Two-dimensional appearance of a relativistic cube

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The two-dimensional or "photographic" appearance of a cube moving at relativistic speeds is investigated. It is shown that the appearance of the cube depends upon whether the camera is held fixed with the optic axis perpendicular to the direction of motion or panned to follow the cube as it moves. The Lorentz contraction, as it is usually defined, is not visible in either case. However, if the camera is rotated, the photographic appearance has some of the characteristics of a picture of a stationary rotated cube.

I. INTRODUCTION

Approximately 20 years ago, Penrose,[1] Terrel,[2] and Weisskopf[3] showed that the appearance of a rapidly moving object differs from what is predicted by a simple contraction of its length in the direction of motion. Since that time controversy has arisen regarding the exact nature of the appearance of such an object.

Consider a cube moving rapidly from left to right past a stationary observer. The rear (left) face becomes visible while the cube still appears to be located to the left of the observer. This effect is caused by the finite speed of light and is distinct from the Lorentz contraction of the side of the cube parallel to the direction of motion. The authors of the articles mentioned above concluded that for large distances between the object and the camera, the resulting appearance would be that of a cube rotated through an angle which depends on the speed of the cube. Even though the term "visual" appearance was sometimes used by these authors, the descriptions actually referred to the "photographic" appearance, i.e., the projection of points from a three dimensional object onto a two-dimensional "photograph."

Scott and van Driel[4] and in more detail, Mathews and Lakshmanan[5] later showed that the appearance of such objects cannot be explained in terms of a simple rotation. Nevertheless, some ambiguities still seem to exist in the literature regarding the description of the two-dimensional picture.[6]

It is the intent of this paper to show that one must consider the orientation of the camera axis with respect to the object when discussing the photographic appearance of rapidly moving objects. If the camera is held in a fixed position perpendicular to the direction of motion of the cube, statements found in the literature such as "planes perpendicular to the x axis remain so" [5] are valid. However, if the camera is panned so that the optic axis always points toward the center of the cube as it moves, then that statement, as applied to the two-dimensional picture, can be misleading. The rotation of the projection plane itself in the latter case leads to an impression of a rotated cube.

II. RAPIDLY MOVING CUBE

For the purpose of this paper, consideration will be restricted to the photographic appearance of a continuously self-luminous cube moving from left to right along the x axis. The cube will be considered to be transparent enough so that the observer or camera can "see" all sides. Only a speed of $\beta = 0.9$ will be discussed, but the results apply for any relativistic speed. Any distortion caused by the rotation of the camera at high speed has been neglected.

The coordinates of any point on the cube in its own frame are related to the coordinates in the stationary or camera frame of reference by the Lorentz transformation if the origins of the two coordinate systems coincide at $t = t' = 0$:

$$x' = \gamma(x - vt),\ y' = y,$$
$$z' = z,\ t' = \gamma(t - vx/c^2),$$
$$\gamma^{-2} = (1 - \beta^2),\ \beta = v/c. \tag{1}$$

Here the primed coordinates refer to a point on the cube in its own frame while the unprimed coordinates locate the same point in the camera frame of reference.

Light which arrives at the camera at time t must have been emitted from different points on the cube at different times owing to the finite speed of light. For a camera located on the z axis at location $(0,0,z_0)$, it has been shown by Scott and Viner[7] and others that the apparent coordinates of the cube at time t are given by

$$x = \gamma\{x' + \beta\gamma ct - \beta[(x' + \beta\gamma ct)^2 + y'^2 + (z' - z_0')^2]^{1/2}\},$$
$$y = y',\ z = z'. \tag{2}$$

McGill[8] has distinguished between "ideal," "stereoscopic," and "two-dimensional" methods of observation of an object. The ideal method requires an "ideal" apparatus capable of determining the exact spatial position of the perceived object. Under these circumstances, Eq. (2) can be used to describe the appearance of the cube. For the ideal method of observation, Yngstrom[9] and others have shown that planes parallel with the x'-y' plane appear parallel with the x-y plane and that a plane whose normal is parallel with the x' axis appears as one sheet of a hyperboloid.

The stereoscopic method of observation has the inherent difficulty that light arriving simultaneously at two spatially separated cameras must have been emitted from the cube at different times. This introduces some distortion which complicates the description of the appearance of the cube.

Finally there is the two-dimensional or photographic technique. The photographic appearance is obtained by considering all of the light rays that arrive simultaneously

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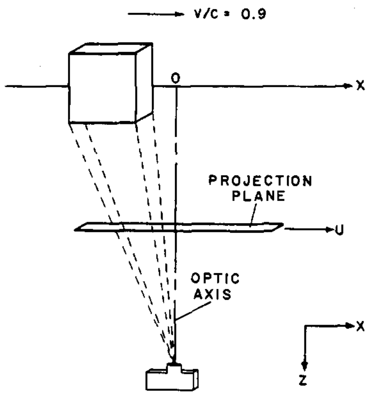

Fig. 1. Camera held fixed with optic axis pointing at origin 0. The u-v plane is parallel with the x-y plane.

at the camera lens and then projecting these rays back through the lens onto a two-dimensional plane. This plane can be arranged so that it is always at a fixed distance from the camera lens.

Two orientations of the projection plane are possible. In Fig. 1 the camera is held fixed such that the optic axis always makes an angle of 90° with the x axis. In this case, the two-dimensional image cannot be represented by a simple rotated cube. The reasons for this will be discussed later. The second orientation, shown in Fig. 2, allows the camera to rotate so that it always points toward the center of the object. In this case the projection plane, which is perpendicular to the optic axis, is inclined at an angle to the x-y plane. As the cube moves along the x axis from left to right, the projection plane rotates about the camera and the resulting "photographs" at each instant have some of the characteristics of a rotated cube for certain orientations of the camera. Terrel's assumption of an "apparent rotation" seems to be valid in this case.

III. PROJECTION IN TWO DIMENSIONS

Shirer and Bartel[10] have considered the general problem of the photographic appearance of a relativistic object, i.e., the projection of the coordinates of a three-dimensional object onto a two-dimensional plane. If u is the vertical coordinate of the projection plane and v is the horizontal coordinate, the relation between the u-v coordinates and the cartesian coordinates is given by

$$u = (z_0 x + x_c z - x_c z_0)/(x_c x - z_0 z + z_0^2),$$
$$v = y(x_c^2 + z_0^2)^{1/2}/(x_c x - z_0 z + z_0^2), \tag{3}$$

when the camera is located at $(0,0,z_0)$. In these equations (x,y,z) are the coordinates of a point on the cube in the camera frame of reference and x_c locates the center of the cube on the x axis. Equation (2) relates these coordinates to the coordinates of the same point in the object frame.

Since $u = u(x,z)$ and $v = v(x,y,z)$ while $x = x(x',y,z)$ for fixed time t, the slope of any line in the u-v plane can be determined from

$$\frac{dv}{du} = \left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial x'}\frac{dx'}{dz} + \frac{\partial v}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial v}{\partial z}\right) \Big/ \left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial x'}\frac{dx'}{dz} + \frac{\partial u}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial u}{\partial z}\right). \tag{4}$$

A. Camera fixed

Figure 1 shows the camera held fixed so that it is always pointed toward the origin. The expression for u and v in this case can be derived from basic principles, but it is simpler to obtain them from Eq. (3) by taking the special case when the center of the cube is at the origin, i.e., $x_c = 0$. In this case, the projection plane is parallel with the x-y plane. The resulting expressions are

$$u = x/(z - z_0), \quad v = y/(z - z_0). \tag{5}$$

Consider a line parallel with the x' axis, i.e., y' and z' are constant. The slope of this line in the projection plane, according to Eq. (4), is zero. Therefore, all lines parallel with the x' axis are parallel with the u axis. Since we are projecting points onto a two-dimensional plane, all planes parallel with the x'-y' plane will appear parallel with the u-v plane.

Lines parallel with the y' axis will project as curved lines on the u-v plane. This result is to be expected because light from the center of the line will arrive at the camera before light from the ends of the line.

For lines parallel with the z' axis, Eq. (4) gives

$$\frac{dv}{du} = \frac{y[(x' + \beta\gamma ct)^2 + y^2 + (z - z_0)^2]^{1/2}}{\beta\gamma(z - z_0)^2 + x[(x' + \beta\gamma ct)^2 + y^2 + (z - z_0)^2]^{1/2}}, \tag{6}$$

where x is given by Eq. (2). This result is given in detail only to show the weak dependence on z that is present. For a distant observer, one for which $z \ll z_0$, the slope of this line on the u-v plane is constant.

Figure 3 shows a sketch of the photographic appearance of a cube approaching and receding from the origin at a speed $\beta = 0.9$. The cube measures two units on each side and the camera is located on the z axis five units from the origin. Since the side of the cube facing the observer (side $ABCD$) always remains parallel with the u-v plane, the appearance is clearly not that of a rotated cube. It is interesting to note that as the cube progresses from left to right, this face contracts in length steadily while the rear face ($ADEF$) becomes more elongated. Therefore, the Lorentz contraction, as defined by Eq.(1), is definitely not visible in this photograph.

Another very interesting effect is that the length, in the direction of motion, of the entire two-dimensional picture decreases steadily after the cube has passed the origin. This is because as the cube approaches, light from edge EF arriving simultaneously at the camera with light from edge BC has traveled a further distance and was emitted at an earlier time. This results in a "stretched-out" appearance.

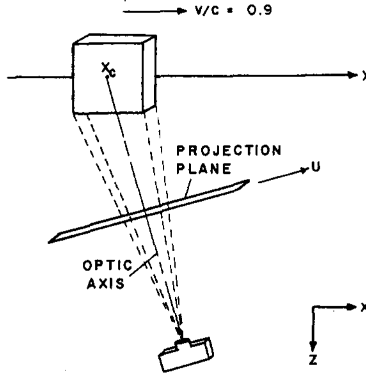

Fig. 2. Camera panned to follow the moving cube. Optic axis always points toward x_c, the center of the cube on the x axis. The v and y axes are parallel but the u and x axes are not.

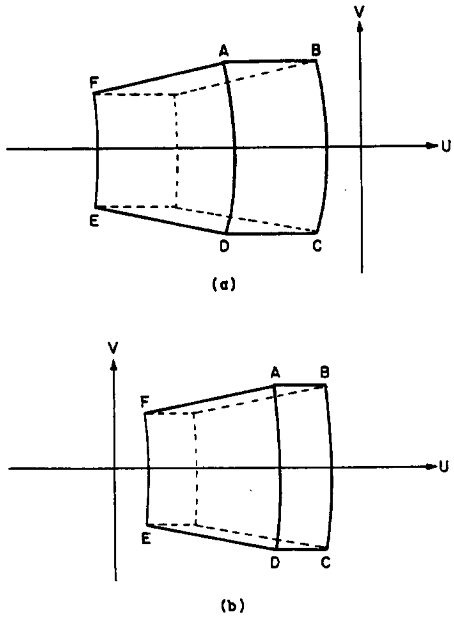

Fig. 3. Two-dimensional appearance of a cube (a) approaching and (b) receding from the origin when the camera is held fixed with the optic axis normal to the x-y plane. Face $ABCD$ remains parallel with the u-v (and x-y) plane and edges AB and CD are parallel with the u axis. Since one face of the cube cannot remain in the same plane under a rotation, the pictures cannot be interpreted in terms of a rotation.

Once the cube passes the origin the same effect in reverse, i.e., light from edge BC travels the greater distance, serves to shorten the two-dimensional appearance.

B. Camera panned

For comparison, consider the case in which the camera is panned so that the optic axis is always directed from the camera lens to the center of the cube. In this case, shown in Fig. 2, the u-v plane is inclined at an angle to the x-y plane at any given time t. As the cube moves to the right, the projection plane rotates in a circular path around the camera. The u and v coordinates of any point on the cube are given by Eq. (3) at any instant.

For this case, lines parallel with the x' axis are not parallel with the u axis as they were for the fixed camera. The slope of a line parallel with the x' axis is

$$\frac{dv}{du}=\frac{-yx_c(x_c^2+z_0^2)^{1/2}}{x_c^2z_0-z_0^2z+z_0^3-x_c^2z}. \qquad (7)$$

It is important to note that in addition to the weak dependence on z, the slope is directly proportional to x_c. Therefore as the cube passes the origin ($x_c = 0$), the slope of a line on the side edge of the cube, such as edge AB, goes from positive to negative. Similarly, the slope of line CD goes from negative to positive. When the center of the cube is at the origin, the slope is zero and the result is identical with that obtained from a fixed camera as considered above.

Lines parallel with the y' axis will appear curved on the u-v plane and lines parallel with the z' axis have a constant slope apart from a weak dependence on z for $z \ll z_0$ similar to that found in Eq. (6).

The photographic appearance of a cube for the case where the optic axis always points x_c is shown in Fig. 4. The dimensions and speed are the same as for the previous case. The progressive decrease in the length of face $ABCD$ and elongation of side $ADEF$ are apparent as they were for a fixed camera. The overall decrease in length of the two dimensional picture is still present but this effect is somewhat reduced by panning the camera to follow the cube's path.

When the cube is approaching the origin [Fig. 4(a)], the flared-out appearance of side $ABCD$ precludes interpretation of the picture in terms of a rotation. However, it is extremely interesting to note that the picture of the receding cube [Fig. 4(b)], apart from curvature effects, is exactly the perspective one would expect for a two dimensional representation of a stationary, rotated cube. Terrel's "apparent rotation" is indeed visible in this instance.

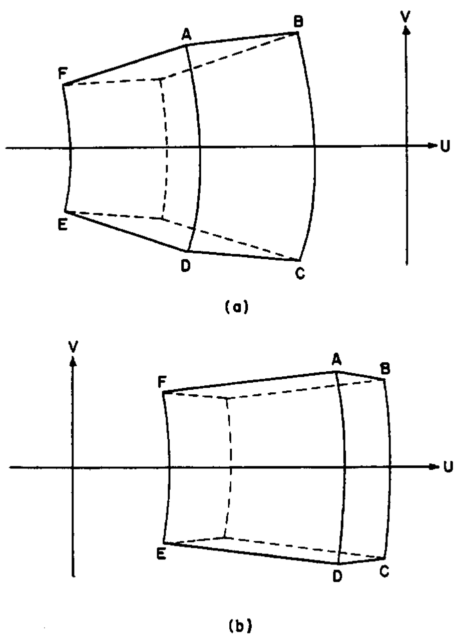

Fig. 4. Two-dimensional appearance of a cube (a) approaching and (b) receding from the origin when the camera is panned to follow the cube. The flared-out appearance of edges AB and CD in (a) precludes interpretation in terms of a rotation. However, the picture of the cube in (b) does have the proper perspective for interpretation in terms of an "apparent rotation" combined with some distortion.

IV. SUMMARY

The photographic appearance of a cube moving from left to right along the x axis at speed $\beta = 0.9$ has been investigated for two cases: (i) camera fixed so that the optic axis always makes an angle of 90° with the x axis and (ii) camera panned so that the optic axis always points toward the center of the cube. Consideration of the slope of lines formed by the edges of the cube has shown that the two cases result in different photographic appearances. Furthermore, in one instance, the appearance is similar to that of a stationary, rotated cube.

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[1]R. Penrose, Proc. Cambridge Philos. Soc. **55**, 137 (1959).

[2]J. Terrel, Phys. Rev. **116**, 1041 (1959).

[3]V. F. Weisskopf, Phys. Tod. **13**, 24 (1960).

[4]G. D. Scott and H. J. VanDriel, Am. J. Phys. **38,** 971 (1970).
[5]P. M. Mathews and M. Lakshmanan, Nuovo Cimento B **12,** 168 (1972).
[6]Many texts present a "rotated cube" argument based on Weiskopf's original paper (Ref. 3). For example, see R. Skinner, *Relativity* (Blaisdell, Waltham, 1969), pp. 69–73. Recently D. Hollenbach, in Am. J. Phys. **44,** 91 (1976), referred readers to this text for a "simple solution" of the appearance of a rapidly moving cube. Further examples of incorrect treatment of fast objects may be found in V. Acosta, C. L. Cowan, and B. J. Graham, *Essentials of Modern Physics* (Harper and Row, New York, 1973) and G. Gamow and J. M. Cleveland, *Physics: Foundations and Frontiers* (Prentice-Hall, Englewood Cliffs, 1976).
[7]G. D. Scott and M. R. Viner, Am. J. Phys. **33,** 534 (1965).
[8]N. C. McGill, Contemp. Phys. **9,** 33 (1968).
[9]S. Yngstrom, Ark. Fys. **23,** 367 (1962).
[10]D. L. Shirer and T. W. Bartel (unpublished) formerly distributed by the AIP Information Pool as CLIP No. 3. The author and G. Hover have also derived these equations using a pedigogically simpler, albeit lengthier procedure. Copies of this derivation are available from the author.