Rotation associated with the product of two Lorentz transformations

C. B. van Wyk
University of the Orange Free State, Bloemfontein, South Africa

(Received 13 July 1983; accepted for publication 15 October 1983)

In the usual presentation of the Lorentz transformation there is an almost complete absence of the use of products of these transformations. On the rare occasions when such products are discussed it is done in the infinitesimal form.[1] This implies that the finite rotation inherent in such a product is not demonstrated to students in an elementary way. It is well known that if two velocities **U** and **V** are added the resultant velocity **W** is not symmetrical in **U** and **V**. The literature merely mentions this fact without further comment.

One of the reasons for the state of affairs described above appears to be the large amount of calculation involved when multiplying the 4×4 matrices of the vector representation of the Lorentz transformation.

In this note the above points are cleared up, at least in part, by using the coordinate-free, two-component spinor representation of rotations and Lorentz transformations, thereby reducing the amount of work considerably. It is shown that if L_1 and L_2 are two finite pure Lorentz transformations then

$$L_1L_2 = RL = L'R,$$

where the finite rotation R and the finite pure Lorentz transformations L and L' are easily found by means of the spinor formulation. A general formula for the axis and finite angle of rotation is obtained for R. Similarly the resultant velocities **W** and **W**$'$ can be derived from L and L', respectively.

Finally it is shown that the theory derived in this note can be applied to Thomas precession in a very simple and direct way.

In the theory of the two-component spinor the coordinate-free pure rotation and Lorentz transformation are given by[2,3]

$$R(\hat{\mathbf{n}},\theta) = \exp(\tfrac{1}{2}i\theta\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}) = \cos(\theta/2) + i\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}\sin(\theta/2), \quad (1)$$

$$L(\hat{\mathbf{u}},\phi) = \exp(\tfrac{1}{2}\phi\boldsymbol{\sigma}\cdot\hat{\mathbf{u}}) = \cosh(\phi/2) + \boldsymbol{\sigma}\cdot\hat{\mathbf{u}}\sinh(\phi/2), \quad (2)$$

respectively, where σ_j with $j = 1,2,3$ are the 2×2 Pauli matrices, $\hat{\mathbf{n}}$ and $\hat{\mathbf{u}}$ are unit vectors, θ is the angle of rotation while

$$\mathbf{U} = c\hat{\mathbf{u}}\tanh\phi, \quad (3)$$

with c the velocity of light, is the relative velocity of the two inertial frames linked by the pure Lorentz transformation.

Consider the decomposition

$$L(\hat{\mathbf{u}},\phi_1)L(\hat{\mathbf{v}},\phi_2) = R(\hat{\mathbf{n}},\theta)L(\hat{\mathbf{w}},\phi) \quad (4)$$

of the product of two pure Lorentz transformations into the product of a pure rotation and a pure Lorentz transformation. We must determine $\hat{\mathbf{n}}$, $\hat{\mathbf{w}}$, θ and ϕ for given $\hat{\mathbf{u}}$, $\hat{\mathbf{v}}$, ϕ_1, and ϕ_2. In the notation

$$\begin{aligned} S_i &= \sinh(\phi_i/2), & C_i &= \cosh(\phi_i/2),\\ S &= \sinh(\phi/2), & C &= \cosh(\phi/2),\\ s &= \sin(\theta/2), & c &= \cos(\theta/2), \end{aligned}$$

we have from (1), (2), and (4)

$$(C_1 + \boldsymbol{\sigma}\cdot\hat{\mathbf{u}}S_1)(C_2 + \boldsymbol{\sigma}\cdot\hat{\mathbf{v}}S_2) = (c + i\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}s)(C + \boldsymbol{\sigma}\cdot\hat{\mathbf{w}}S). \quad (5)$$

Applying the multiplication rule for Pauli matrices

$$(\boldsymbol{\sigma}\cdot\hat{\mathbf{a}})(\boldsymbol{\sigma}\cdot\hat{\mathbf{b}}) = \hat{\mathbf{a}}\cdot\hat{\mathbf{b}} + i\boldsymbol{\sigma}\cdot(\hat{\mathbf{a}}\wedge\hat{\mathbf{b}})$$

and equating coefficients of corresponding Pauli matrices we have from (5) that

$$\begin{aligned} cC &= C_1C_2 + S_1S_2\hat{\mathbf{u}}\cdot\hat{\mathbf{v}},\\ \hat{\mathbf{n}}\cdot\hat{\mathbf{w}} &= 0,\\ cS\hat{\mathbf{w}} - sS\hat{\mathbf{n}}\wedge\hat{\mathbf{w}} &= C_2S_1\hat{\mathbf{u}} + C_1S_2\hat{\mathbf{v}},\\ sC\hat{\mathbf{n}} &= S_1S_2\hat{\mathbf{u}}\wedge\hat{\mathbf{v}}. \end{aligned}$$

Solution for the unknowns yields

$$\tan(\theta/2) = |\hat{\mathbf{u}}\wedge\hat{\mathbf{v}}|S_1S_2/(C_1C_2 + S_1S_2\hat{\mathbf{u}}\cdot\hat{\mathbf{v}}), \quad (6)$$

$$\hat{\mathbf{n}} = \hat{\mathbf{u}}\wedge\hat{\mathbf{v}}/|\hat{\mathbf{u}}\wedge\hat{\mathbf{v}}|, \quad (7)$$

$$S^2 = (\hat{\mathbf{u}}S_1C_2 + \hat{\mathbf{v}}S_2C_1)^2, \quad (8)$$

$$\hat{\mathbf{w}}SC = \hat{\mathbf{u}}S_1C_1 + \hat{\mathbf{v}}[S_2C_2(C_1^2 + S_1^2) + 2S_1C_1S_2^2(\hat{\mathbf{u}}\cdot\hat{\mathbf{v}})]. \quad (9)$$

Equations (6), (7), and (8) follow immediately while (9) requires some effort. According to (4), (6), and (7), θ is the finite angle of the rotation about the axis $\hat{\mathbf{n}}$ contained in the product of the two pure Lorentz transformations. The axis of rotation $\hat{\mathbf{n}}$ is perpendicular to $\hat{\mathbf{u}}$, $\hat{\mathbf{v}}$, and $\hat{\mathbf{w}}$.

After considerable calculation we find

$$\mathbf{W} = \{\mathbf{U} + \mathbf{V}[\gamma + (\gamma - 1)(\mathbf{U}\cdot\mathbf{V})/V^2]\}/[\gamma(1 + \mathbf{U}\cdot\mathbf{V}/c^2)], \quad (10)$$

where

$$\mathbf{U} = c\hat{\mathbf{u}}\tanh\phi_1, \quad \mathbf{V} = c\hat{\mathbf{v}}\tanh\phi_2, \quad \mathbf{W} = c\hat{\mathbf{w}}\tanh\phi$$

and

$$\gamma = (1 - V^2/c^2)^{-1/2}.$$

The physical meaning of (10) is of course that if F, F' and F'' are three Lorentz frames such that the velocity of F'' relative to F' is **U** and the velocity of F' relative to F is **V**, then the velocity of F'' relative to F is **W**.

Note that **W** is unsymmetrical in **U** and **V**. The roles of **U** and **V** in **W** are exchanged if the order in which R and L appear on the right-hand side of (4) is changed. If

$$L(\hat{\mathbf{u}},\phi_1)L(\hat{\mathbf{v}},\phi_2) = L(\hat{\mathbf{w}}',\phi')R(\hat{\mathbf{n}}',\phi'),$$

then solution for $\hat{\mathbf{w}}'$, ϕ', $\hat{\mathbf{n}}'$, θ' in terms of $\hat{\mathbf{u}}$, $\hat{\mathbf{v}}$, ϕ_1, and ϕ_2 yields

$$\theta' = \theta, \quad \hat{\mathbf{n}}' = \hat{\mathbf{n}}, \quad \phi' = \phi$$

while

$$\hat{\mathbf{w}}'SC = \hat{\mathbf{u}}[S_1C_1(C_2^2 + S_2^2) + 2S_2C_2S_1^2(\hat{\mathbf{u}}\cdot\hat{\mathbf{v}})] + \hat{\mathbf{v}}S_2C_2.$$

Hence the lack of symmetry of **W** is associated with the fact that the factors on the right-hand side of (4) do not commute. The literature refers only to the lack of commutativity of the factors on the left-hand side of (4). Commutation of these factors lead to a change in the sign of $\hat{\mathbf{n}}$ and to $\hat{\mathbf{w}}$ or $\hat{\mathbf{w}}'$ depending on the order of the factors on the right-hand side of (4).

The formulae (6)–(10) have been derived by means of the two-component spinor form of the Lorentz transformation. These results also hold for the vector form of the Lorentz transformation.

The above theory can be applied in a very direct way to the Thomas precession. Starting out from (4) in the form[4]

$$L(-\hat{\mathbf{v}},\phi_1)L(\hat{\mathbf{v}}+d\hat{\mathbf{v}},\phi_1+d\phi_1)=R(\hat{\mathbf{n}},d\theta)L(\hat{\mathbf{w}},d\phi),$$

the required angular velocity follows from (6) and (7) in the following very simple way:

$$\hat{\mathbf{n}}d\theta/2=-\hat{\mathbf{v}}_\wedge(\hat{\mathbf{v}}+d\hat{\mathbf{v}})S_1^2$$

or

$$\begin{aligned}\hat{\mathbf{n}}d\theta &= -2\hat{\mathbf{v}}_\wedge d\hat{\mathbf{v}}S_1^2\\ &= -\hat{\mathbf{v}}_\wedge d\hat{\mathbf{v}}(\cosh\phi_1-1)\end{aligned}$$

and finally

$$\hat{\mathbf{n}}\dot{\theta}=\mathbf{\Omega}=-\mathbf{V}_\wedge\dot{\mathbf{V}}(\gamma-1)/V^2,$$

where

$$\gamma=\cosh\phi_1=1/(1-V^2/c^2)^{1/2}.$$

[1]J. T. Cushing, Am. J. Phys. **35**, 858–862 (1967).
[2]See, for example, B. L. van der Waerden, *Group Theory and Quantum Mechanics* (Springer-Verlag, New York, 1974), Chap. III.
[3]J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964), Chap. 2.
[4]J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1975), 2nd ed., p. 544.

Phase of a reflected acoustic wave

C. T. Tindle
Department of Physics, The University of Auckland, Private Bag, Auckland, New Zealand

(Received 15 August 1983; accepted for publication 25 October 1983)

In two introductory physics textbooks,[1,2] it is incorrectly stated that a sound wave reflected at a rigid boundary is 180° out of phase with the incident wave. The error has apparently gone unnoticed by the teaching community as it has persisted in several editions of one of the books.[1] The error occurs because it is easy to overlook the fact that the sign of the particle displacement of a longitudinal wave depends on the direction of travel of the wave.

In most introductory physics textbooks traveling waves are introduced by considering transverse waves on a stretched string. An upright pulse incident on a fixed end is reflected as an inverted pulse in order that the displacement at the fixed end is zero at all times. This inversion is correctly described as a phase change of 180° due to reflection from the fixed end. Thus for a transverse wave the positive or upward displacement of the incident pulse is cancelled at the fixed end by the negative or downward displacement of the reflected pulse. Typical incident and reflected waves are shown in Fig. 1(a).

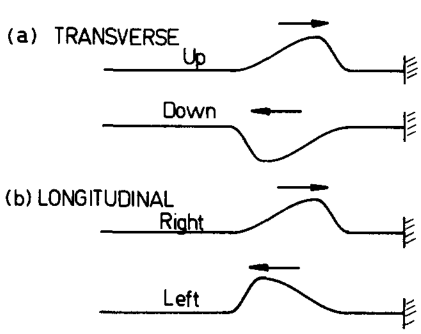

Fig. 1. Reflection at a rigid boundary. (a) Transverse wave; (b) longitudinal wave. The arrows indicate the direction of travel. The words show the direction of displacement.

If we now consider acoustic or longitudinal waves reflected from a rigid boundary it is clear that there can be no particle motion at the boundary and so the particle displacement at the boundary is zero at all times. Since therefore the displacement at the boundary due to the incident wave is cancelled by the displacement due to the reflected wave it is tempting to assume by analogy with transverse waves that the two waves have opposite phase. Unfortunately this is not correct and the reflected wave is actually in phase as is illustrated in Fig. 1(b).

For a longitudinal wave a positive displacement is a displacement in the direction of travel. The incident wave in Fig. 1(b) is traveling to the right and the positive pulse shown represents a displacement to the right. The reflected wave is traveling to the left and the positive pulse shown represents a displacement to the left. Thus as required, the total displacement at the boundary is zero since the displacement to the right due to the incident pulse is cancelled by the displacement to the left due to the reflected pulse.

Thus, contrary to Refs. 1 and 2, there is no phase change on reflection and a longitudinal wave reflected from a rigid boundary is in phase with the incident wave.

[1]D. Halliday and R. Resnick, *Fundamentals of Physics* (Wiley, New York, 1981), 2nd ed., extended version, p. 325.
[2]D. E. Roller and R. Blum, *Physics*, Vol. I (Holden-Day, San Francisco, 1981), p. 491.

