Photographing a relativistic meter stick

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A "gedanken photograph" of a relativistic meter stick is analyzed at a level appropriate for undergraduate students. While an image of the correct Lorentz contracted length can be obtained, that image is distorted, since it does not represent all parts of the stick at one instant of time, but different parts of the stick at different times. Aspects of the photographic process are also analyzed in the frame of the moving meter stick.

A classroom presentation of the Lorentz contraction invariably elicits the question "But what would I really see?" or alternatively, "Can you photograph the contraction?" After explaining the practical impossibility of such an experiment, the question usually persists. Spurred on by a particularly insistent class, I recently investigated the literature on this subject and developed a set of simple problems that lead students to an understanding of some of the curious effects in relativistic photography. I will present here the essential results, with the algebraic details suppressed. (A longer version with complete problem statements and detailed derivations is available from the author.) I have resisted the temptation to generalize my results into algebraic proofs, since for most students greater physical understanding can be achieved through specific examples.

The key to understanding relativistic photography is the fact that a photograph involves the simultaneous arrival at the lens of light rays from different spatial locations. (This is assuming a shutter at the lens. A focal plane shutter is a different case.) These rays will not have been emitted simultaneously if they have traveled different distances to the lens. This is very different from the measurements involved in the Lorentz contraction, simultaneous measurements at different spatial locations. The consequences of this fact were not fully realized until 1959 when J. Terrell[1] showed that a three-dimensional object will appear rotated, not simply contracted. However, Scott and Viner[2] pointed out that the contraction was still present and could be "photographed." In this report I will ignore the third dimension and concentrate on a meter stick of infinitesimal thickness.

To proceed with this development a student needs to understand how Lorentz contraction arises out of the failure of simultaneity, which is in turn a result of the basic postulate of relativity, the invariance of the velocity of light. The measurement of the length of a meter stick will yield its "proper length" if it is measured in the rest frame of the stick. When measured in a frame in which the stick is moving, observers must measure the positions of the two ends of the stick simultaneously, resulting in the contracted length. In the usual notation $L = L_0\sqrt{1 - u^2/c^2} = L_0/\gamma$, where L_0 is the proper length and L is the contracted length, u the stick velocity, and c the speed of light. We will not prove this effect here. We will assume it to be true and concentrate on how the photographic process itself leads to effects that change the image.

Students first need to become familiar with the geometry of the camera system. Figure 1 shows the arrangement. The meter stick will move from left to right along a line, and the camera will always be aimed in a direction perpendicular to the line of motion. The student should prove that stationary meter sticks placed at different locations along the line of motion will all have the same image length, assuming we have no lens distortions. (In other words, treat it like a fast, wide angle, pinhole camera.)

The condition for making a photograph is that light from the two ends of the meter stick must arrive simultaneously at the lens of the camera. We can show qualitatively that this effect will lengthen the image of the approaching stick and shorten the image of the receding stick. The situation is shown in Fig. 1. In this illustration the apparatus is viewed from above. While the meter stick is considered to have no thickness, the front face of the stick has been drawn to clarify the figure. Consider first the stick to the left, approaching the camera axis. In order for light from the two ends of the stick to arrive simultaneously at the lens, light from the leading edge must be emitted at a later time than light from the trailing edge. In the figure, the leading edge was at 2 when light was emitted from 1, the trailing edge. The leading edge moves to 2′ and emits light which arrives simultaneously at the lens with light from 1. So the image must appear elongated. The same effect leads to a shortening of the stick as it recedes from the camera axis to the right, since now the trailing edge is closer to the lens. However, when the middle of the stick is on the camera axis the ends will be equidistant from the lens. Light emitted at this instant from both ends of the stick will arrive simulta-

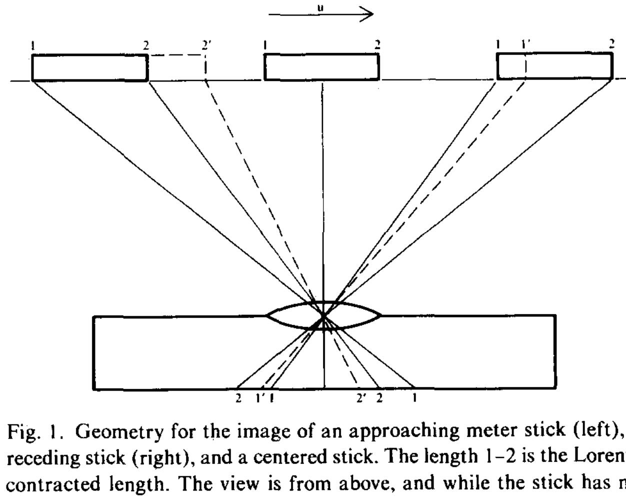

Fig. 1. Geometry for the image of an approaching meter stick (left), a receding stick (right), and a centered stick. The length 1–2 is the Lorentz contracted length. The view is from above, and while the stick has no thickness, the front face is shown for clarity.

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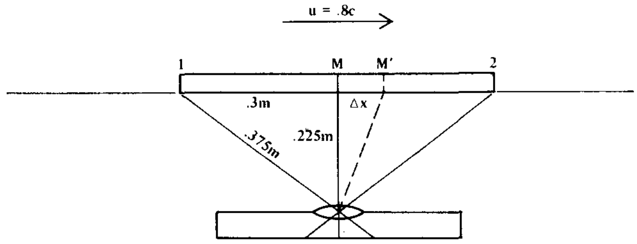

Fig. 2. Apparent shift ΔX of the center of the meter stick for an image formed by light emitted when the ends are equidistant from the camera axis. Light from 1, 2, and M' arrive simultaneously at the lens. The view is from above, but the front face is shown for clarity.

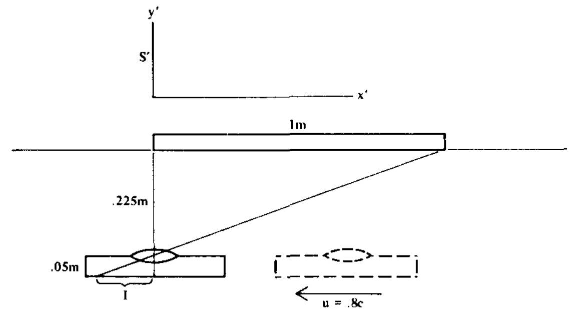

Fig. 4. Photographic process from the point of view of observers in the frame of the moving meter stick. Light rays from the two ends of the stick arrive simultaneously at the lens, but were not emitted simultaneously.

neously at the lens. Of course the moving stick will be contracted, so a contracted image will result.

The maximum length of the photographed stick would occur when the stick is far out to the left, and the minimum length would occur far to the right. While these situations might be hard to photograph, they are easy to calculate. They are just one-dimensional chase problems with constant velocities. We will assume a stick velocity of $0.8c$ for all calculations in this paper. For this choice the Lorentz contracted length is 0.6 m. Then the approaching stick has a maximum apparent length of 3 m, while the receding stick has a minimum apparent length of 0.33 m. (Yes, you can show generally that the maximum and minimum lengths are reciprocals.)

It is interesting to note that at some position the meter stick would have an image length equal to that of a stationary meter stick. This would occur somewhere left of the camera axis as the stick is still approaching. However, this image would not be an acceptable measurement of the length of the stick in either the camera frame or the meter stick frame of reference.

Let us now consider in Fig. 2 the case in which the image shows the two ends of the moving stick equidistant from the camera axis, at ± 0.30 m. Of course, the middle of the stick is closer to the lens than the edges, so it moves over a distance Δx and emits light that will arrive at the lens at the same time as light from the two ends. So even though the image will be the correct contracted length, the image will be distorted with its center toward the right, the direction of motion.

We can calculate Δx, the displacement of the center of the stick, assuming that the camera is 0.225 m from the line of travel. (We will show that this choice will make the geometry particularly simple in the stick frame of reference.) We simply equate the time for light to travel from point 1 to the lens and the time for M to move Δx and send light from M' to the lens. This leads to a quadratic equation with only one physically meaningful root, $\Delta x = 0.1023$ m. The situation is illustrated in Fig. 3, in which the image of the moving stick is compared to that of a stationary stick. Notice that the 50-cm mark of the moving stick image is displaced 10.23 cm to the right. (In Fig. 3 a complete mapping of the moving meter stick is shown.) Of course Δx will decrease if the speed of the stick is decreased or if the distance to the camera is increased. In the limit of large camera distance $\Delta x \rightarrow 0$, and an undistorted, Lorentz contracted image results.

Let us examine the simplest aspect of Fig. 3, the image length, from the viewpoint of observers in the frame of the moving meter stick, S'. If our analysis is correct, these observers must agree with the end result, the image of the S' stick must be shorter than the image of the stick stationary in the camera frame by the factor 0.6.

In order to have an easily visualized geometry in the meter stick frame of reference, we can position the camera such that light that is emitted from the trailing edge of the meter stick at the instant it is 0.30 m from the camera axis arrives at the lens at the same time that the trailing edge reaches the camera axis. In the frame of the meter stick this light would travel perpendicular to the stick. Equating the two times determines the distance from the lens to the line of travel to be 0.225 m, the value used earlier to find Δx.

The S' frame point of view of the photographic process is illustrated in Fig. 4. In this frame the stick is 1 m long and the camera is moving at $0.8c$. Light enters the lens from the two ends simultaneously, but since light rays from the two ends traveled different distances they were not emitted simultaneously. The details of the photographic process after the light enters the camera are shown in Fig. 5. Light from the trailing edge of the stick would hit at B if the camera were not moving. However, during the transit time of this light from the lens to the film plane the camera moves a distance EE' (Δx_1) so the light hits at B' instead of at B.

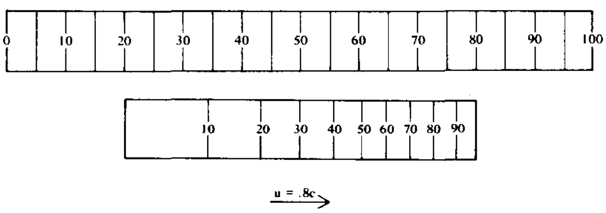

Fig. 3. Contracted, distorted image of the moving meter stick compared to the image of a stick stationary in the camera frame of reference. Vertical lines will be straight if they are much shorter than the distance to the camera.

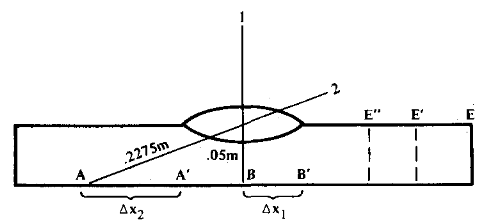

Fig. 5. Detail of the photographic process from the moving stick frame viewpoint. The image would be of length AB for a stationary camera, but is $A'B'$ for the moving camera.

The situation is similar for light from the leading edge of the stick, except that it must travel a longer path inside the camera. Since this takes a longer time, the camera will have moved a longer distance EE'' (Δx_2), and the light strikes at A' rather than at A. So to observers in the meter stick frame the image should have been of length AB for a stationary camera, but was $A'B'$ for the moving camera. But $A'B' = AB + \Delta x_1 - \Delta x_2$, so $A'B'$ is less than AB because $\Delta x_2 > \Delta x_1$. Choosing the distance from the lens to the film plane to be 0.05 m, we can calculate the ratio of the observed image length to the expected image length:

$$A'B'/AB = 0.36 = (0.6)^2 = 1/\gamma^2.$$

The correctness of this result can be seen in the following way. The image, according to camera frame observers, was already shrunk by a factor 0.6. But the image is a proper length in the camera frame. Attempts by meter stick frame observers to measure the length of the image must result in another contraction of 0.6. Of course, stick frame observers would measure a camera frame meter stick contracted by the factor 0.6 and agree with camera frame observers that the image of the moving stick was only six tenths as long as the image of the camera frame stick.

According to stick frame observers the camera shutter was tripped too late. They think that the shutter should have been tripped at the instant that the camera was at the center of the meter stick instead of waiting until it was at the end. In that way the light from the two ends of the stick would have been emitted simultaneously, arrived simultaneously, and traveled equal distances inside the camera body.

While this analysis has been restricted to a two-dimensional meter stick, it would also apply to the face of a three-dimensional object that is parallel to the line of motion, as long as the height of that face is small compared to the distance to the camera. If the object is quite tall, we must consider the different travel times for light from the top, bottom, and middle of the object. Scott and Viner[2] have shown that in this case, vertical lines become hyperbolas. To this analysis must then be added the third dimension, with the interesting effect that the trailing face of the object shows up on the image before the object reaches the camera axis. This is well discussed by Weisskopf[3] at a level appropriate for undergraduates. A simple treatment of the cube is presented by Skinner.[4]

ACKNOWLEDGMENT

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[1] J. Terrell, Phys. Rev. **116**, 1041 (1959).
[2] G. D. Scott and M. R. Viner, Am. J. Phys. **33**, 534 (1965).
[3] V. Weisskopf, Phys. Today **13**, 24 (1960).
[4] R. Skinner, *Relativity* (Blaisdell, Waltham, MA, 1969), p. 69.