A note on rotating coordinates in relativity

T. A. Weber
Department of Physics and Astronomy, Iowa State University, Ames, Iowa 50011-3160

(Received 31 July 1996; accepted 4 December 1996)

In this paper, the rotating disk as an example of a non-Euclidean space is carefully examined; the basic arguments of Einstein are emphasized. A new approach is also presented which resolves the Ehrenfest paradox.

Einstein, in his popular book,[1] *The Meaning of Relativity*, gives a rotating coordinate system as an example of a non-Euclidean space. We reconstruct his arguments while emphasizing the principles involved. We also present a new approach which gives the same result and shows that a material body set into rotational motion experiences stretching tangent to circles centered on the axis of rotation.

Consider a region where there are no effects of gravity. Let K be an inertial frame of reference and let the frame $\bar{K}$ rotate with constant angular velocity ω about a fixed point O in K. For both frames choose cylindrical coordinates with common origin at O and with the z and $\bar{z}$ axes aligned along the axis of rotation. We restrict r by $\omega r < c$ so that the coordinate system of $\bar{K}$ can be realized by markings on a material body. Here, c is the velocity of light in vacuum and ωr is the velocity of a point in $\bar{K}$ as observed by K. In $\bar{K}$ let $\bar{C}$ be a circle of radius $\bar{r}$ centered at O and perpendicular to the axis rotation. Let C be the locus of points in K that coincide with $\bar{C}$ at any given time. C will be a circle of radius r. We know that the circumference of the circle C divided by its radius is 2π since we have Euclidean geometry in inertial frames of reference. Einstein showed that the ratio, $\bar{R}$, of the circumference of $\bar{C}$ as measured in $\bar{K}$ to its radius is greater than 2π. Such a measurement presents a problem since it involves an interpretation of coordinates in a non-inertial reference frame.

We can start with the invariant interval as described by the coordinates in K:

$$ds^2 = c^2\ dt^2 - dr^2 - r^2\ d\phi^2 - dz^2, \tag{1}$$

where the time is measured by a set of synchronized clocks placed at convenient fixed positions in K and cylindrical coordinates are used for the spatial part. Arbitrary transformations to new coordinates will leave the interval unchanged; only the space-time description of events will be changed. Now all of these frames of reference with new space-time coordinates are equally valid. Einstein was led to this conclusion by thoughtful consideration of the local equivalence of an accelerated frame to a gravitational field. Even though all these frames are equally valid we know the form of the laws of nature only in inertial frames where distance is measured with standard rods and time intervals by standard clocks. Even in the case of a space time manifold including gravitational fields,[2] a transformation can be made to a local inertial coordinate system such that the geodesic equation is a straight line, that is, coordinates in which particles behave as if ''free'' of gravitational forces or inertial forces. Einstein refers to observers in such coordinates as freely falling. From the known laws of physics in the inertial system we can then infer the form of the laws in the general system.

There is no need to make any detailed algebraic transformations to apply the above procedure to the measurement of the circumference and the radius of $\bar{C}$ (see Adler, Bazin, and Schiffer[3] for the details of such a transformation). The measurement of the length of any short segment in the plane of $\bar{C}$ can be done in an inertial frame that is instantaneously at rest with respect to the segment. Imagine using many short standard rods placed end to end along a curve to measure its length. Do this along the circumference of $\bar{C}$, with each rod in the instantaneous rest system of the segment it is to measure. As observed simultaneously from K, the rods appear Lorentz contracted and lie end to end along the circle C which is superimposed on $\bar{C}$. Since the rods are shortened as compared to a rod at rest in K, the number along the circumference of $\bar{C}$ is greater by the factor $(1-\omega^2 r^2/c^2)^{-1/2}$ than the number of rods in K needed to measure the circumference of C. Do the same for the radius and find that $r=\bar{r}$ since lengths perpendicular to the motion are not Lorentz contracted. We therefore conclude that the circumference of $\bar{C}$, as measured by freely falling standard rods instantaneously at rest with respect to the segments of $\bar{C}$, divided by its radius is

$$\bar{R} = 2\pi(1-\omega^2 r^2/c^2)^{-1/2}. \tag{2}$$

At this point one might ask about a reciprocal measurement of C as observed from $\bar{K}$. It might be mistakenly thought that the measured circumference of C would be larger than $\bar{C}$. Certainly such reciprocity exists in the measurements of lengths between two inertial frames, but for a rotating system coordinate clocks cannot be synchronized throughout space.[2] Without synchronization of clocks, consistent measurements of moving lengths cannot be made. Thus the measurement can be made from K but not K.

We can also arrive at this same result in a new way. Draw n equally spaced radial lines from O to C in the frame K. Draw the same number of equally spaced radial lines from O to C in the frame K. Note that clocks need not be synchronized for the construction of these equally spaced radial lines in either K or K. For definiteness take the number to be 360. In each frame these radial lines divide the circumference of the corresponding circle into 360 equal intervals. As observed simultaneously from K, the length of the intervals on the circle C must appear to be exactly the same length as the intervals on the circle C. If they appeared otherwise, a count of the intervals on the circle C would not be 360. This is impossible since from K one can observe all the radial lines in K simultaneously. Note that this applies for any angular velocity of the rotating frame, even during an acceleration of the frame from rest to its final angular velocity. For the intervals to appear the same length in spite of the Lorentz contraction, the actual rest length of an interval in K must be greater by the factor $(1-\omega^2 r^2/c^2)^{-1/2}$. This confirms our earlier result and leads to an interesting conclusion. Suppose the coordinate system is realized by markings on a material disk. Then the material within the intervals must physically stretch as the disk, starting at rest, is brought up to rotational speed ω.

Notice that the above arguments, based solely on relativistic kinematics, offers a resolution of the Ehrenfest paradox.[4–6] The paradox is as follows: As observed from the inertial system K, the Lorentz contraction acts only on the periphery and not on the radius of the disk. It is therefore proposed that the ratio of the circumference to the radius for the rotating disk is less than 2π, an apparent violation of the Euclidean geometry of an inertial frame! But, as seen above, an increase in rest length compensates for the Lorentz contraction. This is consistent with the statement of Ø. Grøn[7] that ''relativistic kinematics alone forbids giving the disk a rotation so that the rest lengths of the elements of the periphery remain constant during the period of angular acceleration.'' In detail, consider two closely spaced marks on the periphery of the disk which is initially at rest. As observed from K, the two marks have exactly the same angular acceleration as the disk is brought up to rotational speed ω. It follows from simple kinematics that the distance between the two marks must not change as observed from the inertial frame. Special relativity then implies that the rest length between the two marks has increased. Thus, if the ends of an unstretched spring were initially fastened to the marks, the spring would be elongated by

$$\Delta l = l_0[(1-\omega^2 r^2/c^2)^{-1/2} - 1], \tag{3}$$

where l_0 is the unstretched length of the spring. As stated earlier, the material of the disks within the intervals must physically stretch. This change in rest length is entirely different than the change in size associated with observations made between moving inertial frames in special relativity. Here it is assumed that the disk will elastically deform rather than fragment.

Discussion of the forces involved and the elastic properties of the material of the disk are beyond the scope of the present note. Some dynamical aspects of the rotating disk can be found in the papers by Clark,[8] Cavaller,[9] Brotus,[10] and McCrea.[11]

ACKNOWLEDGMENT

The author thanks the Aspen Centerfor Physics where this paper was written.

[1] A. Einstein, *The Meaning of Relativity* (Princeton U.P., Princeton, NJ, 1955), pp. 55–63.
[2] H. C. Ohanian and R. Ruffini, *Gravitation and Space Time* (Norton, New York, 1994), pp. 317–325.
[3] R. Adler, M. Bazin, and M. Schiffer, *Introduction to General Relativity* (McGraw-Hill, New York, 1975), pp. 120–125.
[4] P. Ehrenfest, ''Gleichförmige Rotation starrer Körper und Relativitätstheorie,'' Phys. Z. **10**, 918 (1909).
[5] N. Sama, ''On the Ehrenfest Paradox,'' Am. J. Phys. **40**, 415–418 (1972).
[6] Ø. Grøn, ''Relativistic description of a rotating disk,'' Am. J. Phys. **43**, 869–876 (1975).
[7] Ø. Grøn, ''Covariant formulation of Hooke's law,'' Am. J. Phys. **49**, 28–30 (1981).
[8] G. L. Clark, ''The mechanics of continuous matter in relativity theory,'' Proc. R. Soc. Edinburgh Sec. A **62**, 434–441 (1948).
[9] G. Cavalleri, ''Solution of Ehrenfest's Paradox for a Relativistic Rotating Disk,'' Nuovo Cimento B **53**, 415–432 (1968).
[10] A. Brotas, ''Sur le problème du disque tournant,'' C. R. Acad. Sci. Paris **267**, 57–60 (1968).
[11] W. H. McCrea, ''Rotating relativistic ring,'' Nature (London) **232**, 399–401 (1971).

SIMPLICITY VS. UGLINESS

Ever since 't Hooft's 1971 paper I had been quite convinced of the correctness of the outlines of this theory, but I regarded the particular version of this theory that Salam and I had constructed as only one specially simple possibility. For instance, there might be other members of the family formed by the photon and the W and Z particles, or other particles related to the electron and neutrino. Pierre Duhem and W. Van Quine pointed out long ago that a scientific theory can never be absolutely ruled out by experimental data because there is always some way of manipulating the theory or the auxiliary assumptions to create an agreement between theory and experiment. At some point one simply has to decide whether the elaborations that are needed to avoid conflict with experiment are just too ugly to believe.

Steven Weinberg, *Dreams of a Final Theory* (Pantheon Books, New York, 1992), p. 125.