." They are also called "conformal diagrams" – conformal is a technical word related to the rescaling of size.

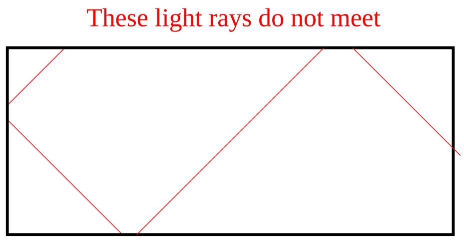
Let's think about how we could draw a Penrose diagram of Minkowski space. For simplicity, let's consider our favorite case of 1+1 dimensional Minkowski space. Would you like to guess what the diagram should look like?

As a first guess, we might try a square or rectangle. However, this guess has a problem associated with the picture below. To see the point, consider any light ray moving to the right in 1+1 Minkowski space, and also consider any light ray moving to the left. Any two such light rays are guaranteed to meet at some event. The same is in fact true of any pair of leftward and rightward moving objects since, in 1 space dimension, there is no room for two objects to pass each other!

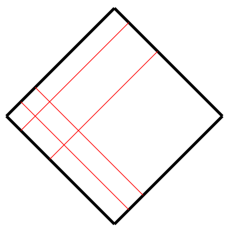
Left- and right- moving objects always collide when space has only one dimension

However, if the Penrose diagram for a spacetime is a square, then there are in fact leftward and rightward moving light rays that never meet! Some examples are shown on the diagram below.

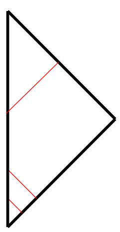
[11] Penrose is a mathematician and physicist who is famous for a number of things.



So, the rectangular Penrose diagram does not represent Minkowski space. What other choices do we have? A circle turns out to have the same problem. After a little thought, one finds that the only thing which behaves differently is a diamond:



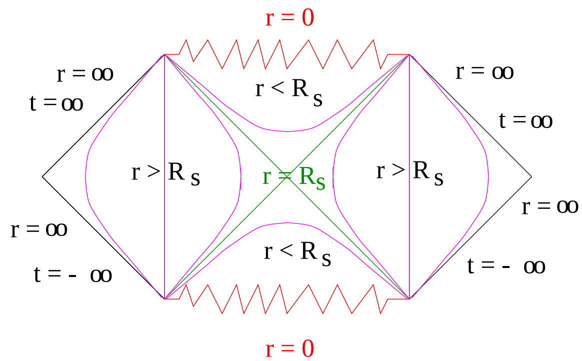
That is to say that infinity (or at least most of it) is best associated not which a place or a time, but with a set of light rays! In 3+1 dimensions, we can as usual decide to draw just the r, t coordinates. In this case, the Penrose diagram for 3+1 Minkowski space is drawn as a half-diamond:



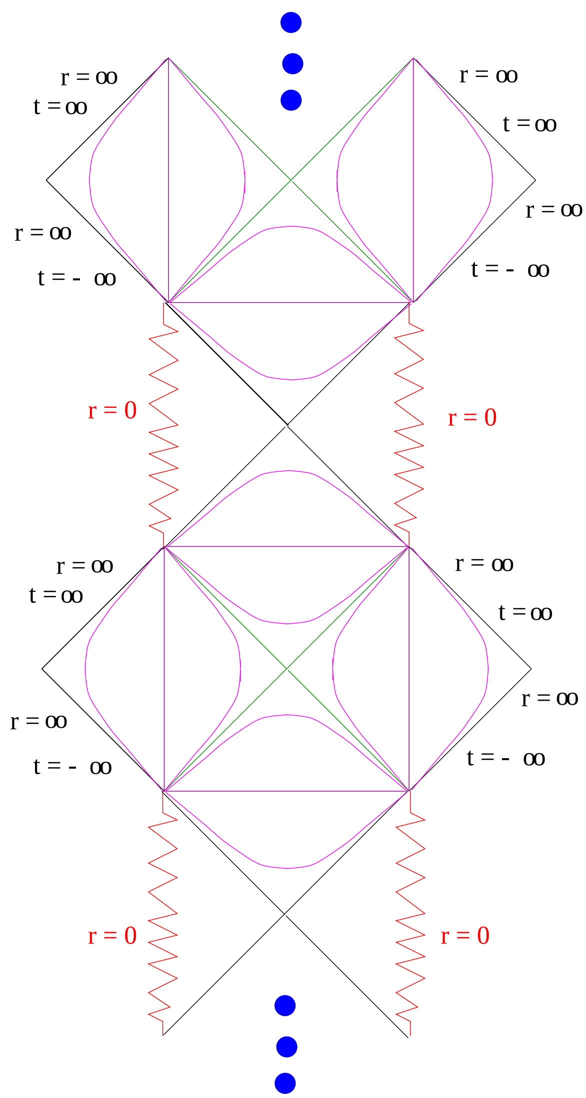
9.6.3 Penrose Diagrams for Black holes

Using the same scheme, we can draw a diagram that shows the entire spacetime for the eternal Schwarzschild black hole. Remember that the distances are no longer represented accurately. As a result, some lines that used to be straight get bent[12]. For example, the constant r curves that we drew as hyperbolae before appear somewhat different on the Penrose diagram. However, all light rays still travel along straight 45 degree lines. The result is:

[12]This is the same effect that one finds on flat maps of the earth where lines that are really straight (geodesics) appear curved on the map.



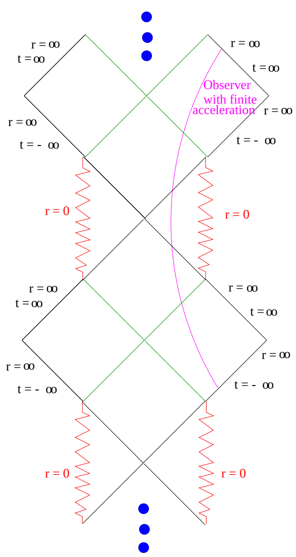
As you might guess, I did not introduce Penrose diagrams just to draw a new diagram for the Schwarzschild black hole. It turns out though that Schwarzschild black holes are not the only kind of black holes that can exist. Recall that the Schwarzschild metric was correct only outside of all of the 'matter' (which means anything other than gravitational fields) and only if the matter was spherically symmetric ('round'). Another interesting case to study occurs when we add a little bit of electric charge to a black hole. In this case, the charge creates an electric field which will fill all of space! This electric field carries energy, and so is a form of 'matter.' Since we can never get out beyond all of this electric field, the Schwarzschild metric by itself is never quite valid in this spacetime. Instead, the spacetime is described by a related metric called the Reissner-Nořdstr∅m (RN) metric. The Penrose diagram for this metric is shown below:



Actually, this is not the entire spacetime.... the dots in the diagram above indicate that this pattern repeats infinitely both to the future and to the past! This diagram has many interesting differences when compared to the Schwarzschild diagram. One is that the singularity in the RN metric is timelike instead of being spacelike. Another is that instead of there being only two exterior regions, there are now infinitely many!

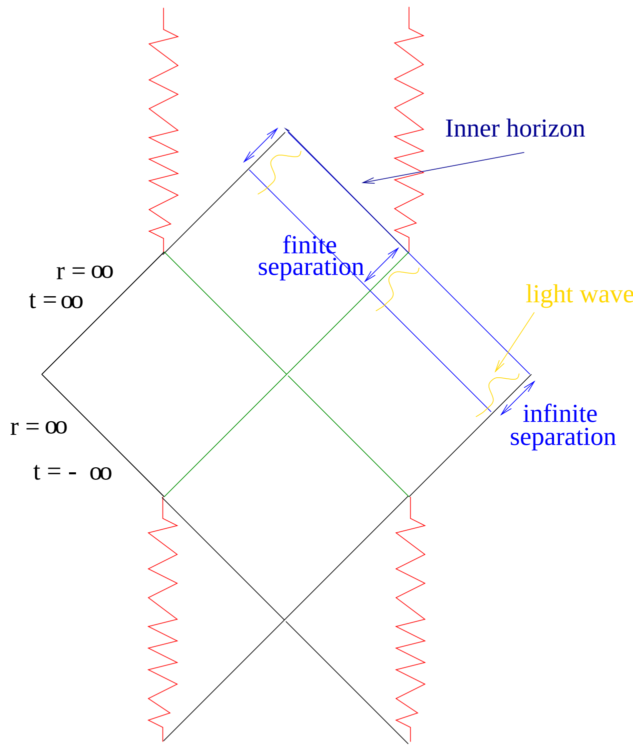
The most interesting thing about this diagram is that there does exist a timelike worldline (describing an observer that travels more slowly than light) that starts in one external region, falls into the black hole, and then comes back out through a 'past horizon' into another external region. Actually, is possible to consider the successive external regions as just multiple copies of the same external region.

In this case, the worldline we are discussing takes the observer back into the same universe but in such a way that they emerge to the past of when the entered the black hole!



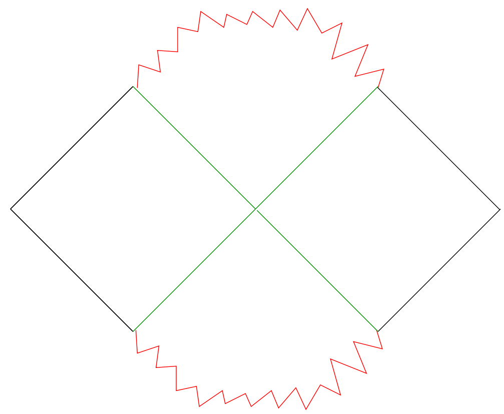
However, it turns out that there is an important difference between the Schwarzschild metric and the RN metric. The Schwarzschild metric is *stable*. This means that, while the Schwarzschild metric describes only an eternal black hole in a spacetime by itself (without, for example, any rocket ships near by carrying observers who study the black hole), the actual metric which would include rocket ships, falling scientists and students, and so on can be shown to be very close to the Schwarzschild metric. This is why we can use the Schwarzschild metric itself to discuss what happens to objects that fall into the black hole.

It turns out though that the RN metric does not have this property. The exterior is stable, but the interior is not. This happens because of an effect illustrated on the diagram below. Suppose that some energy (say, a light wave) falls into the black hole. From the external viewpoint this is a wave with a long wavelength and therefore represents a small amount of energy. The two light rays drawn below are in fact infinitely far apart from the outside perspective, illustrating that the wave has a long wavelength when it is far away.



However, inside the black hole, we can see that the description is different. Now the two light rays have a *finite* separation. This means that that near the light ray marked “inner horizon,” what was a long wavelength light ray outside is now of very short wavelength, and so very high energy! In fact, the energy created by any small disturbance will become infinite at the “inner horizon.” It will come as no surprise that this infinite energy causes a large change in the spacetime.

The result is that dropping even a small pebble into an RN black hole creates a big enough effect at the inner horizon to radically change the Penrose diagram. The Penrose diagram for the actual spacetime containing an RN black hole together with even a small disturbance looks like this:



Some of the researchers who originally worked this out have put together a nice readable website that you might enjoy. It is located at (http://www-theorie.physik.unizh.ch/ droz/inside/).

Actually, I have to admit that no one believes that real black holes in nature will have a significant electric charge. The point is that a black hole with a significant (say, positive charge) will attract other (negative) charges, which fall in so that the final object has zero total charge. However, real black holes do have one property that turns out to make them quite different from Schwarzschild black holes: they are typically spinning. Spinning black holes are not round, but become somewhat disk shaped (as do all other spinning objects....). As a result, they are not described by the Schwarzschild metric. The spacetime that describes a rotating black hole is called the Kerr metric. There is also of course a generalization that allows both spin and charge and which is called the Kerr-Newman metric.

It turns out that the Penrose diagram for a rotating black hole is much the same as that of an RN black hole, but with the technical complication that rotating black holes are not round. One finds the same story about an unstable inner horizon in that context as well, with much the same resolution. I would prefer not to go into a discussion of the details of the Kerr metric because of the technical complications involved, but it is good to know that things basically work just the same as for the RN metric above.

9.6.4 Some Cool Stuff

Other Relativity links: In case you haven't already discovered them, the SU Relativity Group (the group that does research in Relativity) maintains a page of Relativity Links at (http://physics.syr.edu/research/relativity/RELATIVITY.html).
The ones under 'Visualizing Relativity'
(http://physics.syr.edu/research/relativity/RELATIVITY.html#Visualizing Relativity)
can be a lot of fun.

9.7 Homework Problems

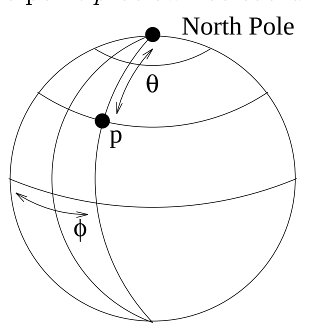
Fun with the Schwarzschild Metric: In the first few problems below, you will explore the Schwarzschild metric – the metric that describes the gravitational field (that is, the shape of spacetime) outside of any spherically symmetric distribution of matter. Thus, this same metric describes the shape of spacetime outside of round planets, round starts, round neutron stars, and round black holes!!! The only differences between these objects are 1) the value of the mass M and 2) how close you can get to the object before you hit the surface of the matter (so that the Schwarzschild metric no longer applies).

In it's full glory, the Schwarzschild metric is:

$$ds^2 = -\left(1 - \frac{R_s}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{R_s}{r}} + r^2 d\theta^2 + r^2 \sin^2\theta \; d\phi^2, \tag{9.48}$$

where $R_S = 2MG/c^2$.

Here, θ and ϕ are coordinates on a sphere as shown below. θ is similar to latitude – it is the angle between your position on the sphere and the north pole – thus, θ goes from zero (at the north pole) to π at the south pole. ϕ is like longitude, running from 0 to 2π around the sphere. These are the standard spherical coordinates used by physicists; however, they are 'backwards' from the coordinates used in many calculus classes (that is, θ and ϕ have been switched). The point p below has coordinates θ, ϕ:



1. Consider a static clock at some value of r.

   (a) In terms of the radial position r, how fast does a static clock run relative to a clock far away (at $r = \infty$)? Hint: calculate the proper time $d\tau$ measured by such a clock.

   (b) Suppose that you are standing on the surface of the Sun. How fast does your clock run relative to a clock far away (at $r = \infty$)? (i.e., calculate the relevant number.) Note that $M_{sun} = 2 \times 10^{30} kg$, $G = 6.7 \times 10^{-11} Nm^2/kg^2$, and $c = 3 \times 10^8 m/s$. Also, at the surface of the sun, $r = 7 \ \times \ 10^8 m$.

(c) In terms of r, what is the acceleration of this clock relative to freely falling observers?

(d) Suppose that you are standing on the surface of the Sun. How heavy do you feel?

2. Let's look at the shape of a 'slice of simultaneity' as defined by the static observers in the Schwarzschild metric. This is a slice with $t = constant$, so that $dt = 0$. The slice is a three dimensional surface labeled by r, θ, and ϕ. For simplicity, let's think about the *two dimensional* surface that goes through the equator – i.e., let's look at the *sub*surface $\theta = \pi/2$. On this *sub*surface, $d\theta = 0$.

   I'd like you to examine the curvature of this subsurface. As we have seen, the easiest way to do this is to compare the circumference of circles with their radius. Here, however, we have a complication: the metric above is only correct *outside* the matter (i.e., outside the star). So, we can't use it to describe what happens at the center.

   One way to get around this problem is to change our focus slightly. Instead of finding the function $C = C(R)$ that related the Circumference C to the Radius R, we can find the derivative dC/dR. This gives the rate of change dC in the circumference as the circle is enlarged by a proper distance dR. Note that, in flat space, $C = 2\pi R$ so $dC/dR = 2\pi$.

   In curved space, the answer will be different. You can calculate dC/dR for the $t = const$, $\theta = \pi/2$ subsurface of the Schwarzschild metric by first finding the circumference $C = C(r)$ for a circle of constant r. You can then compute dC/dr directly. The derivative dC/dR can then be calculated from dC/dr and dr/dR using the chain rule. To find dr/dR, remember that the radius R is the actual length of a line that goes straight out from the center of the circle to the edge.

   (a) What is dC/dR?

   (b) Does C increase too fast or too slow relative to flat space?

   (c) What happens to dC/dR as r gets very big??

3. When we studied the effect of gravity on clocks in section 9.1.1 we only did the computations for *static* clocks. However, we mentioned that other ('moving') clocks would measure time differently – much as was true back in our study of special relativity (flat spacetime). The metric above allows you to calculate how this works.

   Let's look at clocks that are in nice, circular orbits of constant r. A key question is: how fast do such clocks move around the star (relative to static observers)?? Unfortunately, this question is beyond the scope of this course – since orbiting objects are freely-falling, it involves calculating the actual worldlines of geodesics in the Schwarzschild metric. So, let me just give you the answer: an object in a circular orbit at r moves around with an angular velocity of

$$\frac{d\phi}{dt} = \sqrt{\frac{R_s}{2r^3}}. \tag{9.49}$$

That is, it moves *around* the planet, star, or black hole so that $\phi = \phi_0 + t\sqrt{\frac{M}{r^3}}$. In particular, this means that, along its orbit we have,

$$d\phi = dt\sqrt{\frac{R_s}{2r^3}}. \tag{9.50}$$

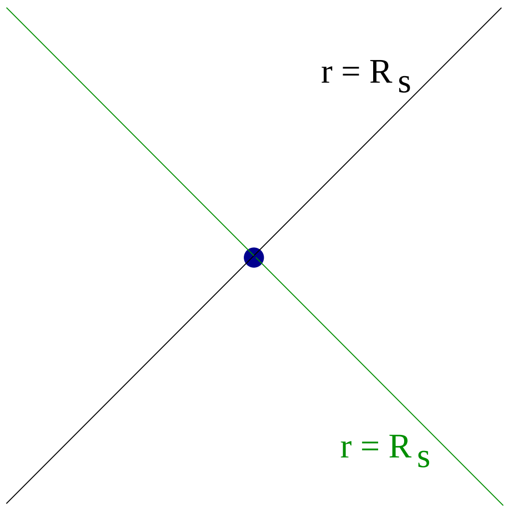
(a) You can now use this to calculate the relationship between the proper time τ measured by the clock in orbit and the time t (which, you may recall [see problem 1], is the proper time measured *by a static clock far away*). Consider a clock in a circular orbit (i.e., at constant r) around the equator (constant θ, $\theta = \pi/2$). You can use the above equation and the Schwarzschild metric to relate $d\tau$ directly to dt. [Remember, $d\tau^2 = -ds^2$.] Solve the resulting equation to express τ in terms of t.

(b) If your calculations above are correct, something interesting should happen at $r = 3R_S/2$. What happens to the relationship of τ and t there?

(c) **[extra credit]** Do you know how to interpret the result you found in (3b)? [Hint: what happens to $d\tau^2$ for $r < 3R_S/2$]?

4. Suppose that you are tossed out of a space ship outside of a black hole and that you fall in. To answer the questions below, use the fact that the spacetime near the horizon of a black hole is just like the region of flat spacetime near an acceleration horizon.

(a) Recall that, near the black hole horizon, a spacetime diagram (in a certain freely falling frame) looks like this:



Sketch a worldline on this diagram describing you falling into the black hole as described above.

(b) Suppose that you send out light rays at regular intervals to signal the rocket. Draw several of these light rays on your diagram.

(c) The people on the rocket watch the light rays you send out through a telescope. Describe what they see. Do they see you age slowly or quickly? When do they see you cross the horizon? If they first sit and watch for awhile, is it possible for them to come and rescue you?

(d) Describe what you *feel* as you fall in. Is there a difference between falling into a 'large' black hole and a 'small' black hole? If so, explain what it is. [Hint: Have you read section 9.4 on tidal forces yet?]

(e) Suppose now that the rocket sends out light rays toward you at regular intervals. Describe what you see if you watch these light rays through a telescope. Do you see people in the rocket age slowly or quickly? Do you see anything special when you cross the horizon? Is there a difference between falling into a 'large' black hole and a 'small' black hole? If so, explain what it is.

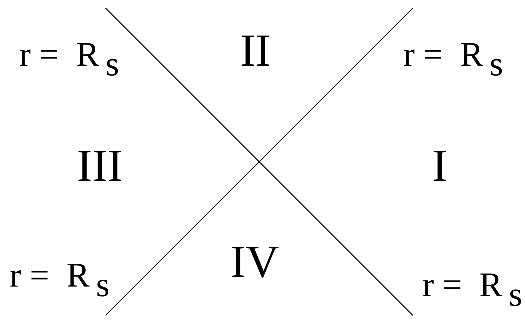
For each of the problems below, the term 'black hole' refers to the round Schwarzschild black hole that we have been studying in class.

5. Use your knowledge of static observers in a gravitational field to answer the following questions.

(a) Is it possible for a rocket to remain static (i.e., to remain at constant r, θ, ϕ) at the photon sphere?

(b) If you were placed in such a rocket, would you remain alive? (Hint: How heavy would you feel?) If the answer depends on the black hole, describe roughly for which black holes you would survive and for which you would not.

6. Remember that, near the horizon of a black hole, a good picture of spacetime looks like:



Here we have drawn an 'eternal' black hole – one that has been present since the Universe began.

   (a) What do the four regions (I, II, III, and IV) represent? (In particular, what is 'outside' and what is 'inside'?)
   (b) Draw another copy of the diagram above. Draw in the singularity on this new diagram.
   (c) Draw in the lines of constant r (the coordinate in the Schwarzschild metric) on this diagram.
   (d) Draw a new diagram showing the lines of constant t (the coordinate in the Schwarzschild metric) on this diagram. (Hint: remember that the static observers near the black hole horizon are just like the uniformly accelerated observers near an acceleration horizon.)
   (e) If you fall into a black hole from the outside (i.e., if you fall past the horizon), can you ever get out? Use your diagram to explain why or why not.
   (f) Can you travel from region I to region III? Use your diagram to explain why or why not.

7. Suppose that you are tossed out of a space ship outside of a black hole and that you fall in. Draw a spacetime diagram (in which light rays move at 45^o to the vertical) showing you falling from the horizon to the singularity. Describe what you feel at various stages as you fall along this worldline.

8. Consider a ball of matter (say, a star) collapsing to form a black hole.

   (a) Draw a spacetime diagram (in which all light rays move at 45^o to the vertical) that describes this process.
   (b) Suppose that we are outside the black hole at some constant value of r. What do we see if we watch the collapse through a telescope?

(c) What would we see and feel if, instead of staying out side, we fell into the black hole? Would we ever see the star's surface reach $r = 0$?

9. Suppose that you are a static observer (perhaps you have built a platform to stand on) far from a black hole and that you tie a long string (of negligible mass) to a $1kg$ rock. You then drop the rock and let it fall toward the black hole, pulling the string with it. You also tie the string to an electric generator so that, as the rock falls, it powers the generator. The generator will take energy from the motion of the rock and turn it into electrical energy.

   Suppose that the rock falls all the way to the photon sphere before stopping. If all of the energy lost by the rock is made into electricity (i.e., a 100% efficiency generator), how much electricity will be made??? Is there a difference between using a 'large' black hole and a 'small' black hole? If so, explain what it is.

Chapter 10

Cosmology: The Study of the Universe as a whole

Read Einstein, ch. 30-32

In the last few chapters we have been talking a lot about the geometry, or shape, of spacetime. In the particular case of the spacetime near a Schwarzschild black hole, we have gone into great detail on this subject. But what about the big picture? What can we say about the shape of the Universe as a whole?

Einstein asked this question very early on. He was motivated by technical problems with the description of the Universe as a whole in Newtonian gravity and he wanted to see if his theory worked better. It did, but not quite in the way that he expected....

10.1 The Copernican Principle and Relativity

Of course, in the early 1900's people did not know all that much about the universe, but they did have a few ideas on the subject. In particular, a certain philosophical tradition ran strong in astronomy, dating back to Copernicus. (Copernicus was the person who promoted the idea that the stars and planets did not go around the earth, but that instead the planets go around the sun.) This tradition held in high esteem the principle that "The earth is not at a particularly special place in the Universe". It was this idea which had freed Copernicus from having to place the earth at the center of the Universe.

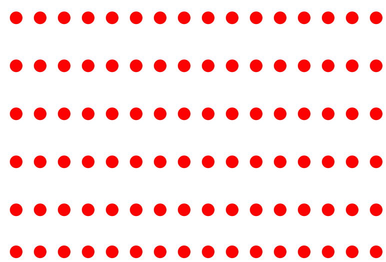
The idea was then generalized to say that, for example "The Sun is not a particularly special star," and then further to *"There is* **no** *special place in the Universe."* Or, said differently, the Copernican principle is that *"Every place in the universe is basically the same."*

So, on philosophical grounds, people believed that the stars were sprinkled more or less evenly throughout the universe. Now, one might ask, is this really true?

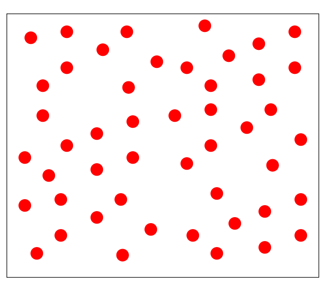
Well, the stars are not in fact evenly sprinkled. We now know that they are clumped together in *galaxies*. And even the galaxies are clumped together a bit. However, if one takes a sufficiently rough average then it is basically true that the clusters of galaxies are evenly distributed. We say that the universe is *homogeneous*. Homogeneous is just a technical word which means that every place in the universe is the same.

10.1.1 Homogeneity and Isotropy

In fact, there is another idea that goes along with every *place* being essentially the same. This is the idea that the universe is the same in every *direction*. The technical word is that the universe is *isotropic*. To give you an idea of what this means, I have drawn below a picture of a universe that is homogeneous but is *not* isotropic – the galaxies are farther apart in the vertical direction than in the horizontal direction:



In contrast, a universe that is both homogeneous and isotropic must look roughly like this:

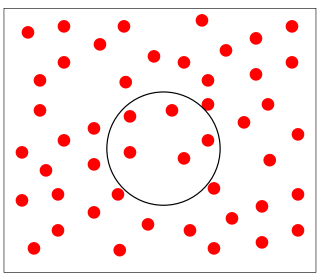


10.1.2 That technical point about Newtonian Gravity in Homogeneous Space

By the way, we can use the picture above to point out that technical problem I mentioned with Newtonian Gravity in infinite space. I will probably skip this part in class, but it is here for your edification.

The point is that, to compute the gravitational field at some point in space we need to add up the contributions from all of the infinitely many galaxies. This is an infinite sum. When you discussed such things in your calculus class, you learned that some infinite sums *converge* and some do not. Actually, this sum is one of those interesting in-between cases where the sum converges (if you set it up right), but it does not converge *absolutely*. What happens in this case is that you can get different answers depending on the order in which you add up the contributions from the various objects.

To see how this works, recall that all directions in this universe are essentially the same. Thus, there is a rotational symmetry and the gravitational field must be pointing either toward or away from the center. Now, it turns out that Newtonian gravity has a property that is much like Gauss' law in electromagnetism. In the case of spherical symmetry, the gravitational field on a given sphere depends only on the total charge inside the sphere. This makes it clear that on any given sphere there must be some gravitational field, since there is certainly matter inside:



But what if the sphere is very small? Then, there is essentially no matter inside, so the gravitational field will vanish. So, at the 'center' the gravitational field must vanish, but at other places it does not.

But now we recall that *there is no center!* This universe is *homogeneous*, meaning that every place is the same. So, if the gravitational field vanishes at one point, it must also vanish at every other point..... This is what physicists call a problem. However, Einstein's theory turns out not to have this problem. In large part, this is because Einstein's conception of a gravitational field is very different from Newton's. In particular, Einstein's conception of the gravitational field is local while Newton's is not.

10.1.3 Homogeneous Spaces

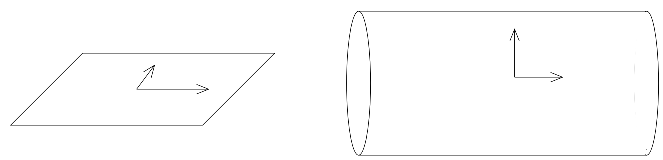
Now, in general relativity, we have to worry about the curvature (or shape) of space. So, we might ask: "what shapes are compatible with the idea that space must be homogeneous and isotropic?" It turns out that there are exactly three answers:

1. A three-dimensional sphere (what the mathematicians call S^3). This can be thought of as the set of points that satisfy $x_1^2 + x_2^2 + x_3^2 + x_4^2 = R^2$ in four-dimensional Euclidean space.

2. Flat three dimensional space.

3. The three dimensional version of the Lobachevskian space.

By the way, it is worth pointing out that option (1) gives us a finite sized universe. The second and third options gives us infinite spaces. However, if we were willing to weaken the assumption of isotropy just a little bit, we could get finite sized spaces that are very much the same. To get an idea of how this works, think of taking a piece of paper (which is a good model of an infinite flat plane) and rolling it up into a cylinder. This cylinder is still flat, but it is finite in one direction. This space is homogeneous, though it is not isotropic (since one direction is finite while the other is not):



Rolling up flat three dimensional space in all three directions gives what is called a 3-torus, and is finite in all three directions. The Lobachevskian space can also be 'rolled up' to get a finite universe. This particular detail is not mentioned in many popular discussions of cosmology.

Actually, these are not just three spaces. Instead, each possibility (sphere, flat, Lobachevskian) represents 3 *sets* of possibilities. To see the point, let's consider option #1, the sphere. There are small spheres, and there are big spheres. The big spheres are very flat while the tiny spheres are tightly curved. So, the sphere that would be our universe could, in principle, have had any size.

The same is true of the Lobachevskian space. Think of it this way: in Escher's picture, no one told us how big each fish actually is. Suppose that each fish is one light-year across. Such a space can also be considered 'big,' although of course any Lobachevskian space has infinite volume (an infinite number of fish). In particular, if we consider a region much smaller than a single fish, we cannot see the funny curvature effects and the space appears to be flat. You may recall that we have to look at circles of radius 2 fish or so to see that C/R is not always 2π. So, if each fish was a light year across, we would have to look really far away to see the effects of the curvature. On the other hand, if each fish represented only a millimeter (a 'small' space), the curvature would be readily apparent just within our class room. The point is again that there is really a family of spaces here labelled by a length – roughly speaking, this length is the size of each fish.

What about for the flat space? After all, flat is flat..... Here, making the universe bigger does not change the geometry of space at all – it simply remains flat. However, it will spread out the galaxies, stars, and such. (The same is, of course, also true in the spherical and Lobachevskian contexts.) So, for the flat space we should think of making the Universe larger as keeping the space itself mostly same and making the density of matter smaller and smaller.

10.2 Dynamics! (a.k.a. Time Evolution)

So, homogeneity and isotropy restrict the shape of space to be in one of a few simple classes. That is to say, at any time (to the extent that this means anything) the shape of space takes one of these forms. But what happens as time passes? Does it maintain the same shape, or does it change? The answer must somehow lie inside Einstein's equations (the complicated ones that we have said rather little about), since they are what control the behavior of the spacetime metric.

Luckily, the assumptions of homogeneity and isotropy simplify these equations a lot. Let's think about what the metric will look like. It will certainly have a dt^2 part. If we decide to use a time coordinate which measures proper time directly then the coefficient of dt^2 will just be -1. We can always decide to make such a choice. The rest of the metric controls the metric for *space*[1], which must be the metric for one of the three spaces described above. Now, the universe cannot suddenly change from, say, a sphere to a Lobachevskian space. So, as time passes the metric for space can only change by changing the the overall size (a.k.a. 'scale') of the space. In other words, the space can only get bigger or smaller.

What this means mathematically is that the metric must take the general form:

$$ds^2 = -d\tau^2 + a^2(t)(metric\ for\ unit-sized\ space). \tag{10.1}$$

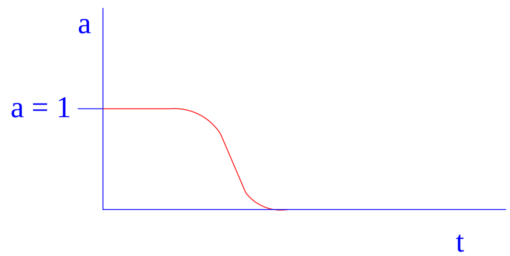
The factor $a(t)$ is called the 'scale factor' or 'size of the universe.' When a is big, all of the spatial distances are very big. When a is small, all of the spatial distances are very small. So, a space with small a will have a highly curved space and very dense matter. Technically, the curvature of space is proportional to $1/a^2$, while the density of matter is proportional to $1/a^3$.

Note that the only freedom we have left in the metric is the single function $a(t)$. Einstein's equations must therefore simplify to just a single equation that tells us how $a(t)$ evolves in time.

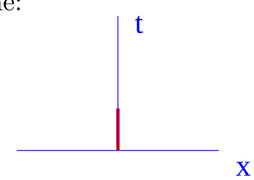
10.2.1 Expanding and Contracting Universes

Before diving into Einstein's equations themselves, let's first take a moment to understand better what it means if a changes with time. To do so, let's consider a case where a starts off 'large' but then quickly decreases to zero:

[1] There can also be terms of the form $dtdx$, $dtdy$, and $dtdz$ that mix time and space. However, these three terms can be thought of as giving the x, y, z components of a vector in space. But space is isotropic, so there is no special direction in which this vector can point. The only vector with no direction is the zero vector, so in fact this vector must vanish. It follows that there are in fact no cross terms of this sort.



This is just a fake model system to better understand what a is. I make no claims that this represents any reasonable solution of Einstein's equations. Nevertheless, let's think about what happens to a freely falling object in this universe that begins 'at rest', meaning that it has zero initial velocity in the reference frame used in equation (10.1). If it has no initial velocity, then we can draw a spacetime diagram showing the first part of its worldline as a straight vertical line:



Now, when a shrinks to zero, what happens to the worldline? Will it bend to the right or to the left? Well, we assumed that the Universe is isotropic, right? So, the universe is the same in all directions. This means that there is a symmetry between right and left, and there is nothing to make it prefer one over the other. So, it does not bend at all but just runs straight up the diagram. In other words, an object that begins at $x = 0$ with zero initial velocity will always remain at $x = 0$.

Of course, since the space is homogeneous, all places in the space are the same and any object that begins at any $x = x_0$ with zero initial velocity will always remain at $x = x_0$. From this perspective it does not look like much is happening.

However, consider two such objects: one at x_1 and one at x_2. The metric ds^2 contains a factor of the scale a. So, the actual proper distance between these two points is proportional to a. Suppose that the distance between x_1 and x_2 is L when $a = 1$ (at $t = 0$). Then, later, when the scale has shrunk to $a < 1$, the new distance between this points is only aL. In other words, the two objects have come closer together.

Clearly, what each object sees is another object that moves toward it. The reason that things at first appeared not to move is that we chose a funny sort of coordinate system (if you like, you can think of this as a funny reference frame, though it is nothing like an inertial reference frame in special relativity). The funny coordinate system simply moves along with the freely falling objects – cosmologists call it the 'co-moving' coordinate system.

It is also worth pointing out what happens if we have lots of such freely falling objects, each remaining at a different value of x. In this case, *each* object sees *all* of the other objects rushing toward it as a decreases. Furthermore, an object which is initially a distance L away (when $a = 1$) becomes only a distance aL away. So, the object has 'moved' a distance $(1 - a)L$. Similarly, an object which is initially a distance $2L$ away becomes $2aL$ away and 'moves' a distance $2(1 - a)L$ – twice as far.

This reasoning leads to what is known (for reasons that we will explain later) as the 'Hubble Law.' This law states that in a homogeneous universe the relative velocity between any two objects is proportional to their distance:

$$v = H(t) \cdot d, \tag{10.2}$$

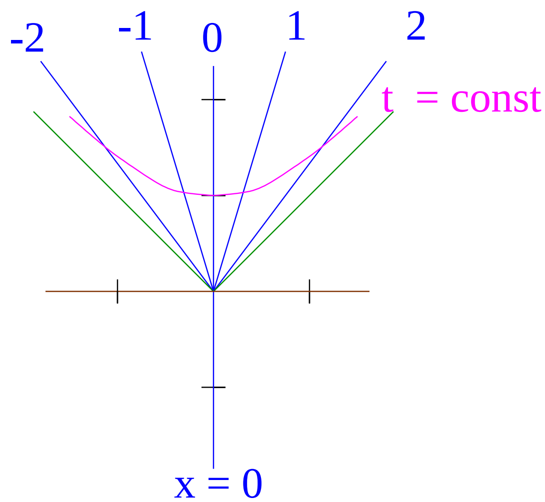
where v is the relative 'velocity', d is the distance, and $H(t)$ is the 'Hubble constant' – a number that does depend on time but does not depend on the distance to the object being considered. Here I have put the word velocity (and related words like 'moves') in quotes for reasons that will become clear below.

It is important to stress again that the Hubble constant is constant only in the sense of being independent of d. There is no particular reason that this 'constant' should be independent of time and, indeed, we will see that it is natural for H to change with time. I have written the above relation using $H(t)$ to emphasize this point. The Hubble constant is determined by the rate of change of a: $H(t) = \frac{1}{a}\frac{da}{dt}$.

As expected, there is no special object that is the 'center' of our collapsing universe. Instead, every object sees itself as the center of the process. As usual, none of these objects is any more 'right' about being the center than any other. The difference is just a change of reference frames.

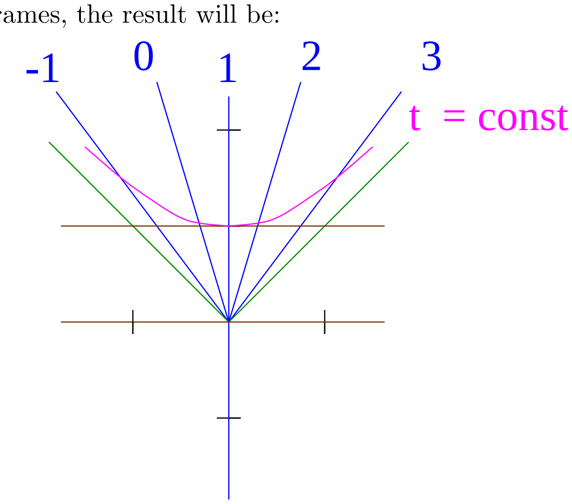
10.2.2 A flat spacetime model

In case this is hard to grasp, it is worth mentioning that you have seen something similar happen even in flat spacetime. Suppose I consider an infinite collection of inertial observers all of whom pass through some special event. Let me suppose that observer #1 differs from observer #0 by the same boost parameter as any other observer $n+1$ differs from observer n. We could draw a spacetime diagram showing these observers as below:



Note that this is **not** the $k = 0$ Universe which has flat *space*. Instead, the entire *spacetime* is flat here when viewed as a whole, but the slice representing space on the above diagram is a hyperboloid, which is most definitely not flat. Instead, this hyperboloid is a constant negative curvature space ($k = -1$). Since the spacetime here is flat, we have drawn the limit of the $k = -1$ case as we take the matter density to *zero*. It is not pysically realistic as a cosmology, but I include it here to give you a diagram that illustrates the co-moving coordiante system used in cosmology. In addition, for $k = -1$ the matter density does become vanishingly small in the distant future (if the cosmological constant vanishes; see below). Thus, for such a case this diagram does become accurate in the limit $t \to \infty$.

Shown here in the reference frame of observer #0, that observer appears to be the center of the expansion. However, we know that if we change reference frames, the result will be:



In this new reference frame, now another observer appears to be the 'center.'

These discussions in flat spacetime illustrate three important points: The first is that although the universe is isotropic (spherically symmetric), there is no special 'center.' Note that the above diagrams even have a sort of 'big bang' where everything comes together, but that it does not occur any more where one observer is than where any other observer is.

The second important point that the above diagram illustrates is that the surface that is constant t in our co-moving cosmological coordinates does not represent the natural notion of simultaneity for *any* of the co-moving observers. The 'homogeneity' of the universe is a result of using a special frame of reference in which the $t = const$ surfaces are hyperbolae. As a result, the universe is not in fact homogeneous in any inertial reference frame (or any similar reference frame in a curved spacetime).

This is related to the third point: When discussing the Hubble law, a natural question is, "What happens when d is large enough that $H(t) \cdot d$ is greater than the speed of light?" Recall that in general relativity measurements that are not local are a subtle thing. For example, in the flat spacetime example above, in the coordinates that we have chosen for our homogeneous metric, the $t = const$ surfaces are hyperbolae. They are **not** in fact the surfaces of simultaneity for any of the co-moving observers.

Now, the distance between co-moving observers that we have been discussing is the distance measured *along the hyperbola* (i.e., along the homogeneous slice), which is a very different notion of distance than we are used to using in Minkowski space. This means that the 'velocity' in the Hubble law is not what we had previously called the relative velocity of two objects in Minkowski space. Instead, in our flat spacetime example, the velocity in the Hubble law turns out to correspond directly to the *boost parameter* θ. However, for the nearby galaxies (for which the relative velocity is much less than the speed of light), this subtlety can be safely ignored (since v and θ are proportional there).

10.2.3 On to the Einstein Equations

So, the all important question is going to be: What is the function $a(t)$? What do the Einstein equations tell us about how the Universe will actually evolve? Surely what Newton called the attractive 'force' of gravity must cause something to happen!

As you might expect, the answer turns out to depend on what sort of stuff you put in the universe. For example, a universe filled only with light behaves somewhat differently from a universe filled only with dirt.

In particular, it turns out to depend on the density of energy (ρ) and on the pressure (P). [You may recall that we briefly mentioned earlier that, in general relativity, pressure is directly a source of gravity.]

For our homogeneous isotropic metrics, it turns out that the Einstein equations can be reduced to the following two equations:

$$\frac{3}{a^2}\left(\frac{da}{dt}\right)^2 = \frac{8\pi G}{c^2}\rho - 3\frac{kc^2}{a^2}, \tag{10.3}$$

$$\frac{3}{a}\frac{d^2a}{dt^2} = -\frac{4\pi G}{c^2}(\rho + 3P) \tag{10.4}$$

In the first equation, the constant k is equal to $+1$ for the spherical (positively curved) universe, $k = 0$ for the flat universe, and $k = -1$ for the Lobachevskian (negatively curved) universe.

We're not going to derive these equations, but let's talk about them a bit. The second one is of a more familiar form. It looks kind of like Newton's second law combined with Newton's law of Universal Gravitation – on the left we have the acceleration d^2a/dt^2 while the right provides a force that depends on the amount of matter present (ρ). Interestingly though, the pressure P also contributes. The reason that Newton never noticed the pressure term is that ρ is the density of energy and, for an object like a planet, the energy is mc^2 which is *huge* due to the factor of c^2. In comparison, the pressure inside the earth is quite small. Nevertheless, this pressure contribution can be important in cosmology.

Recall that when a changes it tells us whether the (co-moving) bits of matter are coming closer together or spreading farther apart. This means that, in the present context, the Einstein equations tell us what the matter is doing as well as what the spacetime is doing. Thought of this way, the second equation should make a lot of sense. The left hand side is an acceleration term, while the right hand side is related to the sources of gravity. Under familiar conditions where the particles are slowly moving, the energy density is roughly c^2 times the mass density. This factor of c^2 nicely cancels the c^2 in the denominator, leaving the first term on the right hand side as G times the density of mass. The pressure has no hidden factors of c^2 and so P/c^2 is typically small. Under such conditions, this equation says that gravity causes the bits of matter to accelerate *toward* one another (this is the meaning of the minus sign) at a rate proportional to the amount of mass around. That sounds just like Newton's law of gravity, doesn't it?

In fact, we see that gravity is attractive in this sense whenever energy density ρ and pressure (P) are positive. In particular, for positive energy and pressure, a *must* change with time in such a way that things accelerate toward each other. Under such conditions it is impossible for the universe to remain static. Now, back in the early 1900's people in fact believed (based on no particular evidence) that the universe had been around forever and had been essentially the same for all time. So, the idea that the universe *had* to be changing really bothered Einstein. In fact, it bothered him so much that he found a way out.

10.2.4 Negative Pressure, Vacuum Energy, and the Cosmological Constant

Physicists do expect that (barring small exceptions in quantum field theory) the energy density ρ will be always be positive. However, the is no reason in principle why the pressure P must be positive. Let's think about what a negative pressure would mean. A positive pressure is an effect that resists something being squeezed. So, a negative pressure is an effect that resists something being stretched. This is also known as a 'tension.' Imagine, for example, a rubber band that has been stretched. We say that it is under tension, meaning that it tries to pull itself back together. A sophisticated relativistic physicist calls such an effect a 'negative pressure.'

Let's look closely at equation (10.4). We see that the universe can in fact 'sit still' and remain static if $\rho + 3P = 0$. If $\rho + 3P$ is negative, then gravity will in fact be repulsive (as opposed to attractive) the various bits of matter will accelerate apart. Now, because ρ is typically very large (since it is the density of energy and $E = mc^2$) this requires a truly huge negative pressure. The kinds of matter that we are most familiar with will never have such a large negative pressure. However, physicists can imagine that their might possibly be such a kind of matter.

The favorite idea along these lines is called "vacuum energy." The idea is that empty space itself might somehow have energy. At first, this is a rather shocking notion. I mean, if it is empty, how can it have energy? But, some reflection will tell us that this may simply be a matter of semantics: given the space that we think is empty (because we have cleared it of everything that we know how to remove), how empty is it really? In the end, like everything else in physics, this question must be answered experimentally. We need to find a way to go out and to measure the energy of empty space.

Now, what is clear is that the energy of empty space must be rather small. Otherwise, it's gravitational effects would screw up our predictions of, for example, the orbits of the planets. However, there is an awful lot of 'empty' space out there. So, taken together it might still have some nontrivial effect on the universe as a whole.

OK, so why should vacuum energy (the energy density of empty space) have negative pressure? Well, an important fact here is that energy density and pressure are not completely independent. Pressure, after all is related to the force required to change the size of a system: to smash it or to stretch it out. On the other hand, force is related to energy: for example, we must add energy to a rubber band in order fight the tension forces and stretch it out. The fact that we must add energy to a spring in order to stretch it is what causes the spring to want to contract; i.e., to have a negative pressure when stretched.

Now, if the vacuum itself has some energy density ρ and we stretch the space (which is just what we will do when the universe expands) then the new (stretched) space has more vacuum and therefore more energy. So, we again have to add energy to stretch the space, so there is a negative pressure. It turns out that pressure is (minus) the derivative of energy with respect to volume $P = -dE/dV$. Here, $E = \rho V$, so $P = -\rho$. Clearly then for pure vacuum energy we have $\rho + 3P < 0$ and gravity is repulsive. On the other hand, combining this with the appropriate amount of normal matter could make the two effects cancel out and could result in a static universe.

Since $P = -\rho$ for vacuum energy, we see that vacuum energy is in fact characterized by a single number. It is traditional to call this number Λ, and to define Λ so that we have

$$\begin{aligned} \rho &= \frac{\Lambda}{8\pi G}, \\ P &= -\frac{\Lambda}{8\pi G}. \end{aligned}$$

Such a Λ is called the 'cosmological constant.' We have, in fact seen it before. You may recall that, during our very brief discussion of the Einstein equations in section 8.4.1, we mentioned that Einstein's assumptions and the mathematics in fact allowed two free parameters. One of these we identified as Newton's Universal Gravitational Constant G. The other was the cosmological constant Λ. This is the same cosmological constant: as we discussed back then, the cosmological constant term in the Einstein equations could be called a funny sort of 'matter.' In this form, it is none other than the vacuum energy that we have been discussing.

We mentioned that Λ must be small to be consistent with the observations of the motion of planets. However, clearly matter is somewhat more clumped together in our solar system than outside. Einstein hoped that this local clumping of normal matter (but not of the cosmological constant) would allow the gravity of normal matter to completely dominate the situation inside the solar system while still allowing the two effects to balance out for the universe overall.

Anyway, Einstein thought that this cosmological constant *had* to be there – otherwise the universe could not remain static. However, in the early 1920's, something shocking happened: Edwin Hubble made detailed measurements of the galaxies and found that the universe is in fact *not* static. He used the Doppler effect to measure the motion of the other galaxies and he found that they are almost all moving away from us. Moreover, they are moving away from us at a rate proportional to their distance! This is why the rule $v = H(t) \cdot d$ is known as the 'Hubble Law.' The universe appeared to be expanding..... The result was that Einstein immediately dropped the idea of a cosmological constant and declared it to be the biggest mistake of his life.

10.3 Our Universe: Past, Present, and Future

OK, so the other galaxies are running away from ours at a rate proportional to their distance from us. The implication is that the universe is expanding, and that it has been expanding for some time. In fact, since gravity is generally attractive, we would expect that the universe was expanding even faster in the past.

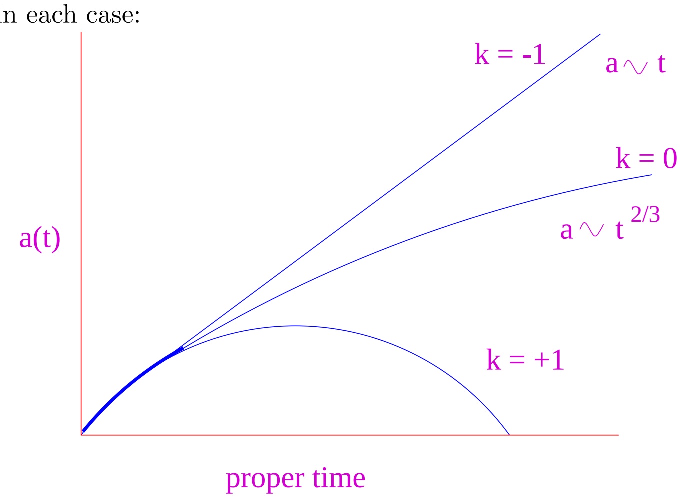
To find out more of the details we will have to look again to the Einstein equations. We will also need to decide how to encode the current matter in the universe in terms of a density ρ and a pressure P. Let's first think about the pressure. Most matter today is clumped into galaxies, and the galaxies are quite well separated from each other. How much pressure does one galaxy apply to another? Essentially none. So, we can model the normal matter by setting $P = 0$.

When the pressure vanishes, one can use the Einstein equations to show that the quantity: $\mathcal{E} = 8\pi G\rho a^3/3$ is independent of time. Roughly speaking, this is just conservation of energy (since ρ is the density of energy and a^3 is proportional to the volume). As a result, assuming that $\Lambda = 0$ the Einstein equations can be written:

$$\frac{1}{c^2}\left(\frac{da}{dt}\right)^2 - \frac{\mathcal{E}}{c^2 a} + k = 0. \tag{10.5}$$

Recall that k is a constant that depends on the overall shape of space: $k = +1$ for the spherical space, $k = 0$ for the flat space, and $k = -1$ for the Lobachevskian space.

In the above form, this equation can be readily solved to determine the behavior of the universe for the three cases $k = -1, 0, +1$. We don't need to go into the details here, but let me draw a graph that gives the idea of how a changes with t in each case:



Note that for $k = +1$ the universe expands and then recontracts, whereas for $k = 0, -1$ it expands forever. In the case $k = 0$ the Hubble constant goes to zero at very late times, but for $k = -1$ the Hubble constant asymptotes to a constant positive value at late times.

Note that at early times the three curves all look much the same. Roughly speaking, our universe is just now at the stage where the three curves are beginning to separate. This means that, the past history of the universe is more or less independent of the value of k.

10.4 Observations and Measurements

So, which is the case for our universe? How can we tell? Well, one way to figure this out is to try to measure how fast the universe was expanding at various times in the distant past. This is actually not as hard as you might think: you see, it is very easy to look far backward in time. All we have to do is to look at things that are very far away. Since the light from such objects takes such a very long time to reach us, this is effectively looking far back in time.

10.4.1 Runaway Universe?

The natural thing to do is to try to enlarge on what Hubble did. If we could figure out how fast the really distant galaxies are moving away from us, this will tell us what the Hubble constant was like long ago, when the light now reaching us from those galaxies was emitted. The redshift of a distant galaxy is a sort of average of the Hubble constant over the time during which the signal was in transit, but with enough care this can be decoded to tell us about the Hubble constant long ago. By measuring the rate of decrease of the Hubble constant, we can learn what kind of universe we live in.

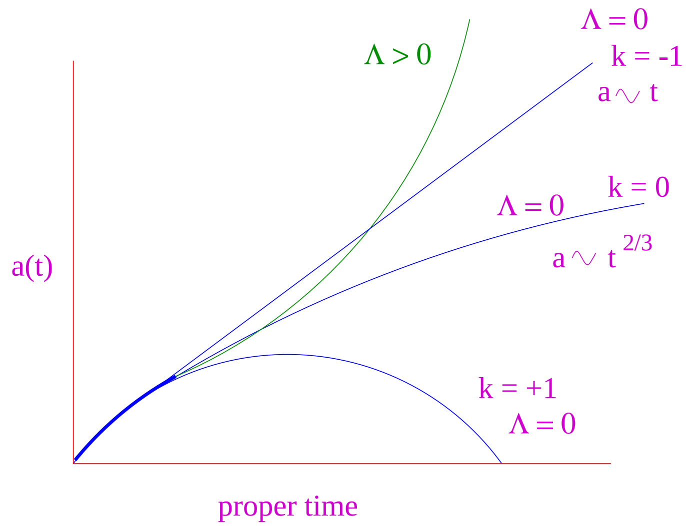
However, it turns out that accurately measuring the *distance* to the distant galaxies is quite difficult. (In contrast, measuring the redshift is easy.) Until recently, no one had seriously tried to measure such distances with the accuracy that we need. However, a few years ago it was realized that there may be a good way to do it using supernovae.

The particular sort of supernova of interest here is called 'Type Ia.' Astrophysicists believe that type Ia supernovae occur when we have a binary star system containing one normal star and one white dwarf. We can have matter flowing from the normal star to the white dwarf in an accretion disk, much as matter would flow to a neutron star or black hole in that binary star system. But remember that a white dwarf can only exist if the mass is less than 1.4 solar masses. When extra matter is added, bringing the mass above this threshold, the electrons in the core of the star get squeezed so tightly by the high pressure that they bond with protons and become neutrons. This releases vast amount of energy in the form of neutrinos (another kind of tiny particle) and heat which results in a massive explosion: a (type Ia) supernova.

Anyway, it appears that this particular kind of supernova is pretty much always the same. It is the result of a relatively slow process where matter is gradually added to the white dwarf, and it always explodes when the total mass hits 1.4 solar masses. In particular, all of these supernovae are roughly the same brightness (up to one parameter that astrophysicists think they know how to correct for). As a result, supernovae are a useful tool for measuring the distance to far away galaxies. All we have to do is to watch a galaxy until one of these supernovae happens, and then see how bright the supernova *appears* to be. Since it's actual brightness is known, we can then figure out how far away it is. Supernovae farther away appear to be much dimmer while those closer in appear brighter.

About two years ago, the teams working on this project released their data. The result came as quite a surprise. Their data shows that the universe is not slowing down at all. Instead, it appears to be accelerating!

As you might guess, this announcement ushered in the return of the cosmological constant. By the way, the cosmological constant has very little effect when the universe is small (since vacuum energy is the same density whether the universe is large or small while the density of normal matter was huge when the universe was small). However, with a cosmological constant, the effects of the negative pressure get larger and larger as time passes (because there is more and more space, and thus more and more vacuum energy). As a result, a cosmological constant makes the universe expand forever at an ever increasing rate. Adding this case to our graph, we get:



The line for $\Lambda > 0$ is more or less independent of the constant k.

So, should we believe this? The data in support of an accelerating universe has held up well for three years now. However, there is a long history of problems with observations of this sort. There are often subtleties in understanding the data that are not apparent at first sight, as the various effects can be much more complicated than one might naively expect. Physicists say that there could be significant 'systematic errors' in the technique. All this is to say that, when you measure something new, it is always best to have at least two independent ways to find the answer. Then, if they agree, this is a good confirmation that *both* methods are accurate.

10.4.2 Once upon a time in a universe long long ago

It turns out that one way to get an independent measurement of the cosmological constant is tied up with the story of the very early history of the universe. This is of course an interesting story in and of itself.

Let's read the story backwards. Here we are in the present day with the galaxies spread wide apart and speeding away from each other. Clearly, the galaxies used to be closer together. As indicated by the curves in our graphs, the early history of the universe is basically independent of the value of Λ or k.

So, imagine the universe as a movie that we now play backwards. The galaxies now appear to move toward each other. They collide and get tangled up with each other. At some point, there is no space left between the galaxies, and they all get scrambled up together – the universe is just a mess of stars.

Then the universe shrinks some more, so that the stars all begin to collide. There is no space left between the stars and the universe is filled with hot matter, squeezing tighter and tighter. The story here is much like it is near the singularity of a black hole: even though squeezing the matter increases the pressure, this does not stop the spacetime from collapsing. In fact, as we have seen, pressure only adds to the gravitational attraction and accelerates the collapse.

As the universe squeezes tighter, the matter becomes very hot. At a certain point, the matter becomes so hot that all of the atoms ionize: the electrons come off and separate from the nuclei. Something interesting happens here. Because ionized matter interacts strongly with light, light can no longer travel freely through the universe. Instead, photons bounce around between nuclei like ping pong balls! It it the cosmic equivalent of trying to look through a very dense fog, and it becomes impossible to see anything in the universe. This event is particularly important because, as we discussed earlier, the fact that it takes light a long time to travel across the universe means that when we look out into the universe *now*, looking very far away is effectively looking back in time. So, this ionization sets a limit on how far away and how far back in time we can possibly see. On the other hand, ever since the electrons and nuclei got together into atoms (deionization) the universe has been more or less transparent. For this reason, this time is also called 'decoupling.' [Meaning that light 'decouples' or 'disconnects' from matter.] As a result, we might expect to be able to see all the way back to this time.

What would we see if we could see that far back? Well, the universe was hot, right? And it was all sort of mushed together. So, we might expect to see a uniform glow that is kind of like looking into a hot fire. In fact, it was quite hot: several thousand degrees.

Another way to discuss this glow is to remember that the universe is homogeneous. This means that, not only was stuff "way over there" glowing way back when, but so was the stuff where we are. What we are saying is that the whole universe (or, if you like, the whole electromagnetic field) was very hot back then. A hot electromagnetic field contains a lot of light.... Anyway, the point about light barely interacting with matter since decoupling means that, since that time, the electromagnetic field (i.e., light) should just have gone on and done its thing independent of the matter. In other words, it cannot receive energy (heat) from matter or loose energy (heat) by dumping it into matter. It should have pretty much the same heat energy that it had way back then.

So, why then is the entire universe today not just one big cosmic oven filled with radiation at a temperature of several thousand degrees? The answer is that the expansion of the universe induces a redshift not only in the light from the distant galaxies, but in the thermal radiation as well. The effect is similar to the fact that a gas cools when it expands. Here, however, the gas is a gas of photons and the expansion is due to the expansion of the universe. The redshift since decoupling is about a factor of 2000, with the result that the radiation today has a temperature of a little over 3 degrees Kelvin (i.e. 3 degrees above absolute zero).

At 3 degrees Kelvin, electromagnetic radiation is in the form of microwaves (in this case, think of them as short wavelength radio waves). This radiation can be detected with what are basically big radio telescopes or radar dishes. Back in the 60's some folks at Bell Labs built a high quality radio dish to track satellites. Two of them (Penzias and Wilson) were working on making it really sensitive, when the discovered that they kept getting a lot of noise coming in, and coming in more or less uniformly from all directions. It appeared that radio noise was being produced uniformly in deep space!

This radio 'noise' turned out to be thermal radiation at a temperature of 2.7 Kelvin. Physicists call it the 'Cosmic Microwave Background (CMB).' It's discovery is one of the greatest triumphs of the 'big bang' idea. After all, that is what we have been discussing. Long ago, before decoupling, the universe was very hot, dense, and energetic. It was also in the process of expanding, so that the whole process bears a certain resemblance (except for the homogeneity of space) to a huge cosmic explosion: a big bang. The discovery of the CMB verifies this back to an early stage in the explosion, when the universe was so hot and dense that it was like one big star.

By the way, do you remember our assumption that the universe is homogeneous? We said that it is of course not *exactly* the same everywhere (since, for example, the earth is not like the inside of the sun) but that, when you measure things on a sufficiently large scale, the universe does appear to be homogeneous. Well, the cosmic microwave background is our best chance to test the homogeneity on the largest possible scales since, as we argued above, it will not be possible to directly 'see' anything coming from farther away. The microwaves in the CMB have essentially traveled in a straight line since decoupling. We will never see anything from farther away since, for the light to be reaching us now, it would have had to have been emitted from an distant object before decoupling – back when the universe was filled with thick 'fog.'

When we measure the cosmic microwave background, it turns out to be *incredibly* homogeneous. The departures from homogeneity in the CMB are only about 1 part in one hundred thousand! I'll give you a handout that includes a 'map' of these tiny inhomogeneities from the first experiment (a satellite called COBE) to measure them.

This, by the way, illustrates an important point about the early universe. It was *not* like what we would get if we simply took the universe now and made all of the galaxies come together instead of rushing apart. If we pushed all of the galaxies together we would, for example, end up with a lot of big clumps (some related to galactic black holes, for example). While there would be a lot of general mushing about, we would not expect the result to be anywhere near as homogeneous as one part in one hundred thousand.

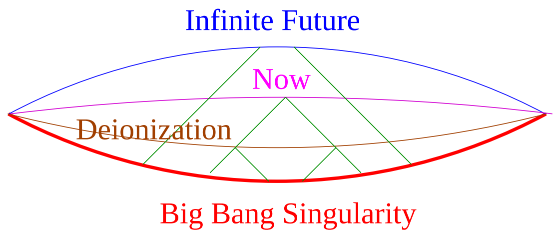
It appears then that the universe started in a very special, very uniform state with only very tiny fluctuations in its density. So then, why are there such large clumps of stuff today? Today, the universe is far from homogeneous on the small scale. The reason for this is that gravity tends to cause matter to clump over time. Places with a little higher density pull together gravitationally and become even more dense, pulling in material from neighboring under-dense regions so that they become less dense. It turns out that tiny variations of one part in one hundred thousand back at decoupling are just the right size to grow into roughly galaxy-style clumps today. This is an interesting fact by itself: Galaxies do not require special 'seeds' to start up. They are the natural consequence of gravity amplifying teeny tiny variations in density in an expanding universe.

Well, that's the rough story anyway. Making all of this work in detail is a little more complicated, and the details do depend on the values of Λ, k, and so on. As a result, if one can measure the CMB with precision, this becomes an independent measurement of the various cosmological parameters. The data from COBE confirmed the whole general picture and put some constraints on Λ. The results were consistent with the supernova observations, but by itself COBE was not enough to measure Λ accurately. A number of recent balloon-based CMB experiments have improved the situation somewhat, and in the next few years two more satellite experiments (MAP and PLANCK) will measure the CMB in great detail. Astrophysicists are eagerly awaiting the results.

10.4.3 A cosmological 'Problem'

Actually, the extreme homogeneity of the CMB raises another issue: how could the universe have ever been so homogeneous? For example, when we point our radio dish at one direction in the sky, we measure a microwave signal at 2.7 Kelvin coming to us from ten billion light-years away. Now, when we point our radio dish in the opposite direction, we measure a microwave signal at the same temperature (to within one part in one hundred thousand) coming at us from ten billion light-years away in the opposite direction! Now, how did those two points so far apart know that they should be at exactly the same temperature?

Ah! You might say, "Didn't the universe used to be a lot smaller, so that those two points were a lot closer together?" This is true, but it turns out not to help. The point is that all of the models we have been discussing have a singularity where the universe shrinks to zero size at very early times. An important fact is that this singularity is spacelike (as in the black hole). The associated Penrose diagram looks something like this:



Here, I have drawn the Penrose diagram including a cosmological constant, but the part describing the big bang singularity is the same in any case (since, as we have discussed, Λ is not important when the universe is small).

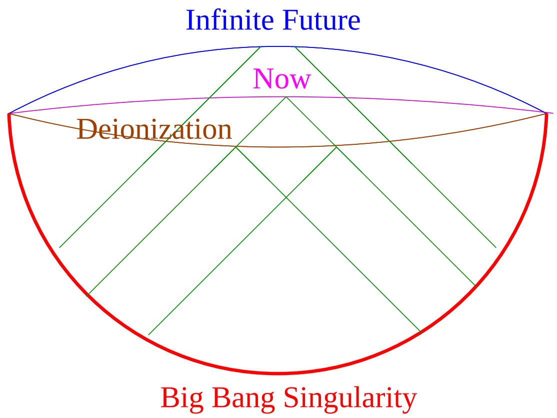
The fact that the singularity is spacelike means that no two points *on* the singularity can send light signals to each other (even though they are zero distance apart). Thus, it takes a finite time for any two 'places' to be able to signal each other and tell each other at what temperature they should be[2]. In fact, we can see that if the two points begin far enough apart then they will *never* be able to communicate with each other, though they might both send a light (or microwave) signal to a third observer in the middle.

The light rays that tell us what part of the singularity a given event has access to form what is called the 'particle horizon' of that event and the issue we have been discussing (of which places could possibly have been in thermal equilibrium with which other places) is called the 'horizon problem.'

There are two basic ways out of this, but it would be disingenuous to claim that either is understood at more than the most vague of levels. One is to simply suppose that there is something about the big bang itself that makes things incredibly homogeneous, even outside of the particle horizons. The other is to suppose that for some reason the earliest evolution of the universe happened in a different way than we drew on our graph above and which somehow removes the particle horizons.

The favorite idea of this second sort is called "inflation." Basically, the idea is that for some reason there was in fact a truly huge cosmological constant in the very earliest universe – sufficiently large to affect the dynamics. Let us again think of running a movie of the universe in reverse. In the forward direction, the cosmological constant makes the universe accelerate. So, running it backward it acts as a cosmic brake and slows things down. The result is that the universe would then be older than we would otherwise have thought, giving the particle horizons a chance to grow sufficiently large to solve the horizon problem. The resulting Penrose diagram looks something like this:

[2] More precisely, they will be unable to interact and so reach thermal equilibrium until some late time.



The regions we see at decoupling now have past light cones that overlap quite a bit. So, they have access to much of the same information from the singularity. In this picture, it is easier to understand how these entire universe could be at close to the same temperature at decoupling.

Oh, to be consistent with what we know, this huge cosmological 'constant' has to shut itself off long before decoupling. This is the hard part about making inflation work. Making the cosmological constant turn off requires an amount of fine tuning that many people feel is comparable to the one part in one-hundred thousand level of inhomogeneities that inflation was designed to explain.

Luckily, inflation makes certain predictions about the detailed form of the cosmic microwave background. The modern balloon experiments are beginning to probe the interesting regime of accuracy, and it is hoped that MAP and PLANCK will have some definitive commentary on whether inflation is or is not the correct explanation.

10.4.4 Looking for mass in all the wrong places

Actually, there is a third chapter in our discussion of the cosmological constant. You see, it turns out that the supernovae results and the CMB do not really measure Λ directly, but instead link the cosmological constant to the overall density of matter in the universe. So, to get a real handle on things, one has to know the density of more or less regular matter in the universe as well.

Before we get into how much matter there actually is (and how we find out), I need to tell you about the somewhat funny language that cosmologists use to discuss this question. To get the idea, I'll need to bring back the Einstein equation, and this time I'll add in a part to describe the cosmological constant. As a change from before though, I'll write it in terms of the Hubble expansion rate $H = \frac{1}{a}\frac{da}{dt}$.

$$H^2 - \frac{8\pi G\rho}{3} - \frac{\Lambda}{3} + ka^{-2}c^2 = 0. \tag{10.6}$$

The cosmologists like to reorganize this equation by dividing by H^2. This gives

$$1 - \frac{8\pi G\rho}{3H^2} - \frac{\Lambda}{3H^2} + \frac{kc^2}{H^2a^2} = 0. \tag{10.7}$$

Now, the three interesting cases are $k = -1, 0, +1$. The middle case is $k = 0$. Look at what happens then: the quantity $\frac{8\pi G\rho}{3H^2} + \frac{\Lambda}{3H^2}$, which measures the overall density of stuff (matter or cosmological constant) in 'Hubble units' *must* be one! So, this is a convenient reference point. If we want to measure k, it is this quantity that we should compute. So, cosmologists give it a special name:

$$\Omega \equiv \frac{8\pi G\rho}{3H^2} + \frac{\Lambda}{3H^2}. \tag{10.8}$$

This quantity is often called the 'density parameter,' but we see that it is slightly more complicated than that name would suggest. In particular, I should point out that (like the Hubble 'constant') Ω will in general change with time. If, however, Ω happens to be *exactly* equal to one at some time, it will remain equal to one. So, to tell if the universe is positively curved ($k = +1$), negatively curved ($k = -1$), or [spatially] flat ($k = 0$), what we need to do is to measure Ω and to see whether it is bigger than, smaller than, or equal to one.

By the way, cosmologists in fact break this Ω up into two parts corresponding to the matter and the cosmological constant.

$$\begin{aligned} \Omega_{matter} &\equiv \frac{8\pi G\rho}{3H^2} \\ \Omega_\Lambda &\equiv \frac{\Lambda}{3H^2} \end{aligned} \tag{10.9}$$

Not only do these two parts change with time, but their ratio changes as well. The natural tendency is for Ω_Λ to grow with time at the expense of Ω_{matter} as the universe gets larger and the vacuum energy becomes more important. Anyway, when cosmologists discuss the density of matter and the size of the cosmological constant, they typically discuss these things in terms of Ω_{matter} and Ω_Λ.

So, just how does one start looking for matter in the universe? Well, the place to start is by counting up all of the matter that we can see – say, counting up the number of stars and galaxies. Using the things we can see gives about $\Omega = 0.05$.

But, there are more direct ways to measure the amount of mass around – for example, we can see how much gravity it generates! Remember our discussion of how astronomers find black holes at the centers of galaxies? They use the stars orbiting the black hole to tell them about the mass of the black hole. Similarly, we can use stars orbiting at the edge of a galaxy to tell us about the total amount of mass in a galaxy. It turns out to be much more than what we can see in the 'visible' matter. Also, recall that the galaxies are a little bit clumped together. If we look at how fast the galaxies in a given clump orbit each other, we again find a bit more mass than we expected.

It turns out that something like 90% of the matter out there is stuff that we can't see. For this reason, it is called 'Dark Matter.' Interestingly, although it is attached to the galaxies, it is spread a bit more thinly than is the visible matter. This means that a galaxy is surrounded by a cloud of dark matter than is a good bit larger than the part of the galaxy that we can see. All of these measurements of gravitational effects bring the matter count up to about $\Omega_{matter} = .4$.

Now, there is of course a natural question: Just what *is* this Dark Matter stuff anyway? Well, there are lots of things that it is *not*. For example, it is not a bunch of small black holes or a bunch of little planet-like objects running around. At least, the vast majority is not of that sort. That possibility has been ruled out by studies of gravitational lensing (a subject on which I wish I had more time to spend). Briefly, recall that general relativity predicts that light 'falls' in a gravitational field and, as a result, light rays are bent toward massive objects. This means that massive objects actually act like lenses, and focus the light from objects shining behind them. When such a 'gravitational lens' passes in front of a star, the star appears to get brighter. When the lens moves away, the star returns to its original brightness. By looking at a large number of stars and seeing how often they happen to brighten in this way, astronomers can 'count' the number of gravitational lenses out there. To make a long story short, there are too few such events for all of the dark matter to be clumped together in black holes or small planets. Instead, most of it must be spread out more evenly.

Even more interestingly, it cannot be just thin gas..... That is, there are strong arguments why the dark matter, whatever it is, cannot be made up of protons and neutrons like normal matter! To understand this, we need to continue the story of the early universe as a movie that we run backward in time. We discussed earlier how there was a very early time (just before decoupling) when the Universe was so hot and dense that the electrons were detached from the protons. Well, continuing to watch the movie backwards the universe becomes even more hot and dense. Eventually, it becomes so hot and dense that the nuclei fall apart.

Now there are just a bunch of free neutrons and protons running around, very evenly spread throughout the universe. It turns out that we can calculate what should happen in such a system as the universe expands and cools. As a result, one can calculate how many of these neutrons and protons should stick together and form Helium vs. how many extra protons should remain as Hydrogen. This process is called 'nucleosynthesis.' One can also work out the proportions of other light elements like Lithium.... (The heavy elements were not made in the big bang itself, but were manufactured in stars and supernovae.) To cut short another long story, the more dense the stuff was, the more things stick together and the more Helium and Lithium should be around. Astronomers are pretty good at measuring the relative abundance of Hydrogen and Helium, and the answers favor roughly $\Omega_{normal\ matter} = .1$, – the stuff we can see plus a little bit more. As a result, this means that the dark matter is *not* made up of normal things like protons and neutrons. By the way, physicists call such matter 'baryonic[3] matter' so that this fact is often quoted as $\Omega_{baryon} = .1$. A lot of this may be in the form of small not-quite stars and such, but the important point is that at least 75% of the matter in the universe really has to be stuff that is not made up of protons and neutrons.

So, what is the dark matter then? That is an excellent question and a subject of much debate. It may well be the case that all of this unknown dark matter is some strange new kind of tiny particle which simply happens not to interact with regular matter except by way of gravity. A number of ideas have been proposed, but it is way too early to say how likely they are to be right.

10.4.5 Putting it all together

The last part of our discussion is to put all of this data together to see what the implications are for Ω_Λ and Ω_{matter}. I will give you a handout with some graphs showing a lot of this data. Many of these graphs (and some other stuff) come from from a talk given by Sean Carroll. The transparencies for his talk are available on the web at: (http://pancake.uchicago.edu/ carroll/talks/ltalk/). You can look them up if you want to see the data before I get around to handing it out.

These graphs show that that each of the three measurements put some kind of constraint on the relationship between Ω_{matter} and Ω_Λ, corresponding to a (wide) line in the $\Omega_{matter} - \Omega_\Lambda$ plane. You can see that, taken together, the data strongly favors a value near $\Omega_{matter} = .4$, $\Omega_\Lambda = .6$. That is, 60% of the energy in the universe appears to be vacuum energy!

Now, what is really impressive here is that any two of the measurements would predict this same value. The third measurement can then be thought of as a double-check. As the physicists say, any two lines in a plane intersect somewhere, but to get three lines to intersect at the same point you have to do something right.

This means that the evidence for a cosmological constant is fairly strong – we have not just one experiment that finds it, but in fact we have another independent measurement that confirms this result. However, the individual measurements are not all that accurate and may have unforeseen systematic errors. So, we look forward to getting more and better data in the future to see whether these results continue to hold up.

We are in fact expecting to get a lot more data over the next few years. Two major satellite experiments (called 'MAP' and 'PLANCK') are going to make very detailed measurements of the Cosmic Microwave Background which should really tighten up the CMB constraints on Ω_{matter} and Ω_Λ. It is also hoped that

[3]Protons and neutrons are examples of a class of particles that physicists call 'baryons.' Baryons are particles that are made up of three quarks.

these experiments will either confirm or deny the predictions of inflation in more detail.

By the way, it is a rather strange picture of the universe with which we are left. There are several confusing issues. One of them is "where does this vacuum energy come from?" It turns out that there are some reasonable ideas on this subject coming from quantum field theory... However, while they are all reasonable ideas for creating a vacuum energy, they all predict a value that is 10^{120} times too large. I will take a moment to state the obvious: 10^{120} is an *incredibly* huge number. A billion is ten to the ninth power, so 10^{120} is one billion raised to the thirteenth power. As a result, physicists are always asking, "Why is the cosmological constant so small?"

Another issue is that, as we mentioned, Ω_Λ and Ω_{matter} do not stay constant in time. They change, and in fact they change in different ways. There is a nice diagram (also from Sean Carroll) showing how they change with time. I'll hand this out too. What you can see is that, more or less independently of where you start, the universe naturally evolves toward $\Omega_\Lambda = 1$. On the other hand, back at the big bang Ω_Λ was almost certainly near zero. So, an interesting question is: "why is Ω_Λ only now in the middle ground ($\Omega_\Lambda = .6$), making it's move between zero and one?" For example, does this argue that the cosmological constant is not really constant, and that there is some new physical principle that keeps it in this middle ground? Otherwise, why should the value of the cosmological constant be such that Ω_Λ is just now making it's debut? It is not clear why Λ should not have a value such that it would have taken over long ago, or such that it would still be way too tiny to notice.

10.5 The Beginning and The End

Well, we are nearly finished with our story but we are not yet at the end. We traced the universe back to a time when it was so hot and dense that the nuclei of atoms were just forming. We have seen that there is experimental evidence (in the abundances of Hydrogen and Helium) that the universe actually was this hot and dense in its distant past. Well, if our understanding of physics is right, it must have been even hotter and more dense before. So, what was this like? How hot and dense was it? From the perspective of general relativity, the most natural idea is that the farther back we go, the hotter and denser it was. Looking back in time, we expect that there was a time when it was so hot that protons and neutrons themselves fell apart, and that the universe was full of things called quarks. Farther back still, the universe so hot that our current knowledge of physics is not sufficient to describe it. All kinds of weird things *might* have happened, like maybe the universe had more than four dimensions back then. Maybe the universe was filled with truly exotic particles. Maybe the universe underwent various periods of inflation followed by relative quiet.

Anyway, looking *very* far back we expect that one would find conditions very similar to those near the singularity of a black hole. This is called the 'big bang singularity.' Just as at a black hole, general relativity would break down there and would not accurately describe what was happening. Roughly speaking, we would be in a domain of quantum gravity where, as with a Schwarzschild black hole, our now familiar notions of space and time may completely fall apart. It may or may not make sense to even ask what came 'before.' Isn't that a good place to end our story?