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Apparent Shape of Large Objects at Relativistic Speeds

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It has been recently recognized that there is a difference between the measured Lorentz contracted shape of an object moving at relativistic speed and the shape as seen by a single observer. The case of an object which subtends a small solid angle at the observer has been discussed by several authors. This paper discusses objects so large or so near that the subtended solid angle cannot be considered small, and gives simple proofs that spheres always present a circular outline and that straight lines may appear curved. These results are applied to revise Gamow's well-known picture of the bicyclist seen by Mr. Tompkins.

UNTIL recently it has been generally believed that the Lorentz contraction in the direction of motion of a body at relativistic speed would be seen visually by a single observer. Several authors have now shown that this is incorrect. Penrose[1] has given several sophisticated proofs that a moving sphere of any size presents a circular (rather than elliptical) outline to all observers; in this paper we shall give an elementary proof of this striking fact. Terrell[2] has considered objects so small or so far away that they subtend an infinitesimal solid angle at the observer, and has shown that with this restriction an object appears to have its rest system shape. In this paper we shall consider the results of removing the restriction to small solid angles. The sphere is then the only geometrical figure which always appears the same shape to all observers. Straight line segments may not only appear either longer or shorter than their rest length, but may appear curved! The distortion of other shapes can be seen qualitatively by thinking of them as made up of a set of line segments.

In order to discuss these ideas, let S be the system in which the observer O is at rest, and let S' be the rest system of the object. Let v be the velocity of S' relative to S along the common x and x' axis, and at $t=t'=0$, let the y and y' axes coincide, and the z and z' axes coincide. Then the Lorentz transformation formulas connecting S and S' are

$$x'=\gamma(x-vt),\quad y'=y,\quad z'=z,\qquad t'=\gamma(t-vx/c^2),\quad \gamma=(1-v^2/c^2)^{-\frac{1}{2}}. \tag{1}$$

First let the object be a sphere of diameter L_0 in its rest system S'. The sphere is at rest with respect to O', and it is easy to describe what O' sees. If we draw tangents to the sphere from O', these form a right circular cone with vertex at O'. The light rays from the sphere to O' are contained in this cone, and O' sees a circular outline all the time and in a fixed direction. What does O see at the instant that O and O'

[1] R. Penrose, Proc. Cambridge Phil. Soc. **55**, 137 (1959).
[2] J. Terrell, Phys. Rev. **116**, 1041 (1959); see also comments on Terrell's paper by V. F. Weisskopf, Phys. Today **13**, 24 (September, 1960); Sci. American **203**, No. 1, 74 (1960).

coincide? (If we want to ask what O sees at other times, we need only consider a whole set of O' observers each at rest with respect to the sphere, and ask, as O passes each of them, what O sees; each O' observer sees a circular outline as above.)

First, it must be realized that the *same* light (same beam of photons, if you like!) is seen by O and O' as they coincide. O and O' will not agree on the direction from which the light comes (aberration), but the same light strikes their eyes (or activates their cameras) as they coincide. In relativistic terminology, the coincidence of O and O' is an event $x=y=z=t=0$, $x'=y'=z'=t'=0$; the event also includes the arrival of a flash of light. Let us trace backwards the history of that flash of light. (It may help our thinking here to imagine the flash of light originating as O and O' coincide and spreading outward instead of converging inward. Turning time backward changes these situations into each other.) Since the velocity of light is c in both systems, the light seen by O and O' at $t=0$ was, at some negative time t', on the sphere in S',

$$x'^2+y'^2+z'^2=c^2t'^2,$$

and at some negative time t on the sphere in S,

$$x^2+y^2+z^2=c^2t^2.$$

The spheres represent the same light, and the Lorentz transformation (1) transforms either one into the other.

Now consider the light in S', the rest system of the sphere. The paths of the light rays from the sphere to O' fill the interior of a right circular cone. Let (Fig. 1) $\mathbf{a}'=$ a unit vector in the direction of the axis of the cone, $\kappa'=$ the cosine of the half-angle of the cone, $\mathbf{r}'=$ the vector from O' to a point on the cone, $t'=$ a parameter, which we recognize as the time. (Remembering that the light arriving at O' at $t=0$ is on its way during negative t' values, we write $-t'$ when we need a positive quantity.) Then the equation of the cone can be written

$$\mathbf{r}'\cdot\mathbf{a}'=|\mathbf{r}'|\,|\mathbf{a}'|\cos(\mathbf{r}',\mathbf{a}')=\kappa' r',$$

or in parametric form

$$\mathbf{r}'\cdot\mathbf{a}'=-\kappa' ct',\quad r'^2=c^2t'^2. \tag{2}$$

The set of r',t' values satisfying these equations

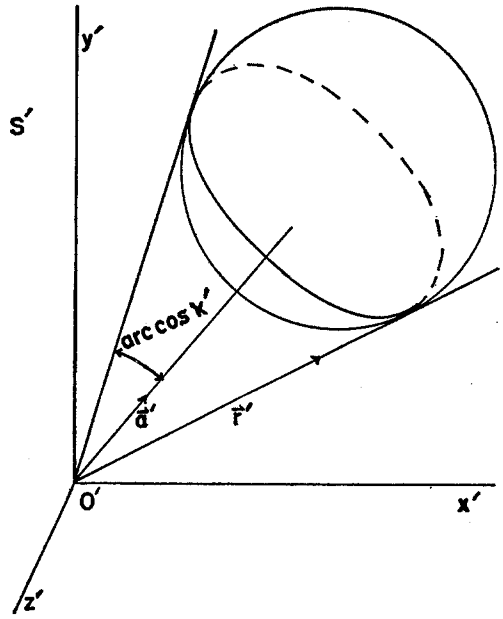

FIG. 1. Cone of light rays from a luminous sphere pictured in the rest system of the sphere.

represents the history of the light which arrives at O' at $t'=0$. The same light is described in the S system by the equations obtained from these by the Lorentz transformation. The equation $r'^2=c^2t'^2$ becomes $r^2=c^2t^2$ as we have previously noted. Before transforming the other equation, it is convenient to let l',m',n' be the components of the unit vector $\mathbf{a}'$; then the first of Eqs. (2) becomes

$$l'x'+m'y'+n'z'=-\kappa' ct'.$$

On applying the Lorentz transformation, Eqs. (1), we get

$$l'\gamma(x-vt)+m'y+n'z=-\kappa' c\gamma(t-vx/c^2).$$

Collecting coefficients of x and t, we have

$$x(l'\gamma-\kappa'\gamma v/c)+m'y+n'z=-ct(\kappa'\gamma-\gamma l'v/c).$$

This may be written as

$$lx+my+nz=-\kappa ct,$$

or

$$\mathbf{a}\cdot\mathbf{r}=-\kappa ct,$$

if $\mathbf{a}$ is a unit vector with components $l,m,n,$ where

$$l=N(l'\gamma-\kappa'\gamma v/c),\quad m=Nm',\quad n=Nn',$$
$$\kappa=N(\kappa'\gamma-\gamma l'v/c),$$

and N is determined so that $l^2+m^2+n^2=1$. The equations $\mathbf{a}\cdot\mathbf{r}=-\kappa ct$ and $r^2=c^2t^2$ (like the corresponding equations in S') represent a right circular cone. In other words, the O observer also sees a circular outline although the sphere appears in a different direction (aberration, $\mathbf{a}\neq\mathbf{a}'$) and of a different size ($\kappa\neq\kappa'$).

It is of interest to ask why the Lorentz contraction is not seen by O. Consider the rays of light seen by O and O' which form the lateral surface of the right circular cone discussed above. These are the rays which determine the apparent shape of the object seen. In S', where the sphere is at rest, all these rays are the same length; hence the light seen by O and O' started from the sphere at one t' instant. Since, by Eqs. (1), $t=\gamma(t'+vx'/c^2)$, and x' takes various values on the sphere, t is not constant over the sphere, and the light seen by O and O' started from the sphere at various t values in the S system. Thus O does not observe the sphere as it is at one instant of time in his own system as is required in the Lorentz contraction measurement. The physicist in system S can reasonably say that the Lorentz contracted length is the real or correct value for system S, and that O does not see the object as it really is. An analogy from everyday life may clarify this. A circular object appears elliptical when viewed from an angle, but we say it is really round (i.e., by measurement). Similarly, the Lorentz contracted sphere is really an oblate spheriod in S (i.e., by measurement), but it appears round to a single observer in S.

Besides the sphere, the only other geometrical figure which appears the same shape to observers in its own rest system viewing it from different angles is a straight line segment. We next investigate the appearance to O of a straight line segment L' fixed in S'. In S', consider the plane determined by L' and O'. Its equation may be written

$$\mathbf{N}'\cdot\mathbf{r}'=0,$$

where $\mathbf{N}'$ is a unit normal to the plane. All the light rays traveling from the line L' to O' lie in this plane. The light which will arrive at O' at $t'=0$ is, at some negative t', on the intersection of this plane with the sphere $r'^2=c^2t'^2$. The same light is, at some negative t, on the intersection of $r^2=c^2t^2$ (obtained by applying the Lorentz transformation to $r'^2=c^2t'^2$), and the surface obtained by transforming $\mathbf{N}'\cdot\mathbf{r}'=0$ by the Lorentz transformation. Let $\mathbf{N}'$ have components l',m',n'. Then $\mathbf{N}'\cdot\mathbf{r}'=l'x'+m'y'+n'z'=0$. This transforms to

$$l'\gamma(x-vt)+m'y+n'z=0$$

or

$$lx+my+nz=bt, \tag{3}$$

where

$$l=Kl'\gamma,\quad m=Km',\quad n=Kn',\quad b=Kl'\gamma v,$$
$$K=(l'^2\gamma^2+m'^2+n'^2)^{-\frac{1}{2}}.$$

For a given (negative) t, this is the equation of a plane in the S system. Its intersection with $x^2+y^2+z^2=c^2t^2$ (for the same t) gives a trace at time t in S of the light which will arrive at O at $t=0$. It may be helpful to think of this trace as painted on the inside of a sphere (sphere of vision) with the observer O at the center; he can then examine at his leisure what he would see instantaneously at $t=0$. It should be realized that when an observer looks at a straight line object at rest in his own system, and sees a straight line, the trace on his sphere of vision of the light from the straight line is an arc of a great circle; when he looks at a sphere and sees a circular outline, the trace on his sphere of vision is a small circle. The same interpretations apply if the object is moving. If the trace on the observer's sphere of vision is a small circle, he sees a circular outline; if it is part of a great circle, he sees a straight line segment.

Now consider, for O, the intersection of the plane, Eq. (3), with the sphere of vision $r^2=c^2t^2$. If $b=0$, the plane passes through the origin, and the trace on the sphere of vision is a great circle. Observer O sees a straight line segment. Looking at Eqs. (3), we see that $b=0$ must mean $l'=0$. If $l'=0$, $\mathbf{N}'$ is perpendicular to the x' axis; then the plane $\mathbf{N}'\cdot\mathbf{r}'=0$, which is perpendicular to $\mathbf{N}'$ and passes through O, contains the x' axis. This case includes, then, any straight line segment fixed in S' lying in a plane containing the line of relative motion (i.e., the common x and x' axis). The simplest examples would be a line segment in the (xy) plane moving parallel to its length, or moving perpendicular to its length. Both of these would appear to O as straight line segments.

The line moving parallel to itself has been discussed in detail previously.[2,3]

The case $b \neq 0$ is more surprising. Consider, say, a straight line segment fixed in O' parallel to the y' axis [but not in the $(x'y')$ plane]. It is then moving perpendicular to its own length in S. Now the plane (3) does not pass through O and its intersection with the sphere of vision gives a small circle; observer O sees a curved arc and not a straight line segment! In fact the top of the "line segment" will appear bent backward from the direction of motion. The reason for this is not difficult to see. The top is farther from the observer than the bottom; hence the observer sees simultaneously the top of the line as it was at one time and the bottom as it was at a later time.

It is amusing to apply these results to Gamow's Mr. Tompkins[4] when he observes the bicyclist. You may recall that Mr. Tompkins visits a city where the velocity of light is very small. He observes a bicyclist coming down the street and "sees" him flattened in accord with the Lorentz contraction. But now we realize that the measured Lorentz contraction is not the same as what is seen by a single observer who makes no corrections for the fact that the light he sees left different parts of the observed object at different times. To simplify matters we consider a skeleton man and bicycle consisting in the rest system S' of: spherical head, straight line vertical body, straight line arms perpendicular to the body and to the direction of motion, circular wheels in the plane of motion, and straight line bicycle frame moving parallel to its length. Let the body, frame, and wheels be in the plane $z = z' = \text{constant} \neq 0$. Let the bicyclist (fixed in S') ride past O from $x = -\infty$ to $x = +\infty$, and consider what O sees. The bicyclist's head will appear round, although it should be remarked that if features are considered, they will appear distorted; in fact, O will see more of the back and less of the front of the head than he "should." The arms and bicycle frame, being essentially straight line segments in a horizontal plane containing the observer and the direction of motion, appear as straight lines of varying lengths depending on position. The bicyclist's body (second straight-line case above) will appear bent backwards. The wheels, to an observer in S' some distance down the street from the bicyclist, but moving with him, appear as ellipses. A vertical diameter (in S') of a wheel (or any vertical line on it) appears to O to be bent backwards. We can then see that the wheels appear to O as if they had first been made elliptical, and then bent backwards (more or less hot-dog shaped).

[3] R. Weinstein, Am. J. Phys. **28**, 607 (1960).

[4] G. Gamow, *Matter, Earth and Sky* (Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1958), p. 187 ff.