Projection of relativistically moving objects on a two-dimensional plane, the 'train' paradox and the visibility of the Lorentz contraction

E B Manoukian and S Sukkhasena

School of Physics, Suranaree University of Technology, Nakhon Ratchasima 30000, Thailand

Abstract
Although many papers have appeared on the theory of photographing relativistically moving objects, pioneered by the classic work of Penrose and Terrell, three problems remain outstanding. (1) There does not seem to exist a general formula which gives the projection of a relativistically moving object, applicable to any object no matter how complicated, on a two-dimensional plane in conformity with Terrell's observation. (2) No resolution seems to have been provided for the associated so-called 'train' paradox. (3) No analytical demonstration seems to have been offered on how the Lorentz contraction may be actually detected on a photograph. This paper addresses all of these three problems. The analysis does not require any more than trigonometry and elementary differentiation.

1. Introduction

Since the early classic work of Penrose [1], Terrell [2] and Weisskopf [3] on the appearance of relativistically moving objects several refinements and extensions of the above work have been carried out (see, e.g., [4–11]). Three problems seem, however, to remain outstanding.

 (1) There does not seem to be in the literature a general formula which provides the projection of a relativistically moving object, applicable to *any* object no matter how complicated, on a two-dimensional plane in conformity with Terrell's observation. Terrell's observation is that different points on the object must 'emit' light at different times in order to reach an observation point simultaneously.
 (2) No resolution seems to have been offered of one of the most puzzling aspects in the above investigations referred to as the 'train' paradox. No much attention has been given to this in the literature. This paradox has its roots in the early work of Terrell [2] and Weisskopf [3] and was emphasized by Mathews and Lakshmanan [8] almost 30 years ago. In its simplest terms, the latter paradox arises in the following manner. One often reads in the above

papers that an object appears to be rotated when in relative motion to an observation frame due to the fact that different points on the object must 'emit' light at different times in order to reach an observation point simultaneously. Such an inference seems to indicate that a rectangular block, for example, sliding on (smooth) rails and the edges of its lower side in contact with them, appears off them due to the relative motion with the rails stationary relative to the observer, and hence the paradox. The same reasoning may be applied, as shown below, to an object with a horizontal flat top touching a 'smooth' flat horizontal plane. Again this would seem to imply, in particular cases, that one end of the object had miraculously broken and gone through the flat plane due to the relative motion.

(3) No analytical demonstration seems to have been given on how the Lorentz contraction may be detected on a photograph. The key point that we discover is that, due to Terrell's observation, the portion of a moving ruler to the left of the observer seems elongated, while the portion of the ruler to the right of the observer seems to be shrunk. These effects work, respectively, against and with the Lorentz contraction, in general masking it. Accordingly, we may locate a critical point on the ruler such that the Lorentz contraction is visible for a small interval on the ruler around this point.

The word 'appears' or the statement 'appears as on a photograph' have caused some confusion over the years. Any such wordings necessarily involve assumptions used in one's analysis. To be precise, the latter are meant in the following manner, as working hypotheses, in the present investigation and are based on taking into account these three points:

(i) Terrell's observation that different points on an object must 'emit' light at different times in order to reach an observation point simultaneously. There is an inherit limitation in considering a point observation site. The removal of such a restriction is certainly a formidable task, which will not be attempted in this paper.

(ii) The Lorentz transformations.

(iii) The piercing by these light rays of an appropriate two-dimensional plane in the observation frame.

In section 2, we provide a complete and elementary derivation of the explicit (nonlinear) transformations arising from the application of the three points (i)–(iii), which may be directly applied to *any* object no matter how complicated. The formulae obtained are easily accessible to students and sufficiently precise to illustrate faithfully the main features of the Terrell effect. These transformations are appropriately referred to as nonlinear Terrell transformations. The closest investigation to these transformations was given by Hickey [9], who, however, applied methods of mapping out the tangents to points on an object. The latter also provides no room for resolving the 'train' paradox. Although the demonstration of the absence of a paradox seems non-trivial, we provide in section 3 a rather *elementary* resolution of the 'train' paradox. The visibility of the Lorentz contraction is studied in section 4, where a critical point on a moving ruler is singled out around which the Lorentz contraction is visible for a partitioning of the ruler around this point. Section 5 deals with our conclusions.

2. Nonlinear Terrell transformations

Consider the relative motion of the proper inertial frame F' of an object to be along the x-axis of an observation frame F with speed v. Let (x', y', z'), (x, y, z) denote, respectively, the labellings of an arbitrary point on the object in the corresponding F' and F frames. The observation point $\bar{0}$ is at a height h, along the z-axis (see figure 1) above the origin 0 of the F-frame. When the origins 0 and $0'$, at $t = 0$ and $t' = 0$, of F and F' frames coincide and the observer at $\bar{0}$ sees this happening he takes a snap shot of the object. Since the observer at $\bar{0}$ sees the origin coincide only at a later time equal to h/c, the time that light was 'emitted' from (x, y, z) to reach $\bar{0}$ is given by
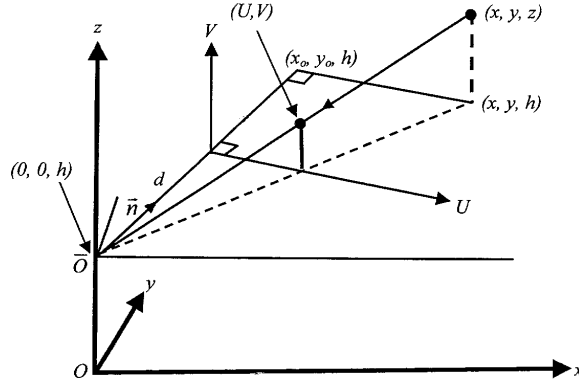
Figure 1. The observation coordinate system. The observer is at $\overline{0}$ at a height h above the origin. The $U–V$ plane is specified by a unit vector $\vec{n}$ chosen parallel to the $x–y$ plane. The origin of the $U–V$ plane is at a distance d from $\overline{0}$. A light ray from (x, y, z), a given point on the object in relative motion as labelled in the observation frame, on its way to $\overline{0}$ pierces the $U–V$ plane at (U, V).

$$t = -\frac{\sqrt{x^2 + y^2 + (z - h)^2}}{c} + \frac{h}{c}. \tag{1}$$

We are interested in all the light rays from the object which reach point $\overline{0}$ simultaneously. The light ray from a point (x, y, z) on the object will pierce a $U–V$ plane specified by a unit vector $\vec{n}$ as shown in figure 1 perpendicular to the former. Here U and V define coordinate axes on the two-dimensional plane on which the object is projected. A point (x', y', z') on the object, as labelled in its rest frame F', will be mapped into a point (U, V) on the two-dimensional plane. d denotes the distance from $\overline{0}$ (the observer) to the origin of the $U–V$ plane. For concreteness $\vec{n}$, emerging from $\overline{0}$ and defining what is called the optic axis, was chosen to be parallel to the $x–y$ plane. The former, in turn, is defined by choosing a point (x_0, y_0, h) as shown in figure 1, and hence may be written as

$$\vec{n} = \left(\frac{x_0}{\sqrt{x_0^2 + y_0^2}}, \frac{y_0}{\sqrt{x_0^2 + y_0^2}}, 0 \right) \equiv (n_1, n_2, 0). \tag{2}$$

From figure 1,

$$(x - x_0)^2 + (y - y_0)^2 + x_0^2 + y_0^2 = x^2 + y^2 \tag{3}$$

or

$$x_0^2 + y_0^2 = (xn_1 + yn_2)^2 \tag{4}$$

and

$$U^2/d^2 = [(x - x_0)^2 + (y - y_0)^2]/(x_0^2 + y_0^2) \tag{5}$$

giving the solution

$$U = d\frac{(xn_2 - yn_1)}{(xn_1 + yn_2)}. \tag{6}$$

Also

$$V/(z - h) = \sqrt{d^2 + U^2}\Big/\sqrt{x^2 + y^2} \tag{7}$$

giving the solution

$$V = d\frac{(z - h)}{(xn_1 + yn_2)}. \tag{8}$$
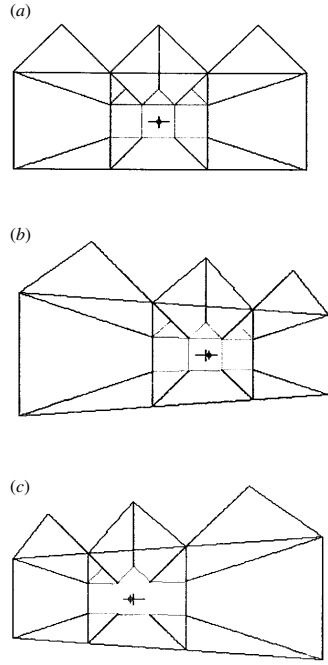
(a)

(b)

(c)

Figure 2. Projection on the U–V plane for $\beta = 0$, (a) $\vec{n} = (0, 1, 0)$, (b) $\vec{n} = (0.0712, 0.9975, 0)$, (c) $\vec{n} = (-0.0712, 0.9975, 0)$. The crossed line denotes a midpoint of the houses. The crossed small circle denotes the centre of the U–V plane.

The transformations (6) and (8), as shown above, follow from the examination of figure 1. These transformations will now allow us to find the mapping of a point (x', y', z') on the object, as labelled in its rest frame F', into the U–V plane. To this end, from the Lorentz transformations $x' = \gamma(x - vt)$, $y' = y$, $z' = z$, using equation (1), we may solve for x and substitute the latter in (6) and (8) to obtain the transformations we are seeking, where we set $v/c = \beta$,

$$U = d \frac{\gamma\big[(x' + \gamma\beta h) - \beta\sqrt{(x' + \gamma\beta h)^2 + y'^2 + (z' - h)^2}\big]n_2 - y'n_1}{\gamma\big[(x' + \gamma\beta h) - \beta\sqrt{(x' + \gamma\beta h)^2 + y'^2 + (z' - h)^2}\big]n_1 + y'n_2} \tag{9}$$

$$V = d \frac{(z' - h)}{\gamma\big[(x' + \gamma\beta h) - \beta\sqrt{(x' + \gamma\beta h)^2 + y'^2 + (z' - h)^2}\big]n_1 + y'n_2} \tag{10}$$

where $\gamma = (1 - \beta^2)^{-1/2}$ and (x', y', z') denotes any given point on the object in its *rest* frame. Unlike the Lorentz transformations $(t, x, y, z) \rightarrow (t', x', y', z')$ for a time–space point, the transformations in (9) and (10), $(x', y', z') \rightarrow (U, V)$, are obviously nonlinear. The latter may be appropriately referred to as nonlinear Terrell transformations.

An explicit application of the transformations (9) and (10) is given in figures 2 and 3, respectively, for $\beta = 0$ and 0.9 for various directions of the optic axis specified by the unit vector $\vec{n}$. Pertinent remarks concerning these figures will be made in section 5.

3. Resolution of the 'train' paradox

Let first $\beta = 0$. Consider a horizontal line parallel to the x'-axis, that is, take y' and z' as fixed. Suppose that the object in question touches this line at some point, say, (a, y', z'). x' takes on
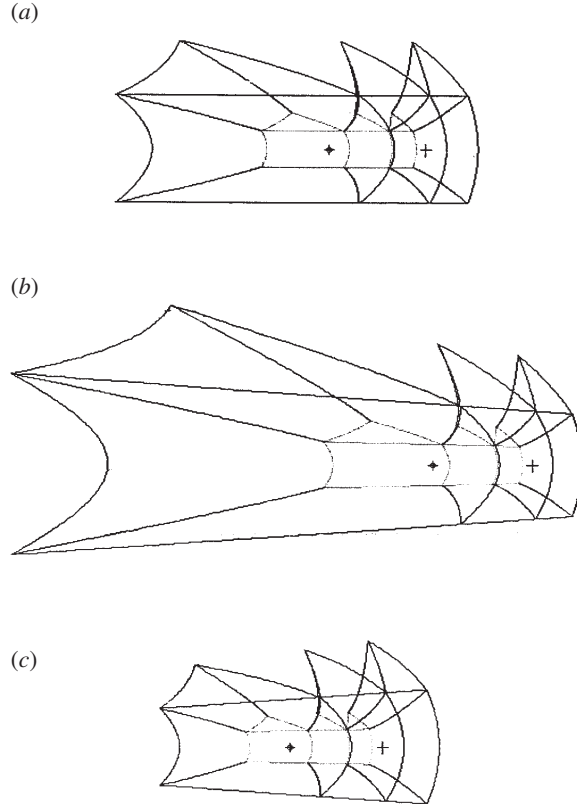
(a)

(b)

(c)

Figure 3. Projection on the U–V plane for $\beta = 0.9$ with $\vec{n}$ in (a), (b), (c), respectively, as in figure 2. The observation frame moves to the left with speed βc.

arbitrary values along the line with $x' = a$ corresponding to the point of contact just mentioned (see figure 4). That is, a is the x' coordinate of a point of contact with the horizontal line (for $\beta = 0$) and also provides a reference point from which other points will be defined as we shall now see. From (6) and (8)

$$U_a = d\frac{(an_2 - y'n_1)}{(an_1 + y'n_2)} \qquad V_a = d\frac{(z' - h)}{(an_1 + y'n_2)} \tag{11}$$

giving

$$U - U_a = d\frac{y'(x' - a)}{(an_1 + y'n_2)(x'n_1 + y'n_2)} \tag{12}$$

$$V - V_a = -d\frac{(z' - h)n_1(x' - a)}{(an_1 + y'n_2)(x'n_1 + y'n_2)}. \tag{13}$$

That is,

$$V - V_a = -\frac{(z' - h)}{y'}n_1(U - U_a) \tag{14}$$

or

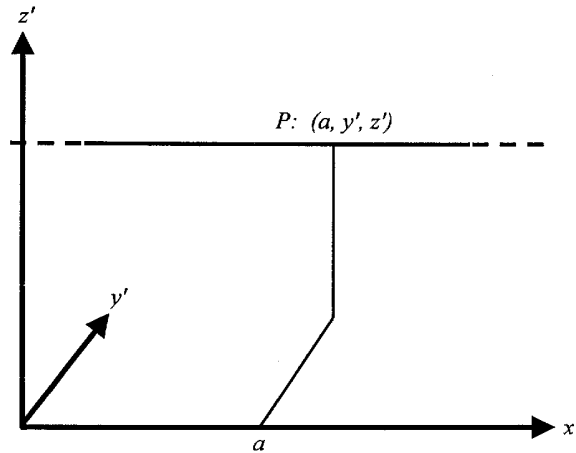$$V = \frac{(z' - h)}{y'}[-n_1 U + n_2 d]. \tag{15}$$

Figure 4. A given horizontal line parallel to the x'-axis. The point (a, y', z') on the object, for $\beta = 0$, touches the line at P.

Hence the above horizontal line is mapped into a straight line in the U–V plane. We recall that (15) was derived for $\beta = 0$. For $\beta \neq 0$, it is readily checked that U and V given, respectively, in (9) and (10) fall on the same line for $x' = a$. That is, for $\beta \neq 0$, U and V also satisfy equation (15). *Hence any point on the object which for $\beta = 0$ lies on the straight line (15) must also lie on the same line for $\beta \neq 0$.* This provides the resolution of the 'train' paradox.

For example, refer to figure 2(*b*) (and (*a*)) and consider an imaginary line joining the tips of the three roofs as drawn in the observation frame. Now consider the corresponding case for $\beta = 0.9$ in figure 3(*b*) with the houses pulled, so to speak, to slide along this line relative to a stationary observer. Here one has the impression that the tip of the roof of the first house on the left has cut through and passed through this line. The above demonstration shows that the tips of the roofs of all the three houses remain *always in contact* for $\beta \neq 0$ as well with the straight line but at different points for $\beta = 0$ and $\beta \neq 0$ due to the relative motion.

4. Visibility of the Lorentz contraction

Consider two rulers of equal proper lengths each moving to the right with speed v, with one approaching and one receding from the observer (see figure 5). The end points of the ruler on the left are labelled by 1, 2, while the ones on the right by 3, 4 (see figure 5). In reference to the ruler on the left (approaching the observer), in order that light 'emitted' from the end points 1 and 2 reach the observer simultaneously, light should be 'emitted' from point 1 first. In the meantime, the ruler moves to the right and the end point 2 reaches some point $2'$ and 'emits' light to reach the observer at the same time as light that was 'emitted' from end point 1 earlier. Accordingly, the ruler seems to be elongated with end points effectively defined by 1 and $2'$. On the other hand, the same reasoning applied to the ruler on the right (receding from the observer) shows that the ruler seems to be shrunk with end points effectively defined by $3'$ and 4. These facts have nothing to do with the Lorentz transformations or the Lorentz contraction, but they would contribute in forming the final projected image on the two-dimensional plane, and would, in general, mask the visibility of the Lorentz contraction, which is of different nature.

The old fundamental and critical question now comes to haunt us. Can we photograph the Lorentz contraction? To answer this question, consider a ruler moving to the right with speed βc such that its lower edge is at $z' = 0$, and $y' \neq 0$ is arbitrary but has some fixed value. We consider the ruler to have a portion of it to the right of the observer and a portion to the left of it.
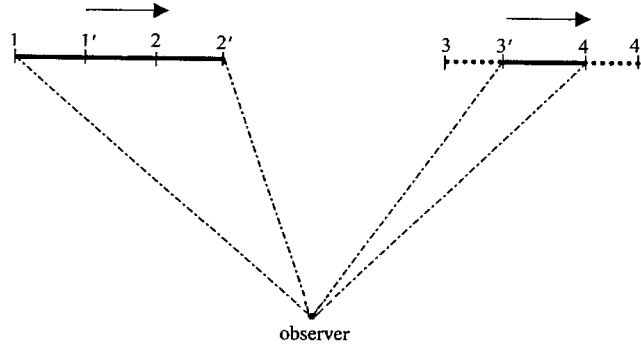
Figure 5. For light 'emitted' from both ends of the moving ruler on the left to reach the observer simultaneously, the ruler labelled by 1, 2 in its proper frame is effectively defined by 1, $2'$ to the observer and seems to be elongated. Similarly, the ruler on the right, which is labelled by 3, 4 in its proper frame, is effectively defined by $3'$, 4 to the observer and seems to be shrunk.

We set the observer at the origin 0 of his coordinate system, i.e. set $h = 0$. We also choose the optic axis to be specified by the unit vector $\vec{n} = (0, 1, 0)$. Then

$$U = \frac{d}{y'} \gamma \left[x' - \beta \sqrt{x'^2 + y'^2} \right] \tag{16}$$

($V = 0$) corresponding to the lower edge of the ruler.

Suppose that the ruler is partitioned with small increments $\Delta x'$. Consider a small change ΔU in (16) along x'-axis, that is along the ruler, corresponding to a small increment $\Delta x'$. By elementary differentiation, (16) gives

$$\Delta U = \frac{d}{y'} \gamma \left[1 - \beta \frac{x'}{\sqrt{x'^2 + y'^2}} \right] \Delta x' \tag{17}$$

which may be conveniently rewritten in the equivalent form

$$\Delta U = \frac{d}{y'} \left[\frac{\Delta x'}{\gamma} + \gamma \beta \left(\beta - \frac{x'}{\sqrt{x'^2 + y'^2}} \right) \Delta x' \right]. \tag{18}$$

Since $\left| x' / \sqrt{x'^2 + y'^2} \right| < 1$, we may infer that for any given $\beta < 1$, around the critical point defined by

$$\frac{x'}{\sqrt{x'^2 + y'^2}} = \beta \tag{19}$$

we have

$$\Delta U = \frac{d}{y'} \frac{\Delta x'}{\gamma} \qquad \left(\Delta U|_{\beta=0} = \frac{d}{y'} \Delta x' \right) \tag{20}$$

which apart from the trivial constant scaling factor d/y', is the famous Lorentz contraction formula. *That is, around a certain point of the ruler, the Lorentz contraction is visible* in the U–V plane on the two-dimensional surface for a small increment of proper length $\Delta x'$ drawn around this critical point.

To find the above critical point in question on the ruler, draw a line making an angle $\theta = \cos^{-1}(\beta)$ with the x'-axis (see figure 6) before setting the ruler to move with a given pre-assigned speed βc. This line will cross a point on the lower edge of the ruler defining the critical point in question.
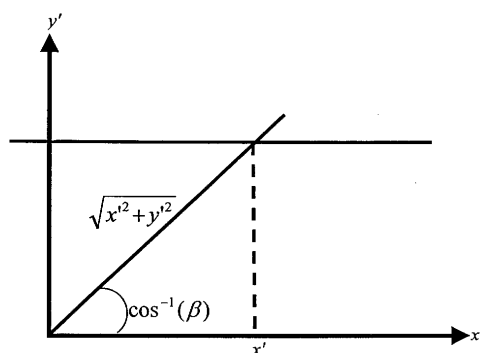
Figure 6. Line making an angle $\theta = \cos^{-1}(\beta)$ with the x'-axis in the ruler's proper frame. This line crosses the ruler at the critical point x' about which the Lorentz contraction is visible *in the* $U-V$ plane for a small interval $\Delta x'$ about x'.

5. Conclusions

That there is contraction on the right-hand side of the houses (figure 3) and relative elongation on their left-hand sides is clear when referring to figure 5 and is due to the fact that light 'emitted' from the houses from different parts must be 'emitted' at different times to reach the observation point $\overline{0}$ simultaneously. We have also seen that all the lines that are parallel to the x'-axis remain straight lines in the $U-V$ plane for all β, and this is the content of the resolution of the 'train' paradox given in section 3. It is readily checked from (9) and (10), that any lines parallel to the y' or the z' axes for $\beta = 0$ become curved in the $U-V$ plane for $\beta \neq 0$. The latter are quite clear in figure 3(b), for example. Finally, the effect of relative contraction and elongation of a ruler discussed at the beginning of this section, which, in general, would mask the Lorentz contraction, becomes, so to speak, 'unmasked' around a particular point on the ruler moving with any given speed β and is easily located. The Lorentz contraction becomes visible for a small interval of length $\Delta x'$ around this critical point.

All of the above observations and deductions follow from our general formulae obtained, which are easily accessible to students and sufficiently precise to illustrate faithfully the main features of the Terrell effect. The next programme in this fundamental problem of relativity is how to generalize the observation site from a pointlike to a non-pointlike one. This is a formidable problem which will not be attempted here and remains to be tackled.

References

[1] Penrose R 1959 *Proc. Camb. Phil. Soc.* **55** 137
[2] Terrell J 1959 *Phys. Rev.* **4** 1041
[3] Weisskopf V F 1960 *Phys. Today* **9** 24
[4] Yngström S 1962 *Ark. Fys.* **23** 367
[5] Scott G D and Viner M R 1965 *Am. J. Phys.* **33** 534
[6] Scott G D and Driel H J van 1970 *Am. J. Phys.* **38** 971
[7] McGill N C 1968 *Contemp. Phys.* **9** 33
[8] Mathews P M and Lakshmanan M 1972 *Nuovo Cimento* **12** 168
[9] Hickey F R 1979 *Am. J. Phys.* **47** 711
[10] Burke J R and Strode F J 1991 *Am. J. Phys.* **59** 912
[11] Howard A, Kitchen L and Dance S 1995 *University of Melbourne preprint*