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On the Ehrenfest Paradox

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Certain aspects of the so-called Ehrenfest paradox are discussed. It is pointed out that while the Ehrenfest description of uniform rotation is not in keeping with relativity, the "paradox" per se is independent of this fact, and rests solely upon an imprecise use of notation.

A consideration of the kinematics pertaining to a uniformly rotating cylinder led Ehrenfest[1] in 1909 to a seeming difficulty that has since become known as "Ehrenfest's Paradox." This paradox is stated, quoting Ehrenfest, as follows:

> . . . Let R' be the radius appearing to the stationary observer during this motion. Then R' must satisfy two conditions that are contradictory to each other:
>
> (a) The circumference of the cylinder must show a contraction relative to the rest state, $2\pi R' < 2\pi R$ since each element of the circumference moves in its own direction with the instantaneous speed $R'\omega$.
>
> (b) If one considers an element of a radius, its instantaneous velocity is perpendicular to its length; thus, an element of a radius cannot show a contraction with respect to the rest state. Therefore, $R' = R$.

Thus Ehrenfest's concern clearly centers about the seeming necessity of concurrently satisfying the equations

$$2\pi R' < 2\pi R \quad \text{and} \quad R' = R. \tag{1}$$

There is the question, however, of what Eqs. (1) say (or should say). If taken at face value, as mathematical statements on some quantities R and R', there is no disputing the fact that an irreconcilable contradiction would be present. But Eqs. (1) are *not* just mathematical statements. On the contrary, they are a summary of a measurement procedure, and it is the purpose of this note to point out that the operational specifics of the intended measurement process are not implicit in these equations, as they should be, but must be carried along as a mental addendum thereto. It is this omission that is at the source of the particular difficulty raised by Eqs. (1). Thus the Ehrenfest paradox *per se* arises not from an inconsistency in relativity, with which it is not actually connected (see below), but from an ambiguous use of notation.

Interestingly, a relativistically valid description of rotation leads to relations that are very similar to Eqs. (1) and that on superficial consideration will still appear to embody a contradiction. It is well known, of course, that there is no real difficulty in this case, unless one insists on viewing the relativistic formulation in terms of nonrelativistic kinematics, and so no details will be gone into here except for a brief comparison later to the Ehrenfest description. What will be done, instead, is to consider only the problem exactly as posed and stated by Ehrenfest. Thus it is important to note that questions as to the geometry and metric[2–4] in a rotating frame are not actually contained in Ehrenfest's problem nor are they essential to its resolution. Therefore, both for simplicity and for purposes of retaining only salient features, the Ehrenfest problem will be discussed in terms of two identical and coaxial hoops, one of which is rotating uniformly about its axis relative to a second (inertial) hoop. It is easy to see that nothing is lost by this substitution, nor anything extraneous added.

The terminology used by Ehrenfest will be followed, with primed and unprimed quantities referring to measurements of the rotating and inertial hoops, respectively, *as made by the inertial hoop* (hereafter denoted by F). Mention is also made of the fact that rotational motion is rigorously describable[4] by special relativity in the inertial frame (i.e., F) in which the hoop center is at rest.

Consider now statement (a) of Ehrenfest's paper: "The circumference . . . must show a contraction relative to the rest state: $2\pi R' < 2\pi R$." This statement is untenable for two reasons. First, the moving circumference is *assumed*, for no good reason, to be represented by $2\pi R'$. Second, the notion of circumference is being conceptually manipulated as if it possessed some intrinsic length ($2\pi R'$), which is subject to a length contraction exactly as if it were some moving rectilinear element contracted relative to its rest state, i.e., $2\pi R' < 2\pi R$. This mixture of relativistic and intuitive considerations is unacceptable because, essentially, one cannot say that a certain curve has an intrinsic length, but must determine this length, in a given kinematical situation, by the utilization of a length standard in a manner appropriate to the situation. Thus it is a fundamental measuring process that is called for here, the application of which leads to the well-known result[4] that the moving circumference *is not* representable as $2\pi R'$ if relativity be correct.

From the point of view of relativity, therefore, there would be no need to pursue the problem any further because it is improperly stated. Nevertheless, it is still possible, and instructive, to continue in the spirit of Ehrenfest for the purpose of showing that even so there is no real inconsistency, and that the difficulty arises solely from an incomplete specification of the quantities R and R' used in Eqs. (1).

Going along then with the assumption that the circumference of the rotating hoop (hereafter denoted by L) is given by $2\pi R'$, one can infer, with Ehrenfest, that since each moving circumferential element is along the direction of motion, each such element and hence L must appear contracted to the inertial frame F, so that $2\pi R' < 2\pi R$. Quantitatively, one would obtain $2\pi R' = 2\pi R/\gamma$, where $\gamma = (1-\omega^2 R^2/c^2)^{-1/2}$, and c is the vacuum speed of light. However, there is hidden in the notation thus used a serious omission: The statement that the moving circumference is $2\pi R'$ is operationally incomplete. More to the point, it must be recalled that the intent is to relate the moving radius to the moving circumference by flexibly placing radial lengths a number of times along the periphery, and it is therefore necessary that the notation reflect this procedure faithfully and in a complete manner, i.e., one must write $L = 2\pi R_{||}'$, where ($||$) denotes the tangential direction. This, together with a similarly simple consideration along the radial direction, demands that Eqs. (1) be rewritten as

$$2\pi R_{\|}' < 2\pi R_{\|} \quad \text{and} \quad R_{\perp}' = R_{\perp}. \tag{2}$$

In this form, the inconsistency vanishes because, as seen by F, $R_{\|}' < R_{\perp}'$, while $R_{\|} = R_{\perp}$. It is obvious that $R_{\|}' < R_{\perp}'$ by exactly the factor necessary to make Eqs. (2) consistent. Thus even within the context of the unsupported assumption that the moving circumference is given by $2\pi R'$, there is no true paradox.

The foregoing explanation of Ehrenfest's paradox may seem somewhat facile, perhaps even a papering over of a basic difficulty by drawing attention to an unimportant lapse in notation. Yet the opposite is true. It is the paradox that is artificial, created where none exists by the notation. The subsequent question of the validity of the kinematics in the Ehrenfest description, or in any other, is properly a matter for physical investigation and as mentioned above, of no present concern. It may or may not turn out to be correct, but it can never be a paradox. Thus the most serious objection to Ehrenfest's description is epistemological. For if his statement of rotation were valid, he would have succeeded, by way of a genuine contradiction, in proving without recourse to any experiment that the ratio of circumference to radius cannot depend on the state of rotation of the disk. This is of course an impossibility, and so it was a foregone conclusion that his description had to be faulty somewhere.

Some comparisons of the Ehrenfest and relativistic descriptions of rotation will now be made. The relativistic description of the rotating circumference is given by[2–4]

$$N(C') = 2\pi N(R')\gamma = N(C)\gamma, \tag{3}$$

$$N(R') = N(R), \tag{4}$$

where γ is as given above, and $N(C')$ and $N(R')$ are the measure numbers of the moving circumference and radius, respectively, obtained in terms of a given length standard carried in the rotating frame, while $N(C)$ and $N(R)$ are the corresponding measure numbers obtained in the rest frame by use of the "same" length standard. It will be noted that there is no notion of an intrinsic peripheral length in Eq. (3), and that it is not at all a comparison of the "lengths" of the moving and stationary peripheries. It says only that the moving and stationary length standards (which are identical when at relative rest) fit into the moving and stationary peripheries, respectively, in multiples related by (3). This is *all* one can say as far as relativity is concerned, and so the comparison of the two circumferences as lengths has no meaning in the theory.

One can now easily see the distinction between Ehrenfest's description of the problem and the relativistic one. The first of Eqs. (1) would yield the result that the measure numbers for both peripheries are the same, when they are measured in their respective frames in terms of their respective length standards. This is, of course, to be expected since with the Ehrenfest premise of the circumference as a definite length undergoing a contraction effect, there is necessarily concomitant the apparent contraction, in the same proportion, of the tangentially oriented moving length standard. Hence the measure number remains the same. Equation (3), on the contrary, says that the measure number in the relativistic case must be a function of ωR.

The preceding observations on measure numbers may be summarized by simply saying that the Ehrenfest scheme is Euclidean, whereas the relativistic one is not. The former statement may seem surprising, but if it be kept in mind that the geometry is determined by the properties of geometrical figures as actually measured, rather than as mentally compared, then there is nothing surprising about it at all. Thus, to say that the circumference "contracts," $2\pi R' < 2\pi R$, while the radius "remains the same," $R' = R$, and to impute to this a change in the ratio of circumference to radius is to totally disregard the all-important details pertinent to this comparison. One might say that there is too much conceptual intrusion into a purely physical process, a situation that seems to have generated a common[2,5–7] and erroneous view that the Ehrenfest statement signals the existence of a non-Euclidean geometry.[8] One author[6] in particular has rather speciously "shown" that the ratio L/R differs from 2π in the Ehrenfest scheme.

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Complex Potentials in Classical Mechanics and Geometrical Optics

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It has been shown that the method of complex potentials widely used in hydrodynamics and electrostatics can be used in two-dimensional problems of particle mechanics and geometrical optics too.

I. INTRODUCTION

Analytical potentials of form $\Omega(z)=\xi(z)-i\eta(z)$ $(z=x+iy)$ have been widely used in two-dimensional problems of mathematical physics (Table I).[1] Recently, I taught a course in analytical mechanics and hydrodynamics, and a question arose about the use of complex potentials in the description of kinematical behavior of particles in a given two-dimensional force field.

This paper presents a possible way to use complex potentials in classical mechanics and geometrical optics. We have to limit problems to two-dimensional ones, and force fields must be conservative. This method is in all cases a little backward since we start with known $\Omega(z)$ and try to find out the problem for which this function is a solution.[2]

II. PARTICLE IN CONSERVATIVE FIELD

If we have a particle moving in two-dimensional field $U(x, y)$ we can use the principle of least action in the form presented by Jacoby,

$$\delta\int_A^B \{2m[E-U(x,y)]\}^{1/2}ds=0, \tag{1}$$

where we have to take integration between the endpoints A and B and where E is the conserved total energy. Now we can define two functions ξ and η so that

$$\xi(x,y)=\ln v(x,y), \tag{2}$$

$$\tan[\eta(x,y)]=dy/dx, \tag{3}$$

where $v(x, y)$ is the local velocity of the particle and dy/dx the slope of its trajectory. The differential ds of trajectory can be expressed as

$$\begin{aligned} ds&=(dx^2+dy^2)^{1/2}\\ &=[1+(dy/dx)^2]^{1/2}dx. \end{aligned} \tag{4}$$