Measurements on a rotating frame in relativity, and the Wilson and Wilson experiment

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(Received 12 August 1996; accepted 15 May 1997)

Pellegrini and Swift have recently suggested that the use of special relativity in the calculation of the electric dipole moment of a moving magnetic dipole cannot be applied to the classic experiment of Wilson and Wilson, which used rotational motion. This paper contests that view. The disagreement arises in the choice of coordinates used to represent physical quantities measured in the rotating frame. The arguments of this paper are based on Einstein's discussion of the validity of arbitrary coordinates and the difficulty in their interpretation. Because of the lack of synchronization of clocks, caution must be used in assigning values to physical quantities in the usual coordinates that describe a rotating frame. This paper gives the detailed transformations to an inertial rest frame, where the interpretation of measurements is assured. Other aspects of the rotating frame are also discussed.

I. INTRODUCTION

In 1908, Einstein and Laub[1] suggested that a measurement of the electric dipole moment of a moving magnetic dipole be used as a test of special relativity. In 1913, Wilson and Wilson[2] performed the experiment on a magnetic insulator rotating in an external magnetic field. A clear account of this experiment is given by Pellegrini and Swift.[3] There are two effects of the magnetic field. First, the field induces magnetization in the material. This moving magnetization, according to relativity, produces an electric polarization as measured in the laboratory. Second, the magnetic field exerts a force on the moving bound charges, thereby inducing an electric polarization in the material. In the experimental setup of Wilson and Wilson these two contributions to the electric polarization must be added. Their experimental results agree with the calculated values based on special relativity. Pellegrini and Swift[3] have recently challenged the conclusion that this experiment is consistent with special relativity and insist that when the electric polarization is properly calculated in a rotating system, the result does not agree with experiment. They suggest that the theory may have to be modified or the experiment is wrong. This author disagrees. In this paper, only the contribution to the electric polarization due to the motion of the magnetized material is discussed since this is where the disagreement lies. For this purpose, the insulator is taken to have a permanent magnetization.

A magnetized slab of material in uniform motion perpendicular to its magnetization has, according to special relativity, an electric polarization as measured in the laboratory. The electric polarization of the magnetized slab, if uniform, can be described by a positive bound surface charge density on one side and a negative bound surface charge density on the other side. Pellegrini and Swift, however, assert that if this slab is part of a rotating cylindrical shell, no charge density is induced so that a result at odds with special relativity is obtained. The following physical reasoning casts doubt on this conclusion.

Suppose the radius of the cylindrical shell of magnetized material is taken to be extremely large and the angular velocity to be small, such that the speed of a segment of the shell matches the experimental value. If the radius is large enough, it would be difficult to distinguish, in a short time interval, the rotational motion of a finite segment from rectilinear motion. To claim that the two motions are qualitatively different in that one induces surface charge while the other doesn't, should bother one's physical intuition. It appears physically unreasonable to have zero induced surface charge density in a motion that can be made arbitrarily close to rectilinear motion, for which everyone agrees surface charge is induced. It should also be noted that the inertial frames used to interpret experiments are only approximate and invariably are part of a rotating system.

A more rigorous treatment follows in Sec. II, where the methods for the determination of physical quantities as one would observe them in a rotating frame are laid out according to the reasoning of Einstein, and in Secs. III and IV the necessary transformations are given and applied to an ideal experiment. In Sec. V, some questions and objections to the method of calculation are discussed. Finally, in the conclusion, the effect of the differential centrifugal force is discussed.

II. THE PRINCIPLE

In the mathematical analysis of this experiment it is assumed that the laboratory is an inertial frame with invariant line interval

$$ds^2 = c^2 dt^2 - dr^2 - r^2 d\phi^2 - dz^2, \tag{1}$$

where the spatial part is described by cylindrical coordinates. Transformation to arbitrary coordinates leaves this interval unchanged; only the space–time description of events will be different. Einstein,[4] basing his arguments on the principle of equivalence, concluded that all coordinates are equally valid. But the laws of nature are known only in inertial frames, where length is measured by standard rods and time is measured by standard clocks. Einstein suggests the use of a freely falling observer whose frame will be inertial. Einstein states: ''We can therefore always regard an infinitesimally small region of the space-time continuum as Galilean. For such an infinitely small region there will be an inertial system relative to which we are to regard the laws of the special theory of relativity as valid.'' In general, even with a metric that describes a gravitational field, a transformation can always be made to a local inertial frame such that the geodesic equation is a straight line, that is, material particles behave as if ''free'' of gravitational or inertial forces. The word local is used since the transformation is strictly valid only at one point in space–time. Even so, the inertial system will be an extremely good approximation if attention is confined to a sufficiently small neighborhood of the space–time point.

It should be noted that Einstein was not suggesting the restriction of relativity to local inertial frames but, on the contrary, his discussions were aimed at the development of the general theory. He argues that all arbitrary frames of reference are equally valid and the laws of physics are to be written in generally covariant form. The important point is that the laws of physics must reduce to their familiar form in a local inertial frame. In such a frame one has confidence in assigning values to physically measurable quantities.

Although the explicit transformation is given below, there is no need for it in the case at hand. The correct frame in which to measure the electric charges and currents of a small segment of the rotating cylindrical shell is the inertial frame which is instantaneously at rest with respect to the segment. Then special relativity applies. Wilson and Wilson used special relativity to calculate the induced electric polarization for the rotating material and verified the results experimentally.

Since the transformation to an inertial frame is valid for a small region of space–time, only physical relationships that are local can be described. Maxwell's equations in differential form are local in that they relate the fields and their sources at any space–time point. On the other hand, such things as the radiation of an accelerated charge is global in that it requires the determination of fields on surfaces that are at large distances to the charge. Such global problems in arbitrary coordinates are extremely difficult, if not impossible, to handle. Fortunately, the problem at hand requires the determination of the current and charge densities, both local quantities.

III. TRANSFORMATION BETWEEN THE ROTATING FRAME AND THE LABORATORY

The needed coordinate transformations are most easily given in terms of matrices. Before looking at the details of these transformations some background material will be presented. Four vectors are represented by column matrices. In particular, the displacement four vector is

$$\mathbf{dx}=\begin{pmatrix} c\,dt \\ dr \\ d\phi \\ dz \end{pmatrix}, \tag{2}$$

and its inner product with itself gives the invariant line element,

$$ds^2=\mathbf{dx}^T\mathbf{G}\mathbf{dx}, \tag{3}$$

where $\mathbf{dx}^T$ is the row matrix formed by taking the transpose of $\mathbf{dx}$, and the metric is represented by the symmetric matrix,

$$\mathbf{G}=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{4}$$

It is easily verified that the matrix multiplication in Eq. (3) yields the correct line interval. The current four vector, which transforms like the displacement four vector under change of coordinates, is

$$\mathbf{J}=\begin{pmatrix} c\rho \\ J^1 \\ J^2 \\ J^3 \end{pmatrix}, \tag{5}$$

where ρ is the charge density and J^i are the components of the current density. The time component $c\rho$ is denoted as the zeroth component while the Latin index i ranges over the spatial components 1, 2, and 3. The invariant line interval is usually written as

$$ds^2=g_{\alpha\beta}dx^\alpha\, dx^\beta, \tag{6}$$

where $g_{\alpha\beta}$ are the components of the symmetric matrix $\mathbf{G}$. Repeated Greek indices are summed over 0, 1, 2, and 3.

Consider now the following situation: A uniformly magnetized material in the form of a cylindrical shell rotates about its symmetry axis, which is oriented along the z axis. The magnetization is also along this axis. Let M be the magnetization of the material when it is at rest in the laboratory, as measured in the laboratory. Two questions arise.

First, what is the magnetization measured by an observer at rest with respect to the rotating cylinder? In Sec. IV it is shown that this measured value is M/γ, where γ is the usual relativistic factor. Typically, magnetic dipole moments are described by current loops, that is, current times the area of the loop. In the present case, because of the cylindrical symmetry, the magnetization (dipole moment per unit volume) can be described by cylindrical sheets of current in the increasing and decreasing ϕ directions. No electric charge density is needed.

The second question is the most important and is the basis of the disagreement with Pellegrini and Swift, that is, to what coordinate system should one ascribe the measured values of the current density? Notice that this is not the same as the situation of determining the components of a vector in a coordinate system by transforming from a system where the components are known. The following development addresses this last question.

To relate the current four vector in the rotating frame to the measured values in the laboratory, three changes of coordinates will be used. The first transformation goes from the laboratory to the rotating frame; the other two transformations are simply changes of coordinates in this frame. Each of the transformations will be discussed separately and then the overall result will be obtained by applying the three in succession. None of the transformations will change the radial coordinate r, so the following notation will be used throughout. The velocity of a point is

$$v=\omega r, \tag{7}$$

$$\beta\equiv v/c, \tag{8}$$

and

$$\gamma\equiv(1-\beta^2)^{-1/2}. \tag{9}$$

It should be kept in mind that two points with different r coordinates will have different velocities.

Let the coordinates in the rotating system be denoted by overbars. The transformation from the laboratory to the rotating frame is

$$t=\bar{t},\quad r=\bar{r},\qquad \phi=\bar{\phi}+\omega\bar{t},\quad z=\bar{z}. \tag{10}$$

In terms of these barred coordinates the invariant line interval given by Eq. (1) is

$$ds^2=\gamma^{-2}c^2d\bar{t}^{\,2}-2c\beta\bar{r}d\bar{\phi}d\bar{t}-d\bar{r}^{\,2}-\bar{r}^{\,2}d\bar{\phi}^{\,2}-d\bar{z}^{\,2}. \tag{11}$$

Even though the new time coordinate is equal to the old, one should not interpret this to mean that the new clocks are the same as the ones at rest in the laboratory system. Coordinate clocks are usually considered at rest with respect to the corresponding spatial coordinates. The invariant line interval in Eq. (11) shows that the proper time interval at a fixed position in the new coordinates is given by $\Delta\tau=\Delta\bar{t}/\gamma=\Delta t/\gamma$. This demonstrates the time dilation of a standard clock at rest in the rotating frame when compared to the clocks of the laboratory.

The transformation, Eq. (10), can be written in matrix form as

$$\mathbf{dx}=\bar{\mathbf{T}}\mathbf{d}\bar{\mathbf{x}}, \tag{12}$$

where

$$\bar{\mathbf{T}}=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ \omega/c & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} \tag{13}$$

is the transformation matrix. The current four vector follows the same transformation rule as the displacement four vector, that is:

$$\mathbf{J}=\bar{\mathbf{T}}\,\bar{\mathbf{J}}. \tag{14}$$

This is the first of the three transformations needed. Notice that if the charge density in the barred system, $\bar{\rho}$, were zero, then no charge density is obtained in the laboratory frame. Since only a surface current density is needed to describe the magnetization, Pellegrini and Swift assumed the charge density to be zero. This is the mathematical basis of their claim. But care must be taken in ascribing meaning to objects expressed in these new coordinates. As explained in Sec. II, the safest way to do this is to transform to a local inertial rest system where the physical meaning of the coordinates is assured. But even without doing this, there is an obvious difficulty with the invariant interval for the rotating system: With the cross-term $d\bar{t}\,d\bar{\phi}$, clocks in the rotating system are not synchronized. This follows directly from the method of synchronization by sending a light signal back and forth between two clocks. Because of the cross term, light appears to propagate differently in the positive ϕ direction compared to the negative ϕ direction, thereby requiring an adjustment to bring the clocks into synchronization. See the discussion of the Sagnac effect in Sec. V.

To synchronize[5,6] two clocks, say clock B to clock A, the time on clock B must be adjusted by the amount

$$c\Delta\bar{t}=c\int_{A}^{B}(\bar{g}_{0i}/\bar{g}_{00})d\bar{x}^{i}=-2\int_{A}^{B}\beta\gamma^2\bar{r}d\bar{\phi}, \tag{15}$$

where the integration is from the location of clock A to the location of B over some chosen path between the two clocks. Here, $\bar{g}_{00}$ is the 00 component of the matrix **G** and the repeated Latin indices are summed over the spatial coordinates. This expression can be used to synchronize clocks along an open curve. Start with clock B infinitesimally close to clock A and adjust clock B according to Eq. (15). Then move the label B to successive clocks along the chosen curve, adjusting each clock in turn. The clocks will then be synchronized for light traveling on the chosen path, but if a different path between the end clocks is taken, these end clocks may appear unsynchronized.

It is apparent from Eq. (15) that a transformation eliminating the cross term $\bar{g}_{0i}$ will automatically synchronize the clocks. The transformation should be such that the clocks are reset by varying amounts depending on their positions but with no changes in their spatial coordinates. The infinitesimal transformation[7] that resets the clocks is

$$c\,d\bar{t}=c\,dt^*+\beta\gamma^2r^*d\phi^*,\quad \bar{r}=r^*,\qquad \bar{\phi}=\phi^*,\quad \bar{z}=z^*. \tag{16}$$

In these new coordinates the invariant line interval reads as

$$ds^2=\gamma^{-2}c^2dt^{*2}-dr^{*2}-\gamma^2r^{*2}d\phi^{*2}-dz^{*2}. \tag{17}$$

Unfortunately, the expression for $d\bar{t}$ in Eq. (16) is not integrable, that is, no function of the sort

$$\bar{t}=\bar{t}(t^*,r^*,\phi^*,z^*) \tag{18}$$

exists. Therefore clocks cannot be synchronized throughout space. But all that is needed in the analysis is the synchronization of clocks in a local region of a spatial point as given in Eq. (16). In matrix form,

$$\mathbf{d}\bar{\mathbf{x}}=\mathbf{T}^*\mathbf{dx}^*, \tag{19}$$

where

$$\mathbf{T}^*=\begin{pmatrix}1 & 0 & \lambda & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} \tag{20}$$

and

$$\lambda\equiv\beta\gamma^2r^*. \tag{21}$$

This is the second transformation needed.

Even though the starred coordinate system is not an inertial frame, there is no difficulty in interpreting the metric. For example, the circumference of a circle of radius r^* is

$$C^*=\int_0^{2\pi}\gamma r^*d\phi=2\pi\gamma r^*, \tag{22}$$

which shows that the spatial part of the metric is not Euclidean. The application of Eq. (17) appears to violate the local restriction placed on the transformation. But this metric implies, however, that the measurement can be carried out by measuring, at rest, successive lengths on the circumference of the circle, each measurement made locally and each measurement identical to the others. The marked off lengths are summed as indicated in Eq. (22). Further discussion of the geometry of the rotating disk will be found in Sec. V.

Since the metric is diagonal, it is easy to transform to Minkowski coordinates at a given space–time point by a change in scale of all the coordinates. This will be the third and final change in coordinates. These new Minkowski coordinates, denoted by a tilde, represent the local inertial rest system in which measurements are to be made. The transformation is

$$dt^* = \gamma d\tilde{t}, \quad dr^* = d\tilde{x}, \qquad d\phi^* = (\gamma r^*)^{-1} d\tilde{y}, \quad dz^* = d\tilde{z}, \tag{23}$$

or, in terms of matrices,

$$\mathbf{dx}^* = \tilde{\mathbf{T}}\, \mathbf{d}\tilde{\mathbf{x}}, \tag{24}$$

where

$$\tilde{\mathbf{T}} = \begin{pmatrix} \gamma & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & (\gamma r^*)^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{25}$$

With this transformation the Minkowski line interval,

$$ds^2 = c^2 d\tilde{t} - d\tilde{x}^{\,2} - d\tilde{y}^{\,2} - d\tilde{z}^{\,2}, \tag{26}$$

is obtained for the point considered. It should be emphasized that with this last change of coordinates, the new metric is a restricted description of the rotating frame valid locally at a given point. It should also be noted that the overall transformation from the lab to this local coordinate patch is a Lorentz transformation.

The current four vector in the Minkowski coordinates must be determined so that it is the source of the observed magnetization of the material. This can be done with confidence since these coordinates describe an inertial frame of reference in which the laws of physics are known. What is needed is a current density in the $d\tilde{y}$ direction and no charge density. With this component of the current density, denoted by $\tilde{J}_y$, the four current is written as

$$\tilde{\mathbf{J}} = \begin{pmatrix} 0 \\ 0 \\ \tilde{J}_y \\ 0 \end{pmatrix}. \tag{27}$$

The remaining chore of transforming this vector to the laboratory system by the series of transformations is as follows:

$$\mathbf{J}^* = \tilde{\mathbf{T}}\, \tilde{\mathbf{J}}, \tag{28}$$

$$\overline{\mathbf{J}} = \mathbf{T}^* \mathbf{J}^*, \tag{29}$$

$$\mathbf{J} = \overline{\mathbf{T}\, \mathbf{J}}. \tag{30}$$

Overall, the transformation is

$$\mathbf{J} = \overline{\mathbf{T}}\, \mathbf{T}^* \tilde{\mathbf{T}}\, \tilde{\mathbf{J}}, \tag{31}$$

where, from Eqs. (13), (20), and (25),

$$\overline{\mathbf{T}}\, \mathbf{T}^* \tilde{\mathbf{T}} = \begin{pmatrix} \gamma & 0 & \beta\gamma & 0 \\ 0 & 1 & 0 & 0 \\ \omega\gamma/c & 0 & (\beta r)^{-1} + \beta\gamma\omega/c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{32}$$

Applying this transformation matrix to $\tilde{\mathbf{J}}$ given in Eq. (27) yields

$$\mathbf{J} = \begin{pmatrix} \beta\gamma\tilde{J}_y \\ 0 \\ [(\gamma\tau)^{-1} + \beta\gamma\omega/c]\tilde{J} \\ 0 \end{pmatrix}. \tag{33}$$

In particular, the charge density as measured in the laboratory is

$$c\rho = \beta\gamma\tilde{J}_y. \tag{34}$$

Since the current density is confined to a surface, so also is the charge density. With κ and σ denoting the surface current and the surface charge densities, respectively, the final result is

$$\sigma = \beta\gamma\kappa/c. \tag{35}$$

This is exactly what would be obtained by a special Lorentz transformation between two inertial frames with relative velocity v in the direction of the current density. It follows that the calculation made by Wilson and Wilson using special relativity does indeed give the correct result.

It may be objected that the result given in Eq. (34) implies a creation of charge by rotation. This, of course, would be impossible. According to Eq. (34), a circular current loop, neutral in the rotating frame as seen by the local comoving observer, will have a net charge density as observed from the laboratory. By symmetry, the charge density is the same at each point of the loop and adds to give a nonzero total charge. It appears that by merely changing the angular velocity, the net charge in the laboratory is changed. On the contrary, it must be concluded that if the angular velocity changes, the charge as observed in the laboratory remains unchanged. Therefore, if the rotating frame is brought to rest, then the charge density, initially zero as observed in the rotating frame, must increase until it agrees with the laboratory value when the rotation ceases. Several examples supporting this view are given in Sec. V.

IV. AN IDEAL EXPERIMENT

Let us now calculate the induced charge distributions of a permanently magnetized insulator in the shape of a cylindrical shell which is rotating. The first task is to determine the appropriate value of magnetization to be associated with the instantaneous rest frame of the magnetized material.

Let M be the magnetization (magnetic dipole moment per unit volume) of the material when it is at rest in the laboratory, as observed in the laboratory. The cylinder is now given an angular acceleration and brought up to the final angular velocity. It is usually assumed that the physical properties of atoms are not altered by their past history. Therefore, the dipole moment of the atoms in the instantaneous rest frame of a segment of the rotating cylinder, as observed from that frame, is taken to have the same value as it had when at rest in the laboratory. But the periphery of the shell as measured in the rotating frame has increased by the factor γ according to Eq. (22). Therefore the density of atoms is decreased by the factor $1/\gamma$, giving a magnetization of M/γ as measured by an observer in the rotating frame of the shell. It should be emphasized that in both cases, the atom is at rest with respect to the frame of reference: in the first case, with respect to the laboratory and in the second case, with respect to the rotating frame. And the magnetizations given are those that would be observed in the rest frames of the material.

As an interesting aside, one could ask a different question, that is, what is the magnetization of the rotating material as observed from the laboratory if the measured value is M/γ in the rest frame of the material? Taking account of the fact that the motion is perpendicular to the magnetization and that magnetization is part of the electromagnetic field tensor, the Lorentz transformation gives the value M for the magnetization as observed from the laboratory. Of course, one also obtains an electric polarization, which is the basis of the experiment of Wilson and Wilson. Although relevant, the discussion of this paragraph is not needed in the following analysis.

Consider a cylindrical shell of inner radius a and outer radius b, rotating about its symmetry axis. Specifically, let the angular velocity point in the positive z direction, that is, follow the right-hand rule with thumb in the direction of the angular velocity and fingers pointing in the direction of rotation, taken as the increasing ϕ direction. Divide this shell of finite thickness into elemental shells of infinitesimal thickness dr. The magnetization of the elemental shell, M/γ in the z direction, is described by a surface current density in the positive ϕ direction on the outside and by a surface current density of the same magnitude in the negative ϕ direction on the inside. Then, on the inside and the outside of the shell of finite thickness, the current densities are

$$K_b = M/\gamma_b, \quad K_a = -M/\gamma_a. \tag{36}$$

The surface charge densities, determined by Eq. (35), are

$$\sigma_b = \beta_b M/c, \quad \sigma_a = -\beta_a M/c. \tag{37}$$

Since the magnitude of the charge density on the inner surface is less than on the outer surface it appears as if the rotating cylinder becomes positively charged. This is impossible since the cylindrical shell was assumed neutral while it was at rest in the laboratory. Actually, there is a negative volume charge density throughout the material given by

$$\rho(r) = -d\sigma(r)/dr - \sigma(r)/r, \tag{38}$$

where

$$\sigma(r) = \beta M/c = \omega r M/c^2. \tag{39}$$

This result can easily be obtained by first noting that the only nonzero component of the electric polarization of an elemental shell is the radial component. This component is easily found to be $\sigma(r)$, the surface charge of an elemental shell as given by Eq. (39). This also gives the polarization throughout the material as a function of the radial coordinate. The volume charge density, as given by Eq. (38), then follows directly from minus the divergence of the polarization. Substituting Eq. (39) into Eq. (38) gives

$$\rho(r) = -2\omega M/c^2. \tag{40}$$

This charge density, when integrated over the volume of the finite shell, gives a charge that exactly cancels the surface charges.

Gauss's law shows that the electric field within the cylindrical shell points in the negative r direction and has radial component in SI units,

$$E = -\omega r M/(\epsilon_0 c^2). \tag{41}$$

This result also follows directly from the electric polarization, $\sigma(r)$. With this electric field, the difference in potential between the outer and the inner surfaces is

$$\Delta V = \omega M(b^2 - a^2)/(2\epsilon_0 c^2). \tag{42}$$

(In the Wilson and Wilson experiment, the value of the potential difference consists of the sum of this term plus the contribution due to the motion of the insulator through the external magnetic field. This magnetic field also induces the magnetization, which can be written in terms of the permeability.)

V. RIDDLES AND ENIGMAS

In order to preserve the continuity of the development given in the preceding sections, discussion of possible objections to the calculation and other issues related to rotating coordinates are gathered into this section.

A. Synchronization of clocks

Pellegrini and Swift propose that the clocks in the rotating system be synchronized to a clock (call it the ''big'' clock) on the axis of rotation, which reads laboratory time t. This can be done, for a clock fixed in the rotating system at some radial distance r, by sending a light signal from the big clock at the origin to the clock at radius r and then back to the big clock. The clock at radius r is synchronized by setting the time of arrival of the signal from the big clock equal to the time midway between the sending and the receiving of the signal at the big clock. This is precisely how Eq. (15) was derived and, for the metric of Eq. (11), it shows that for clocks on the same radial line no adjustment is necessary, that is, all the clocks in the rotating system read laboratory time t, and are synchronized to the big clock on the axis of rotation. But are the clocks synchronized with respect to one another? That is, suppose one takes two nearby clocks, say A and B, fixed in the rotating system at the same distance from the axis of rotation, and checks their synchronization by sending a light signal back and forth directly between them. According to Eq. (15) the time on clock B must be adjusted or changed from time t in order for it to be synchronized with clock A. Thus we have the apparent paradox that two clocks synchronized with a third are not necessarily synchronized with each other.

Clocks can be synchronized along an open curve by following the prescription of Eq. (15). Suppose that clocks are placed all along the curve and adjacent clocks are adjusted according to this prescription. Everything works nicely unless one tries to close the curve. Since the integrand is not an exact differential the integration around a closed path may not be zero. So if one starts at clock A and synchronizes the clocks along a closed path back to A one may find that clock A must be adjusted to be synchronized with itself, which of course is nonsense. Now the question arises how one should synchronize two adjacent clocks at the fixed radius r. Should one follow the radial path to the big clock and back or should one take a direct path between the two clocks. In this paper the direct path between the clocks is chosen. This leads to a synchronization necessary for measurements of lengths and volumes in the vicinity of the clocks. Synchronization of clocks is at the core of the differences between this paper and the paper of Pellegrini and Swift. In the following, several examples are given which support the synchronization used in this paper.

As an example of a transformation that is similar to the one used to go from the laboratory to the rotating frame, consider the Galilean transformation,

$$dx = d\overline{x} + v\, d\,\overline{t}, \quad dt = d\,\overline{t}, \tag{43}$$

on the two-dimensional Minkowski space described by

$$ds^2 = c^2 dt^2 - dx^2, \tag{44}$$

yielding

$$ds^2 = c^2 \gamma^{-2} d\overline{t}^{\,2} - d\overline{x}^{\,2} - 2v\; d\overline{x}\, d\overline{t}. \tag{45}$$

This is a perfectly valid coordinate description of the original two-dimensional Minkowski space. This is a transformation to a new moving reference frame since the transformation of the x coordinate contains the time. Note that in the new coordinates, the time is taken to be the original lab coordinate t in parallel with the transformation between the lab and the rotating system given by Eq. (10). Also notice the cross term indicates that the clocks using the lab time $t = \overline{t}$ are not synchronized in the new frame. The cross term can be eliminated by adjusting the readings of the clocks by varying amounts depending on their positions:

$$c\,d\,\overline{t} = c\,dt^* + \beta\gamma^2 dx^*, \quad d\,\overline{x} = dx^*, \tag{46}$$

which then gives the line interval

$$ds^2 = c^2 \gamma^{-2} dt^{*2} - \gamma^2 dx^{*2}. \tag{47}$$

Now rescale the coordinate by

$$dt^* = \gamma d\,\tilde{t}, \quad dx^* = \gamma^{-1} d\tilde{x}, \tag{48}$$

to again obtain the Minkowski line interval

$$ds^2 = c^2 d\tilde{t}^{\,2} - d\tilde{x}^{\,2}. \tag{49}$$

Putting together all the transformations, one obtains the special Lorentz transformation between the coordinate frames (ct, x) and $(c\,\tilde{t}, \tilde{x})$ with relative velocity v as expected. With the new time $\tilde{t}$, the clocks are synchronized but no one would have said that the clocks in the new frame with interval given by Eq. (45) were synchronized with the old lab time $t = \overline{t}$. Notice that this calculation exactly parallels the development in Sec. III, except that the $d\,\overline{t}$ in Eq. (46) is integrable in terms of t^* and x^*, unlike the case of rotating coordinates.

B. The geometry of the rotating disk

It may be claimed that the geometry of the rotating disk is Euclidean and that the circumference of a circle divided by the radius is 2π, and not greater. The original space–time is flat and the new coordinate description does not change this fact. The Riemann curvature tensor is zero in either set of coordinates. But when it is said that the space of the rotating disk is non-Euclidean, one is talking about a subspace of the original flat space–time, that is, the subspace defined by dt^* set equal to zero. This is similar to taking the ordinary three-dimensional flat space described by spherical polar coordinates and obtaining the curved subspace consisting of the surface of a sphere by setting r equal to a constant. Thus, in the subspace in which the clocks are synchronized, it is found that the circumference of a circle divided by its radius is $2\pi\gamma$, as given by Eq. (22). Einstein first talked about this as an example of a curved space. Over the years it has been written about by many authors including Ø. Grøn, who obtains[8] the same spatial geometry of the metric given in Eq. (17). Ø. Grøn also states,[9] in regard to a rotating disk, that ‘‘relativistic kinematics alone forbids giving the disk a rotation so that the rest lengths of the elements of the periphery remain constant during the period of the angular acceleration.’’

This geometry of rotating coordinates explains an apparent creation of charge with the initiation of a current in a wire. Suppose a copper wire is bent into a circle of radius r. The copper has charge density ρ_0 of electrons and an equal and opposite charge density of positive ions which remain at rest. The wire is neutral. Now apply an electric field so that the electrons move. The four-current density[10] associated with the moving electrons is

$$\mathbf{J} = \gamma_d\, \rho_0' \begin{pmatrix} c \\ v_d \end{pmatrix}, \tag{50}$$

where ρ_0' is the charge density in the frame at rest with respect to the moving electrons. For simplicity, the random motion of the electrons is neglected, and it is assumed all the electrons travel at the same velocity, the drift velocity v_d. The relativistic factor $\gamma_d = (1 - v_d^2/c^2)^{-1/2}$ compensates for the contraction of lengths in the rest system of the electrons as measured by an observer in the laboratory. It seems reasonable to take the rest density of the moving electrons to be the same as the original rest density as measured in the lab, that is, $\rho_0' = \rho_0$. But then there is the creation of charge in the amount

$$\Delta Q = \rho_0 (\gamma_d - 1) 2\pi r A, \tag{51}$$

where A is the cross-sectional area of the wire. The resolution to this paradox is found in the realization that the circumference of the circle in the rotating frame in which the charge carriers are at rest has increased by γ_d. With the same amount of charge distributed over this increased length, the charge density in the rest frame of the moving electrons is

$$\rho_0' = \rho_0 / \gamma_d, \tag{52}$$

so that the density of the moving charges as observed from the laboratory remains unchanged.

This result can be seen another way. Consider two adjacent electrons with angular separation $\Delta\phi$ before the application of the electric field. Let them both start at rest simultaneously and have exactly the same angular acceleration as observed from the laboratory. Then elementary kinematics tells us that the angular separation does not change so that the charge density as observed from the laboratory does not change. The proper length of each element of the arclength between the charges, when they have reached the drift velocity, must be increased by the factor γ_d to compensate for the Lorentz contraction; that is, the arclength between the charges in the rotating frame in which they are at rest is $\gamma_d r \Delta\phi$ rather than $r\Delta\phi$. This is an alternate way to see that the circumference of a circle divided by its radius is greater than 2π for a rotating frame.

The problem of two electrons with the same angular acceleration parallels the problem posed by Dewan and Beran.[11] Consider two identical rockets at rest in an inertial frame S. Let them face the same direction and be situated one behind the other. A thin thread links the two rockets and is just long enough to span the distance between the rockets (center to center, say). The rockets are then fired simultaneously and have identical acceleration programs. As observed from S, the rockets remain displaced from each other by a fixed distance (center to center). What happens to the thread? J. S. Bell[12] relates the humorous story about the discussion of this problem that once took place at the CERN canteen. ''A distinguished experimental physicist refused to accept that the thread would break, and regarded my assertion, that indeed it would, as a personal misinterpretation of special relativity. We decided to appeal to the CERN Theory Division for arbitration, and make a canvas of opinion... There emerged a clear consensus that the thread would not break!''

C. Charges and neutral currents

Contrary to Pellegrini and Swift, this paper claims the existence of charge density as measured in the lab for a rotating neutral current. For simplicity, take the current to be in a circular hoop with axis of rotation though the center of the hoop perpendicular to its plane. The observer on the rotating hoop claims to measure no charge density at all but only a current density, say in the direction of rotation and the same all around the hoop. This observer also claims that the reason the lab observer measures a charge density is because the lab observer, in measuring the charges in a volume, does not measure the sides of the volume simultaneously and, in this time difference as measured by the rotating observer, a net charge has flowed into or out of the volume. That a charge density is measured in the lab and not in the rotating system is simply due to the disagreement between the two observers on the simultaneity of events. The charge that is measured in the lab has all the properties of charge; that is, the electric flux through a closed surface containing the rotating neutral current is nonzero. This charge cannot change as the rotational speed of the hoop changes. One must conclude that, a charge density must be measured by an observer on the hoop as his reference frame changes with the slowing of rotation. The following examples may help to clarify this issue.

(1) This example supports the contention that a neutral current cannot be maintained as the system is brought to rest. Instead of a rotating hoop, consider a long continuous belt. The path of the belt has a long straight section, then follows a semicircle around a spindle into another long straight section which parallels the first, and finally follows a semicircle around another spindle completing the loop. The belt is driven by the spindles at any desired speed. Suppose at a certain speed an observer riding the belt reports a neutral current density in the same direction the belt is moving. An observer in the lab reports no charge density on the curved sections (according to Pellegrini and Swift) but a positive charge density on the straight portions according to special relativity. It seems strange that the charge density on the belt suddenly disappears on the portion of the belt that reaches a spindle. Even so, let us take the charge on the curved portions to be zero. Now it appears that charge can be created or destroyed on the straight portions of the belt by simply changing the speed of the belt if one insists that no charge density is ever measured by the observer on the belt. The way out of this paradox is not to claim that there is no observed charge in the lab but to discard the supposition that the observer on the belt always measures a neutral current. This is reasonable since the observer is changing reference frames as the belt slows down. Besides, it rescues special relativity.

(2) In this example a neutral current density as observed in a rotating system will be constructed and shown to give a charge density in the lab when the rotating system is brought to rest. The relativistic dynamics necessary to do this with an actual current in a wire is much too complicated. For example, the inertial forces tangent to the wire as the rotation is reduced to zero must be included. Fortunately, the effect being studied is purely kinematic. To simplify, the currents will be produced by rotating charged hoops of radius r. Start with two hoops at rest in the lab, the first one with charge density ρ_1 in the lab and the second hoop with charge density ρ_2. The charges are fixed in the hoops and it is assumed that the two hoops are superimposed. The hoops are now rotated about their centers such that hoop one has tangential velocity v_1 and hoop two has a greater tangential velocity $v_2 > v_1$. Thus an observer at rest with respect to hoop one will see a current due to the relative motion of hoop two. The charge densities ρ_1 and ρ_2 must be chosen such that this observer measures no net charge. Quantities measured by the observer on hoop one will be denoted by a tilde. From the discussion in Sec. V. B it follows that the observer on hoop one measures the stationary charge density of that hoop to be ρ_1/γ_1 and for neutrality the charge density of hoop two as observed from hoop one must be

$$\tilde{\rho}_2 = -\rho_1/\gamma_1. \tag{53}$$

The observed current density is $\tilde{\rho}_2 v_d$, where v_d is the drift velocity which, in this case, is the tangential velocity of hoop two as observed from hoop one. Now the four-current density is given by Eq. (50), where the relativistic factor γ_d accounts for the Lorentz contraction of volumes and ρ_0' is the charge density in the rest system of hoop two, that is, ρ_2/γ_2. Therefore the charge density of hoop two as observed from hoop one is

$$\tilde{\rho}_2 = \gamma_d \rho_2/\gamma_2. \tag{54}$$

Equating $\tilde{\rho}_2$ from Eqs. (53) and (54) gives

$$\rho_2 = -\frac{\gamma_2}{\gamma_1}\frac{\rho_1}{\gamma_d}. \tag{55}$$

Hence, the observed charge density in the lab is

$$\rho_{\text{lab}} = \rho_1 + \rho_2 = \rho_1\left(1 - \frac{\gamma_2}{\gamma_1\gamma_d}\right). \tag{56}$$

Now eliminate γ_2 by noting that the velocity v_2 is the relativistic addition of v_d and v_1,

$$v_2 = \frac{v_1 + v_d}{1 + v_1 v_d/c^2}, \tag{57}$$

so that

$$\gamma_2 = \left(1 + \frac{v_1 v_d}{c^2}\right)\gamma_1\gamma_d. \tag{58}$$

Then the observed charge density in the lab is

$$\rho_{\text{lab}} = -\rho_1 v_d v_1/c^2. \tag{59}$$

This charge density is written in terms of the current density as observed on hoop one,

$$\tilde{J}_1 = \tilde{\rho}_2 v_d = -\rho_1 v_d/\gamma_1, \tag{60}$$

to finally obtain, for the charge density as observed in the lab,

$$c\rho_{\mathrm{lab}} = \gamma_1 \beta_1 \tilde{J}_1 . \tag{61}$$

This result is consistent with Eq. (34) obtained by the series of transformations in Sec. III.

D. Sagnac effect

The experimentally observed asymmetry in the propagation of light around a closed path in a rotating system certainly distinguishes it from an inertial frame and is referred to as the Sagnac effect.[13] Setting the line interval of Eq. (11) equal to zero, the time interval as measured in the lab for a light signal, traveling at a fixed radius in either direction, to go completely around is

$$T^{\pm} = \frac{2\pi\gamma^2 R}{c} (1 \pm \beta), \tag{62}$$

where the plus sign is for the light signal going in the direction of rotation and the minus against. (Note that the integration is through an angle 2π when integrating in the direction of rotation and -2π against.)

This result shows that clocks cannot be synchronized in the large. Consider the synchronization of clocks in the usual way by sending a light signal back and forth between the clocks. Now try to synchronize a clock with itself by sending a light signal in the direction of rotation at a fixed radius around the disk to the clock and the return signal against the rotation back to the clock. Label the initial sending of the signal as event 1 occurring at time t. Label the reception of the signal after it has gone around once along with its emission in the opposite direction as event 2. Label the final reception of the signal as event 3. From Eq. (62) it is seen that the times of the events are

$$\begin{aligned} t_1 &= t, \\ t_2 &= t + \frac{2\pi\gamma^2 R}{c} (1+\beta), \\ t_3 &= t + \frac{4\pi\gamma^2 R}{c}. \end{aligned} \tag{63}$$

For synchronization of the clock with itself, t_2 must be adjusted to read

$$t_{2_{\mathrm{syn}}} = \frac{t_1 + t_3}{2} = t + \frac{2\pi\gamma^2 R}{c}, \tag{64}$$

which is in conflict with the actual reading of the clock given in Eq. (63). The adjustment in the time,

$$t_{2\mathrm{syn}} - t_2 = -2\pi\gamma^2 R\beta/c, \tag{65}$$

can be obtained from Eq. (15) by integrating around a closed path of radius R and identifying clock B with clock A. As noted earlier, there is no contradiction in synchronizing clocks on an open path. For this paper it is not necessary to synchronize clocks in the large but only to synchronize adjacent clocks infinitesimally separated. Measurements of physical quantities are made locally and by symmetry apply to the measurements made anywhere on the circle.

VI. CONCLUSION

Caution must be used in assigning physical values to quantities described in arbitrary coordinates. In particular, clocks in a rotating frame are not synchronized and this renders the interpretation of the components of four vectors difficult. But Einstein clearly shows how to assign physical sense to arbitrary coordinates by considering freely falling observers. The above analysis follows Einstein's prescription for assigning meaning to arbitrary coordinates and confirms the valid use of special relativity in the experiment of Wilson and Wilson. As shown in Sec. V, the analysis, based on relativistic kinematics, is simple and self-consistent. There is no need to propose ad hoc mechanisms to resolve the paradox as presented by Pellegrini and Swift. It should also be noted that these comments apply not only to the interpretation of the current four vectors but also to the electric and magnetic fields.

There is one further aspect of rotating systems that has not been discussed. By transforming to an inertial frame that is instantaneously at rest with respect to a segment of the rotating shell, a frame of reference is obtained that accounts for the centrifugal force at a single point. But no account is made for differential forces, that is, the difference in force at two different spatial points. For example, the difference in centrifugal force between two charge carriers both of mass m but with radial separation distance Δr is

$$\Delta F = m\omega^2 \Delta r. \tag{66}$$

If an electric dipole is oriented along the radial direction and is modeled by two equal but opposite charges q separated a distance Δr, the dipole moment,

$$p = q\Delta r, \tag{67}$$

is subject to the differential force,

$$\Delta F = m\omega^2 p/q, \tag{68}$$

pulling the charges apart. This effect is negligibly small in the Wilson and Wilson experiment.

ACKNOWLEDGMENTS

I want to thank Jim Griffin and Marshall Luban for stimulating discussions on the various problems addressed by this paper.

[1]A. Einstein and J. Laub, ''Uber die electromagnetischen Grundgleichungen für bewegte Körper,'' Ann. Phys. **43**, 869–876 (1908).

[2]M. Wilson and H. A. Wilson, ''On the electric effect of rotating a magnetic insulator in a magnetic field,'' Proc. R. Soc. London, Ser. A **89**, 99–106 (1913).

[3]G. N. Pellegrini and A. R. Swift, ''Maxwell's equations in a rotating medium: Is there a problem?'' Am. J. Phys. **63**, 694–705 (1995).

[4]Albert Einstein, *The Meaning of Relativity* (Princeton U.P., Princeton, 1955), pp. 55–63.

[5]H. Ohanian and R. Ruffini, *Gravitation and Space Time* (Norton, New York, 1994), pp. 324–325.

[6]L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Pergamon, Oxford, 1975), pp. 234–238.

[7]R. Adler, M. Bazin, and M. Schiffer, *Introduction to General Relativity* (McGraw-Hill, New York, 1975), pp. 120–131.

[8]Ø. Grøn, ''Relativistic description of a rotating disk,'' Am. J. Phys. **43**, 869–876 (1975).

[9]Ø. Grøn, ''Covariant formulation of Hooke's law,'' Am. J. Phys. **49**, 28–30 (1981).

[10]D. J. Griffiths, *Introduction to Electrodynamics* (Prentice-Hall, Englewood, NJ, 1989), p. 503.

[11]E. Dewan and M. Beran, ''Note on stress effects due to relativistic contraction,'' Am. J. Phys. **31**, 517–518 (1959).

[12]J. S. Bell, *Speakable and Unspeakable in Quantum Mechanics* (Cambridge U.P., Cambridge, 1987), pp. 67–68.

[13]E. J. Post, ''Sagnac effect,'' Rev. Mod. Phys. **39**, 475–493 (1967).