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Rest frames for a point particle in special relativity

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The idea of a rest frame for a point particle is most useful when it is defined as a proper-time-dependent basis of the Minkowski vector space with time axis along the tangent vector to the particle's worldline. There are many such rest frames; the different possibilities rotate with respect to each other. Methods are developed to make simple and workable the concepts and techniques to deal with the relative rotations. The Thomas precession becomes clear and easy to calculate. Applications are made to spin equations.

I. INTRODUCTION

The Thomas precession[1–3] has surprised, not to say confused, several generations of students of physics. In fact, this reaction could serve as a definition of the phenomenon, for the phrase has been applied to almost any relativistic rotational motion whose existence is surprising from a nonrelativistic point of view.

The purpose of the present paper is to make clear the new distinctions that the relativistic point of view calls for, to develop simple and effective mathematical tools for dealing with the phenomena, and (it is hoped) thereby to remove the surprises. The treatment involves considerable mathematics, but no more than is absolutely necessary. Unfortunately, physical intuition, grounded on nonrelativistic phenomena, fails in this area. It could hardly be otherwise, for the Thomas angular velocity ω_T, of magnitude va/c^2 in household units, vanishes in the nonrelativistic limit $c\rightarrow\infty$. But, this fact is no cause for despair. It gives us an opportunity to improve our understanding and to develop a new relativistic physical intuition.

The search for a suitable relativistic understanding has never stopped. In contrast to the geometrical approach of the present paper, the Clifford algebra formalism is expounded in reference,[4] and a new groupoid structure is developed in Ref. 5; both these references have extensive bibliographies. However, notation is a serious problem in a subject whose foundations are not yet set. Two references seldom share a common notation (even if they are by the same author![6]).

In the nonrelativistic kinematics of an accelerating point particle, given an inertial frame of reference, there is at each instant a well-defined instantaneous rest frame (a Euclidean triad). This is chosen at each instant to be parallel to the given frame (the lab, say). The instantaneous rest frames, for different instants, are parallel to each other. These simple facts are the basis for our nonrelativistic expectations about the properties of rest frames.

The relativistic generalisation of the ideas in the previous paragraph shows that new distinctions have to be made. If the point particle is not moving uniformly (in which case its instantaneous rest frame could be a fixed inertial frame), then it is not possible for its rest frame (now a tetrad rather than a triad) to remain parallel to a given frame: the timelike basis vector, required to be parallel to the spacetime velocity of the point, must change under acceleration; and the spatial triad, required to be orthogonal to the timelike basis vector, must change too.

The simplest universal idea of parallelism having failed, separate relativistic concepts must be introduced to embody separate features of the nonrelativistic notion.

Two inertial frames of reference, K and K', whose coordinates (we use $x^0=ct$ and units such that $c=1$) are related by

$$\begin{aligned} x'^0 &= \gamma(x^0 - Vx^1) \\ x'^1 &= \gamma(x^1 - Vx^0) \\ x'^2 &= x^2, \quad x'^3 = x^3 \end{aligned} \tag{1}$$

have two sets of corresponding parallel basis vectors, $\mathbf{E}_2' = \mathbf{E}_2, \mathbf{E}_3' = \mathbf{E}_3$. But, $\mathbf{E}_1' = \gamma(\mathbf{E}_1 + V\mathbf{E}_0) \neq \mathbf{E}_1$; not all the spatial axes are parallel in spacetime because $\mathbf{E}_0' = \gamma(\mathbf{E}_0 + V\mathbf{E}_1)$. Such "partially parallel," or "quasiparallel" (the name used by Schwartz[7]), frames traditionally are again simply called "parallel" in much of the literature of special relativity. The latter name is also used for pairs of frames related to K and K' by a common (subjective) rotation of their spatial axes. There is perhaps no logical reason why this name should not be used, but the danger of psychological drag from the nonrelativistic idea must be guarded against.

The idea of partially parallel frames is a concept which helps us get beyond nonrelativistic intuition. Also, for a continuous sequence of frames one can analyse more carefully the ideas of rotation. Among sequences of frames the instantaneous rest frames are the most interesting.

Two types of instantaneous rest frame have proved to be especially useful. The first, called a Fermi–Walker or "nonrotating" frame, is defined by differential equations governing its development in proper time. The intuitive idea that the equations implement is that the frame changes in such a way that at each instant it is parallel (that is, "partially parallel," as above) to itself at the just previous instant. This condition can also be understood as requiring that the angular velocity of the frame with respect to the rest space should vanish. The Fermi–Walker frame appears to have great physical importance; rotational equations of motion take much their simplest forms with respect to it.

The second type of instantaneous frame is the so-called "boosted frame," a rest frame which at each instant is parallel ("partially parallel") to a given fixed inertial frame, the lab.

The two frames just mentioned differ from each other, at one instant, by a spatial rotation. As time passes they rotate with respect to each other; this motion is the commonest instance of a Thomas precession. It is no wonder that the idea is hard to grasp from a nonrelativistic perspective: We must contemplate the relative rotation of two frames both of which "ought to be" fixed.

Precisely the rotation above arises if we specify the "fixed" spatial direction of a gyroscope, carried by a moving point, by

$$\frac{d^f\mathbf{S}}{d\tau} = \mathbf{0}. \tag{2}$$

(The notation, which will be developed later, in Sec. VII, means that the spin, the vector $\mathbf{S}$, does not change with respect to the Fermi–Walker frame.) When Eq. (2) is decomposed with respect to the *boosted* rest frame, however, the spin vector $\mathbf{S}$ is seen to precess with respect to it. It is therefore vital to keep clear the difference between the various rest frames, and useful, to say the least, to use a notation that makes this easy, something that has not always been done in the literature.

To help make the important distinctions we keep quite separate spacetime $\mathscr{M}$ (a collection of points, with the set of inertial coordinate systems) on the one hand, and the space of vectors (a four-dimensional vector space $\mathscr{V}$ with many orthonormal basis sets, called frames) on the other. Any orthonormal basis will be the special coordinate basis for many coordinate systems, differing only by translations, but despite this we do not think of vectors in terms of coordinate differences. Vectors are always just elements of the abstract space $\mathscr{V}$ and can be decomposed in any basis. Referring to the spin vector of (2), we might write

$$\mathbf{S} = S_f^m \mathbf{f}_m = S_b^m \mathbf{b}_m = S_E^\mu \mathbf{E}_\mu \tag{3}$$

to exhibit the decompositions with respect to the Fermi–Walker basis, the boost basis, or an inertial frame basis. We never use a concept of vectors tied to one basis, such as $(S_E^0, S_E^1, S_E^2, S_E^3)$.

The Fermi–Walker frame is the rest frame with the greatest physical importance; it arises in the theoretical development of rotational equations of motion for particles. However, it requires the solution of differential equations to make it explicit. The boosted rest frame is calculationally more accessible, and this fact accounts for its use. The inertial frames, on the other hand, are part of the furniture of spacetime, indeed part of the definition of spacetime; they are immediately available to all. But, as (3) shows, using inertial frames can introduce redundant components. One can get some of the advantages of the Fermi–Walker frame, together, with accessibility of the inertial frames, by using an "unboosted" Fermi–Walker frame in an inertial frame, i.e., a representative of the former in the inertial frame. By the same process, an unboosted $\mathbf{S}$ gives a representative $\boldsymbol{\sigma}$ in the inertial frame with no redundant component.

In the sections that follow the ideas mentioned above are developed: in Sec. II a description of the conceptual framework of relativity, the distinction between $\mathscr{M}$ and $\mathscr{V}$; in Sec. III, the notation for relative velocities and accelerations of a moving point; the idea of basis frame in Sec. IV, together with generalised angular velocity; in Secs. V and VI, the definitions of the Fermi–Walker frames and the boost frames; relative angular velocity, as appears implicitly in (2), is defined in Sec. VII; the unboosted Fermi–Walker frame is defined in Sec. VIII; spin equations for a magnetic dipole in a general external field are discussed in Sec. IX, and the frames are explicitly displayed for the case of a uniform magnetic field in Sec. X. The methods are also applicable to discrete inertial frames, as arise in the general addition-of-velocities formula or representations of the Poincaré group. This is discussed in Sec. XI.

II. Conceptual Framework of Special Relativity

In this section, we lay out the bare bones of our representation of special relativity. It is not meant to be a serious development of the full theory, but just a setting down of sufficient relations so that the important distinctions are made, and so that ambiguities due to unexplained notations and unrevealed conventions are avoided.

We consider the spacetime of special relativity to be a four-dimensional manifold $\mathscr{M}$ whose points $P,Q,\ldots$ are the mathematical counterpart of idealized physical events.

Mappings $K,K',\ldots$ of $\mathscr{M}$ onto R^4 provide coordinate systems for $\mathscr{M}$. Typically,

$$K:\mathscr{M}\to R^4,\quad K:P\mapsto x_P^\mu=(x_P^0,x_P^m)\in R^4. \tag{4}$$

If K and K' are the Minkowskian coordinate systems associated with inertial frames of reference, then for every P, the K and K' coordinates are related by a Poincaré transformation

$$x_P'^\mu=a^\mu_{\ \nu}x_P^\nu+a^\mu, \tag{5}$$

where

$$a^\mu_{\ \alpha}a^\nu_{\ \beta}\eta_{\mu\nu}=\eta_{\alpha\beta} \tag{6}$$

with

$$\eta_{\alpha\beta}\ \text{diagonal, diagonal elements}=(-1,+1,+1,+1). \tag{7}$$

We are using spacetime units with $c=1$. Time orientation and space orientation are fixed by restricting ourselves to systems such that

$$a^0_{\ 0}>0,\quad \det(a^\mu_{\ \nu})=+1 \tag{8}$$

for the transformations between them. We may think of time increasing into the future and all spatial coordinate systems as right-handed.

The important point at this stage is that the theory exists at a coordinate-independent level. There is a distinction between the geometrical, or "absolute" $\mathscr{M}$ with its points $P,Q,\ldots$ and the "relative" representations of them in R^4 given by $x_P^\mu, x_Q^\mu,\ldots$. We understand concepts best when they are expressed in geometrical terms.

Spacetime vectors are defined, in the first instance, as translation vectors in $\mathscr{M}$, a special class of mappings of $\mathscr{M}$ onto itself. They form a new space, a four-dimensional vector space $\mathscr{V}$. Two points P,Q define a vector, which determines the mapping connecting any pair of points P_1, Q_1 in the same relation to each other as P,Q

$$\overrightarrow{PQ}\in\mathscr{V}:\mathscr{M}\to\mathscr{M},\quad \overrightarrow{PQ}:P\mapsto Q,\quad \overrightarrow{PQ}:P_1\mapsto Q_1,\ldots\ . \tag{9}$$

Translations are only the beginning of vectors, but all others get their physical significance by their relation (an isomorphism) to the translations. We use the same notation for all (bold-face type except for the translations written $\overrightarrow{PQ}$). Introducing vectors in such a relatively abstract way actually makes them more physical and so more *concrete*: they are defined independently of any coordinate system.

Nonetheless, the coordinate systems determine useful basis systems for $\mathscr{V}$. For each inertial coordinate system K, we define four basis vectors $\mathbf{E}_\mu$ by

$$\mathbf{E}_\mu:P(\text{with } K \text{ coordinates } x_P^\nu)\mapsto Q(x_Q^\nu=x_P^\nu+\delta^\nu_\mu). \tag{10}$$

Then,

$$\overrightarrow{PQ}=(x_Q^\mu-x_P^\mu)\mathbf{E}_\mu=(x_Q'^\mu-x_P'^\mu)\mathbf{E}'_\mu. \tag{11}$$

The relation between the bases $\{\mathbf{E}_\mu\}$ and $\{\mathbf{E}'_\mu\}$ is given in terms of the matrix in (5) by

$$\mathbf{E}_\mu=a^\nu_{\ \mu}\mathbf{E}'_\nu. \tag{12}$$

(The bases associated with inertial frames will consistently be denoted by an uppercase $\mathbf{E}$. The noninertial moving rest frames, to be introduced later, will be denoted by lower case letters $\mathbf{b}_\mu$, $\mathbf{f}_\mu$, $\mathbf{r}_\mu$, $\mathbf{e}_\mu$.) Other noninertial frames $\mathbf{F}_\mu$, $\mathbf{R}_\mu$ also appear, which are not rest frames but whose time axis is fixed.

Because of (6) and (12), the scalar product

$$\mathbf{E}_\mu\cdot\mathbf{E}_\nu=\eta_{\mu\nu} \tag{13}$$

has the same form for all inertial bases.

Dyadics will prove to be very useful in the subsequent development. They relate different bases, express generalised angular velocities, represent the electromagnetic field. They form a sixteen dimensional vector space denoted by $\mathscr{V}\otimes\mathscr{V}$.

Without serious loss they may be considered to be linear transformations of $\mathscr{V}$ into itself. For any two ordered vectors $\mathbf{a}$ and $\mathbf{b}$ there is an elementary dyadic $\mathbf{a}\otimes\mathbf{b}$ for which the corresponding transformation is

$$\mathbf{a}\otimes\mathbf{b}:\mathscr{V}\to\mathscr{V},\quad \mathbf{a}\otimes\mathbf{b}:\mathbf{x}\mapsto\mathbf{a}(\mathbf{b}\cdot\mathbf{x})\equiv(\mathbf{a}\otimes\mathbf{b})\cdot\mathbf{x}. \tag{14}$$

Compound dyadics are linear combinations of the elementary ones. (Because $\mathscr{V}$ is a finite-dimensional vector space, with a scalar product, there are several equivalent definitions of dyadics and more general tensors. They may, for example, simply be considered to be formal sums of primitive elements $\mathbf{a}\otimes\mathbf{b}$, but it is very convenient to be able to relate them to the already-familiar concept of linear transformation of vectors. The scalar product that appears is a version of a linear functional of vectors. A discussion of various definitions (in a very general setting) and their equivalence is given in Penrose and Rindler.[8])

For each basis $\{\mathbf{E}_\mu\}$ of $\mathscr{V}$ there is a basis $\{\mathbf{E}_\mu\otimes\mathbf{E}_\nu\}$ of $\mathscr{V}\otimes\mathscr{V}$. A general dyadic (sans serif letters are used to denote single symbol dyadics) may be decomposed

$$\mathsf{U}=\mathsf{U}^{\mu\nu}\mathbf{E}_\mu\otimes\mathbf{E}_\nu=\mathsf{U}'^{\mu\nu}\mathbf{E}'_\mu\otimes\mathbf{E}'_\nu, \tag{15}$$

where the components $\mathsf{U}^{\mu\nu}$, $\mathsf{U}'^{\mu\nu}$ are related by the tensor transformation law.

The unit dyadic is written

$$\eta=\eta^{\mu\nu}\mathbf{E}_\mu\otimes\mathbf{E}_\nu=-\mathbf{E}_0\otimes\mathbf{E}_0+\mathbf{E}_m\otimes\mathbf{E}_m, \tag{16}$$

and satisfies $\eta\cdot\mathbf{x}=\mathbf{x}$ (in the notation of (14)).

The dot product of dyadics works like the composition of the corresponding linear transformations and is exemplified by

$$(\mathbf{a}\otimes\mathbf{b})\cdot(\mathbf{c}\otimes\mathbf{d})=(\mathbf{b}\cdot\mathbf{c})\mathbf{a}\otimes\mathbf{d}.$$

The transpose U^T of the dyadic U in (15) is defined by

$$\mathsf{U}^T=\mathsf{U}^{\mu\nu}\mathbf{E}_\nu\otimes\mathbf{E}_\mu=\mathsf{U}^{T\mu\nu}\mathbf{E}_\mu\otimes\mathbf{E}_\nu,$$

so $\mathsf{U}^{T\mu\nu}=\mathsf{U}^{\nu\mu}$. We use a special notation for the antisymmetric combination

$$\mathbf{a}\wedge\mathbf{b}\equiv\mathbf{a}\otimes\mathbf{b}-\mathbf{b}\otimes\mathbf{a}.$$

(The wedge product notation comes from the exterior product, but none of its properties will be needed beyond the built-in antisymmetry.)

Finally, we introduce the dual of an antisymmetric dyadic $\mathsf{F}=-\mathsf{F}^T$. The definition is given using a specific inertial basis, in terms of which

$$\mathsf{F}=\mathsf{F}^{\mu\nu}\mathbf{E}_\mu\otimes\mathbf{E}_\nu,\quad \mathsf{F}^{\mu\nu}=-\mathsf{F}^{\nu\mu},$$

but the result is actually basis independent. We need the completely antisymmetric symbol $\epsilon^{\mu\nu\lambda\rho}$, fixed by $\epsilon^{0123}=+1$, and the covariant components of F,

$$\mathsf{F}_{\lambda\rho}=\eta_{\lambda\mu}\eta_{\rho\nu}\mathsf{F}^{\mu\nu}=\mathbf{E}_\lambda\cdot\mathsf{F}\cdot\mathbf{E}_\rho.$$

Then define the dual of F to be

$$\mathsf{F}^*=\mathsf{F}^{*\mu\nu}\mathbf{E}_\mu\otimes\mathbf{E}_\nu,\quad \mathsf{F}^{*\mu\nu}=\tfrac{1}{2}\epsilon^{\mu\nu\lambda\rho}\mathsf{F}_{\lambda\rho}.$$

For the dual of $\mathbf{a}\wedge\mathbf{b}$ we also use a second notation

$$\begin{aligned}(\mathbf{a}\wedge\mathbf{b})^*&=\mathbf{E}_\mu\otimes\mathbf{E}_\nu\epsilon^{\mu\nu\lambda\rho}(\mathbf{a}\cdot\mathbf{E}_\lambda)(\mathbf{b}\cdot\mathbf{E}_\rho)\\ &\equiv\epsilon(\mathbf{a},\mathbf{b})=\mathbf{E}_\mu\otimes\mathbf{E}_\nu\epsilon^{\mu\nu\lambda\rho}a_\lambda b_\rho.\end{aligned}\tag{17}$$

The notation can be extended to give a vector which is a linear, antisymmetric function of three others

$$\epsilon(\mathbf{a},\mathbf{b},\mathbf{c})\equiv\mathbf{E}_\mu\epsilon^{\mu\nu\lambda\rho}a_\nu b_\lambda c_\rho.\tag{18}$$

If $\mathbf{a}=\mathbf{E}_0$, and $\mathbf{b}\cdot\mathbf{E}_0=\mathbf{c}\cdot\mathbf{E}_0=0$, then

$$\epsilon(\mathbf{E}_0,\mathbf{b},\mathbf{c})=\mathbf{E}_\mu\epsilon^{\mu\nu\lambda\rho}(\eta_{\nu0})(b_\lambda c_\rho)=\mathbf{E}_m\epsilon^{mlr}b_l c_r\equiv\mathbf{b}\times\mathbf{c}\tag{19}$$

gives a workable formulation in $\mathscr{V}$ of the three-dimensional vector product in a frame with time axis $\mathbf{E}_0$. Both the factors and the product itself are spatial in the special frame. The same notation is used with spatial vectors in the rest frame. (Note: *spatial* is frame dependent, $\mathbf{a}$ is spatial with respect to $\mathbf{E}_0$ if $\mathbf{a}\cdot\mathbf{E}_0=0$; *spacelike* is frame independent, $\mathbf{a}\cdot\mathbf{a}>0$.)

Because we have vectors (in $\mathscr{V}$) and dyadics (in $\mathscr{V}\otimes\mathscr{V}$) available at an abstract, "absolute" level, equations involving them are not tied to one basis. One can routinely work, without difficulty or ugliness, with equations using different bases in different terms, or different bits of one term, or no basis at all. This is especially valuable while one is trying to unravel the definitions of, and relations between, the various possible rest frames.

III. WORLDLINE OF A POINT PARTICLE

The worldline of a point particle may be specified by a τ-dependent translation vector from some fixed point O in spacetime:

$$\mathbf{z}(\tau)\in\mathscr{V}:O\mapsto P(\tau).\tag{20}$$

The spacetime velocity vector $\mathbf{v}(\tau)$, the tangent vector at $P(\tau)$ to the curve in (20), is

$$\mathbf{v}(\tau)=\frac{d\mathbf{z}}{d\tau}.\tag{21}$$

The worldline must be timelike, which requires $\mathbf{v}\cdot\mathbf{v}<0$, and the parameter τ is proper time (as we assume) if

$$\mathbf{v}(\tau)\cdot\mathbf{v}(\tau)=-1.\tag{22}$$

With this parameterization, the spacetime acceleration,

$$\mathbf{a}(\tau)=\frac{d\mathbf{v}}{d\tau},\tag{23}$$

satisfies

$$\mathbf{v}\cdot\mathbf{a}=0.\tag{24}$$

We choose τ to increase into the future.

Decomposing $\mathbf{z}$ with respect to an inertial frame basis, we have

$$\mathbf{z}=Z^\mu\mathbf{E}_\mu=T\mathbf{E}_0+Z^m\mathbf{E}_m,\tag{25}$$

where $Z^0=T$, two alternative notations for time in the frame. (The notational scheme that we try to adhere to, when other conventions do not have priority, is: upper case letters for *relative*, inertial frame dependent vectors and scalars; lower case letters for basis-independent geometric quantities.) Then,

$$\mathbf{v}=\frac{d\mathbf{z}}{d\tau}=\frac{dZ^\mu}{d\tau}\mathbf{E}_\mu=\frac{dT}{d\tau}\left(\mathbf{E}_0+\frac{dZ^m}{dT}\mathbf{E}_m\right),\tag{26}$$

where

$$\mathbf{V}\equiv\frac{dZ^m}{dT}\mathbf{E}_m\tag{27}$$

is the relative velocity of the particle with respect to the inertial frame ($\mathbf{V}$ is spatial in the inertial frame, $\mathbf{E}_0\cdot\mathbf{V}=0$; the derivative is with respect to T, the time in the frame). The condition (22) requires that $\gamma\equiv-\mathbf{v}\cdot\mathbf{E}_0$ satisfies

$$\gamma=-\mathbf{v}\cdot\mathbf{E}_0=\frac{dT}{d\tau}=+(1-\mathbf{V}\cdot\mathbf{V})^{-1/2}.\tag{28}$$

In the new notation, the spacetime velocity is

$$\mathbf{v}=\gamma(\mathbf{E}_0+\mathbf{V})\tag{29}$$

when decomposed relative to the frame K.

Differentiating (29), we get the K-frame decomposition of the spacetime acceleration

$$\mathbf{a}=\frac{d\mathbf{v}}{d\tau}=\dot\gamma(\mathbf{E}_0+\mathbf{V})+\gamma\frac{d\mathbf{V}}{d\tau}=\gamma^4\mathbf{V}\cdot\mathbf{A}(\mathbf{E}_0+\mathbf{V})+\gamma^2\mathbf{A},\tag{30}$$

where

$$\mathbf{A}=\frac{d\mathbf{V}}{dT},\quad \mathbf{A}\cdot\mathbf{E}_0=0\tag{31}$$

is the relative acceleration. The magnitude of $\mathbf{a}$ is given by

$$\mathbf{a}\cdot\mathbf{a}=\gamma^4\mathbf{A}\cdot\mathbf{A}+\gamma^6(\mathbf{V}\cdot\mathbf{A})^2.\tag{32}$$

IV. MOVING FRAMES OF REFERENCE

We consider a set of τ-dependent orthonormal basis vectors, that is, $\{\mathbf{e}_\mu(\tau)\in\mathscr{V}\}$ such that

$$\mathbf{e}_\mu(\tau)\cdot\mathbf{e}_\nu(\tau)=\eta_{\mu\nu}. \tag{33}$$

We call such a collection a moving frame. When the basis vectors are related to a particle worldline, τ will be its proper time; till then it is just a real parameter. We suppose that at each τ the basis $\mathbf{e}_\mu(\tau)$ coincides with one of the inertial frames of Sec. II, so $\mathbf{e}_0$ may be considered to be future pointing and the $\mathbf{e}_m$ right-handed.

Denoting the derivative with respect to τ by an overdot, differentiating (33) gives

$$\dot{\mathbf{e}}_\mu\cdot\mathbf{e}_\nu+\mathbf{e}_\mu\cdot\dot{\mathbf{e}}_\nu=0. \tag{34}$$

If we define the dyadic $\Omega_e\in\mathscr{V}\otimes\mathscr{V}$ by

$$\Omega_e=\eta^{\mu\nu}\dot{\mathbf{e}}_\mu\otimes\mathbf{e}_\nu=-\dot{\mathbf{e}}_0\otimes\mathbf{e}_0+\dot{\mathbf{e}}_m\otimes\mathbf{e}_m, \tag{35}$$

then

$$\dot{\mathbf{e}}_\mu=\Omega_e\cdot\mathbf{e}_\mu, \tag{36}$$

and this will lead us to an understanding of Ω_e as a generalised angular velocity dyadic. Equation (34) may be written

$$\mathbf{e}_\nu\cdot\Omega_e\cdot\mathbf{e}_\mu+\mathbf{e}_\mu\cdot\Omega_e\cdot\mathbf{e}_\nu=0. \tag{37}$$

This means that Ω_e is antisymmetric

$$\Omega_e+\Omega_e^T=0. \tag{38}$$

Any antisymmetric dyadic can be decomposed in the form

$$\Omega_e=\mathbf{e}_0\wedge\boldsymbol{\alpha}+(\mathbf{e}_0\wedge\boldsymbol{\beta})^* \tag{39}$$

with vectors $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ orthogonal to $\mathbf{e}_0$; in fact (39) holds with

$$\boldsymbol{\alpha}=\Omega_e\cdot\mathbf{e}_0,\quad \boldsymbol{\beta}=-\Omega_e^*\cdot\mathbf{e}_0. \tag{40}$$

[To prove these relations it is enough to split up the sums in

$$\Omega_e=\Omega_e^{\mu\nu}\mathbf{e}_\mu\otimes\mathbf{e}_\nu=\Omega_e^{0n}\mathbf{e}_0\wedge\mathbf{e}_n+\Omega_e^{mn}\mathbf{e}_m\otimes\mathbf{e}_n$$

and check with the definition of the dual in (17). Any unit timelike vector can be used in place of $\mathbf{e}_0$, though of course the $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ will depend on the choice.] It is just such a decomposition of the electromagnetic field dyadic F that produces the (relative) electric and magnetic fields in the frame with time basis vector $\mathbf{e}_0$.

The basis $\{\mathbf{e}_\mu(\tau)\}$ is a rest frame for the worldline $\mathbf{z}(\tau)$ if

$$\mathbf{e}_0(\tau)=\mathbf{v}(\tau)=\dot{\mathbf{z}}(\tau). \tag{41}$$

In this case, (36) and (40) give

$$\dot{\mathbf{e}}_0=\dot{\mathbf{v}}=\mathbf{a}=\Omega_e\cdot\mathbf{e}_0=\boldsymbol{\alpha}. \tag{42}$$

The remaining equations in (36) are then

$$\dot{\mathbf{e}}_m=\Omega_e\cdot\mathbf{e}_m=\mathbf{v}(\mathbf{a}\cdot\mathbf{e}_m)+(\mathbf{v}\wedge\boldsymbol{\beta})^*\cdot\mathbf{e}_m. \tag{43}$$

By (17) and (18), the second term on the right may be written

$$\boldsymbol{\epsilon}(\mathbf{e}_m,\mathbf{v},\boldsymbol{\beta})=\boldsymbol{\epsilon}(\mathbf{v},\boldsymbol{\beta},\mathbf{e}_m)=\boldsymbol{\beta}\times\mathbf{e}_m, \tag{44}$$

as in (19). When the acceleration is zero, $\dot{\mathbf{e}}_m=\boldsymbol{\beta}\times\mathbf{e}_m$, showing that $\boldsymbol{\beta}$ is an angular velocity in the frame with time axis $\mathbf{v}=\mathbf{e}_0$ (a rest frame). To make the notation more suggestive, we write $\boldsymbol{\omega}$ in place of $\boldsymbol{\beta}$ so (39) becomes, for a rotating rest frame,

$$\Omega_e=\mathbf{v}\wedge\mathbf{a}+(\mathbf{v}\wedge\boldsymbol{\omega})^*, \tag{45}$$

with $\boldsymbol{\omega}=-\Omega_e^*\cdot\mathbf{v}$.

V. FERMI–WALKER (NONROTATING) FRAME

The Fermi–Walker (F–W) frame is a particular case of the moving rest frames of the previous section. (A useful reference is the book Gravitation.[9]) It is defined to be a rest frame for the worldline $\mathbf{z}(\tau)$ such that the angular velocity $\boldsymbol{\omega}(\tau)$ of (45) is zero. This accounts for the alternative designation, "nonrotating." If an orthonormal basis changes, a minimum of two vectors must change. The minimum case is therefore one timelike and one spacelike or two spacelike; the former is "nonrotating" (despite one spatial vector changing), the latter is pure rotation in a plane. This is a geometrical (absolute) characterisation of a changing basis. But of course the F–W frame can rotate with respect to other rest frames (a simple example will be given in Sec. IX). In a sense, it is the frame in special relativity which comes closest to playing the role of the Newtonian inertial frames of nonrelativity [compare the simple spin equations (112) and (113)]. With respect to the F–W frame, rotational equations for particles take their simplest form, simpler even than with the geometrical derivatives based on inertial frames, because the worldline determines a privileged rest space and the F–W frame is able to take maximum advantage of it [compare the more complicated spin equations involving radiation reaction (132), (133) with (134)].

The F–W frame is denoted by a special letter, f. The frame $\{\mathbf{f}_\mu\in\mathscr{V}\}$ is defined by the differential equations that express the vanishing of the angular velocity with respect to the rest space:

$$\dot{\mathbf{f}}_\mu=\Omega_f\cdot\mathbf{f}_\mu,\quad \Omega_f=\mathbf{v}\wedge\mathbf{a},\quad \mathbf{f}_0=\mathbf{v}; \tag{46}$$

alternatively, we may write simpler but equivalent equations in which the structure is somewhat concealed,

$$\dot{\mathbf{f}}_0=\mathbf{a},\quad \dot{\mathbf{f}}_m=\mathbf{v}(\mathbf{a}\cdot\mathbf{f}_m). \tag{47}$$

There is very little freedom in the solutions to (46). Given a worldline $\mathbf{z}(\tau)$, $\mathbf{v}(\tau)$, and $\mathbf{a}(\tau)$ are determined. The equations have a unique solution for each choice of initial vectors $\mathbf{f}_0(0)=\mathbf{v}(0)$, $\mathbf{f}_m(0)$. A different choice of initial vectors $\mathbf{f}'_0(0)=\mathbf{v}(0)$, $\mathbf{f}'_m(0)=R_{mn}\mathbf{f}_n(0)$, a spatial rotation of the original choice, leads to the trivially different $\mathbf{f}_0(\tau)=\mathbf{f}_0(\tau)=\mathbf{v}(\tau)$, $\mathbf{f}'_m(\tau)=R_{mn}\mathbf{f}_n(\tau)$ with the same τ-independent rotation.

A complete solution of (46) requires a knowledge of the worldline $\mathbf{z}(\tau)$. But, we can see what condition the equations impose by looking at the infinitesimal change from proper time zero (say) to proper time $\delta\tau$,

$$\delta\mathbf{f}_\mu=\delta\tau(\mathbf{v}\wedge\mathbf{a})\cdot\mathbf{f}_\mu. \tag{48}$$

At $\tau=0$ we have $\mathbf{v}(0)$ and $\mathbf{a}(0)$ and, because of the rotational freedom, we can suppose $\mathbf{a}(0)=a\mathbf{f}_1(0)$. Then, Eqs. (46), at proper time zero, reduce to

$$\dot{\mathbf{f}}_0=a\mathbf{f}_1,\quad \dot{\mathbf{f}}_1=a\mathbf{f}_0,\quad \dot{\mathbf{f}}_2=\dot{\mathbf{f}}_3=0, \tag{49}$$

and, therefore,

$$\begin{aligned}\mathbf{f}_0(\delta\tau)&=\mathbf{f}_0(0)+\delta\tau a\mathbf{f}_1(0)\\ \mathbf{f}_1(\delta\tau)&=\mathbf{f}_1(0)+\delta\tau a\mathbf{f}_0(0)\\ \mathbf{f}_2(\delta\tau)&=\mathbf{f}_2(0),\quad \mathbf{f}_3(\delta\tau)=\mathbf{f}_3(0).\end{aligned} \tag{50}$$

These equations express the change from the frame $\mathbf{f}_\mu(0)$ to a new frame $\mathbf{f}_\mu(\delta\tau)$ moving at (small) relative velocity $\delta\tau a\mathbf{f}_1(0)$ with respect to the former. The fact that two spatial basis vectors remain unchanged identifies the transformation as nonrotating. It is, in fact, just the small V case of (1), which connected "partially parallel" frames. So, over an infinitesimal interval, nonrotating frames are partially parallel. We also describe Eq. (50) as a small active boost (see next section). If we do not arrange the special relation between the acceleration and the F–W frame at $\tau=0$, we have the general case $\mathbf{a}(0)=a^m\mathbf{f}_m(0)$ and then

$$\mathbf{f}_0(\delta\tau)=\mathbf{f}_0(0)+\mathbf{a}(0)\delta\tau, \quad \mathbf{f}_n(\delta\tau)=\mathbf{f}_n(0)+a^n\delta\tau\mathbf{f}_0(0). \tag{51}$$

A solution to (49), regarded as differential equations for all τ, is simple and celebrated. It gives the bases associated with so-called hyperbolic motion. The basis vectors of the frames K and K' of Eq. (1) provide the solution. Writing $\mathbf{f}_\mu$ in place of $\mathbf{E}'_\mu$, we have, according to (12),

$$\begin{aligned}\mathbf{f}_0&=\gamma_V(\mathbf{E}_0+V\mathbf{E}_1)\\ \mathbf{f}_1&=\gamma_V(\mathbf{E}_1+V\mathbf{E}_0)\\ \mathbf{f}_2&=\mathbf{E}_2, \quad \mathbf{f}_3=\mathbf{E}_3, \quad \gamma_V=(1-V^2)^{-1/2}.\end{aligned} \tag{52}$$

Differentiating with respect to τ, we find that Eq. (49) is satisfied if $a=\gamma_V^2\dot{V}$. In this case, (52) provides a nonrotating F–W frame for a worldline accelerating for all τ, but because two spatial axes are fixed throughout it is a very special nonrotating frame.

In general, for an arbitrary worldline, in the absence of an analytic solution of (46) for the F–W frame, one may intuitively picture it as built up in a succession of small steps each of the form (50) or (51).

VI. BOOSTED FRAME

The boosted rest frame arises in a quite different way from the F–W frame. At a given point $\mathbf{z}(\tau)$ on the worldline, when the spacetime velocity is $\mathbf{v}$, the Fermi–Walker frame is determined by the history of an initial rest frame at $\tau=0$ and differential equations that require the frame to be nonrotating along the worldline up to the point. But the boosted rest frame is at each point determined by the relation of the worldline to a special inertial frame K (the lab, say, with associated basis $\mathbf{E}_\mu$). The boosted rest frame $\{\mathbf{b}_\mu(\tau)\in\mathscr{V}, \mathbf{b}_0(\tau)=\mathbf{v}(\tau)\}$ is determined, when $\mathbf{v}=\gamma(\mathbf{E}_0+\mathbf{V})$, to be the frame moving with relative velocity $\mathbf{V}$ with respect to K which is partially parallel to K. This characterization is determined by the relation at one instant between the worldline and the lab frame K, the relevant parameters being $\mathbf{E}_0$ and $\mathbf{V}$, or alternatively, $\mathbf{E}_0$ and $\mathbf{v}$.

If $\mathbf{V}=V\mathbf{E}_1$, so $\mathbf{v}=\gamma(\mathbf{E}_0+V\mathbf{E}_1)$, then the case is exactly the same as for (1) and the boost basis for this point coincides with the F–W basis in (52). The general formulae are obtained by rotating the spatial lab basis vectors $\mathbf{E}_m$ and the boosted spatial basis vectors by the same (subjective) rotation. [That this procedure really implements partial parallelism will be shown presently, after Eq. (59).] For the general case the relative velocity is $\mathbf{V}=V^m\mathbf{E}_m$. We can convert (52) to the general boost by replacing, on the right hand sides of the equations, $\mathbf{E}_m$ by $R_{mn}\mathbf{E}_n$ and on the left hand sides, $\mathbf{f}_m$ by $R_{mn}\mathbf{b}_n$, where R_{mn} is a rotation matrix. If we choose R_{mn} so that $V^m=VR_{1m}$, this gives

$$\begin{aligned}\mathbf{v}=\mathbf{b}_0&=\gamma(\mathbf{E}_0+V^n\mathbf{E}_n)\\ R_{1n}\mathbf{b}_n&=\gamma(R_{1n}\mathbf{E}_n+V\mathbf{E}_0)\\ R_{2n}\mathbf{b}_n&=R_{2n}\mathbf{E}_n, \quad R_{3n}\mathbf{b}_n=R_{3n}\mathbf{E}_n.\end{aligned}$$

Rearranging the spatial equations (using $R_{km}R_{kn}=\delta_{mn}$), we get

$$\mathbf{b}_m=\mathbf{E}_m+\frac{\gamma^2}{\gamma+1}V^mV^n\mathbf{E}_n+\gamma V^m\mathbf{E}_0. \tag{53}$$

The relation between $\mathbf{E}_\mu$ and $\mathbf{b}_\mu$ is most easily handled by using the boost dyadic $\mathsf{B}(\mathbf{E}_0\mapsto\mathbf{v})\in\mathscr{V}$ given by

$$\mathsf{B}=-\mathbf{b}_0\otimes\mathbf{E}_0+\mathbf{b}_m\otimes\mathbf{E}_m=\eta^{\mu\nu}\mathbf{b}_\mu\otimes\mathbf{E}_\nu. \tag{54}$$

The transpose,

$$\mathsf{B}^T=\eta^{\mu\nu}\mathbf{E}_\mu\otimes\mathbf{b}_\nu, \tag{55}$$

satisfies

$$\mathsf{B}\cdot\mathsf{B}^T=\mathsf{B}^T\cdot\mathsf{B}=\eta=\eta^{\mu\nu}\mathbf{b}_\mu\otimes\mathbf{b}_\nu=\eta^{\mu\nu}\mathbf{E}_\mu\otimes\mathbf{E}_\nu, \tag{56}$$

where η is the unit dyadic (16), so the dyadic does transform an orthonormal basis to an orthonormal basis.

The dyadic provides the connections between the two bases in various forms:

$$\mathbf{b}_\mu=\mathsf{B}\cdot\mathbf{E}_\mu=\mathbf{E}_\mu\cdot\mathsf{B}^T \tag{57}$$

and

$$\mathbf{E}_\mu=\mathsf{B}^T\cdot\mathbf{b}_\mu=\mathbf{b}_\mu\cdot\mathsf{B}. \tag{58}$$

Equation (52), with $\mathbf{f}_\mu\to\mathbf{b}_\mu$ is the simplest boost; if we substitute from it in (54) we get

$$\mathsf{B}=\eta-2\mathbf{b}_0\otimes\mathbf{E}_0+\frac{(\mathbf{b}_0+\mathbf{E}_0)\otimes(\mathbf{b}_0+\mathbf{E}_0)}{\gamma+1}. \tag{59}$$

In fact, this is the general form of the dyadic which is true for all boosts. It is easy to see that $\mathsf{B}\cdot\mathbf{E}_0=\mathbf{b}_0$, and if $\mathbf{b}_0=\gamma(\mathbf{E}_0+V^m\mathbf{E}_m)$, we can verify that $\mathsf{B}\cdot\mathbf{E}_m$ reproduces $\mathbf{b}_m$ of (53). The dyadic makes the structure of the boost transformation particularly clear. Only vectors in the plane of $\mathbf{E}_0$ and $\mathbf{v}=\mathbf{b}_0$ are affected by it; those orthogonal to the plane are unchanged. The invariant spatial plane is the characteristic of the boost and verifies its partial parallelism.

A significant advantage of the boosted rest frame $\mathbf{b}_\mu=\mathsf{B}\cdot\mathbf{E}_\mu$ is that it is completely known as soon as $\mathbf{b}_0=\mathbf{v}$ is known. No differential equations have to be solved as for the F–W frame. But the generalized angular velocity Ω_b for the boosted frame is nonetheless of great interest. In fact, the relative angular velocity of the F–W frame with respect to the boost frame provides probably the simplest way of getting an understanding of the motion of the F–W frame. This relative angular velocity proves to be $\Omega_f-\Omega_b$ and will be discussed in the next section.

We can find Ω_b in terms of B from

$$\dot{\mathbf{b}}_\mu=\dot{\mathsf{B}}\cdot\mathbf{E}_\mu=\dot{\mathsf{B}}\cdot\mathsf{B}^T\cdot\mathbf{b}_\mu\equiv\Omega_b\cdot\mathbf{b}_\mu. \tag{60}$$

From (59), with $\mathbf{b}_0=\mathbf{v}$ and $\gamma=-\mathbf{v}\cdot\mathbf{E}_0$,

$$\dot{\mathsf{B}}=-2\mathbf{a}\otimes\mathbf{E}_0+\frac{\mathbf{a}\otimes(\mathbf{v}+\mathbf{E}_0)+(\mathbf{v}+\mathbf{E}_0)\otimes\mathbf{a}}{\gamma+1}+\frac{\mathbf{a}\cdot\mathbf{E}_0}{(\gamma+1)^2}(\mathbf{v}+\mathbf{E}_0)\otimes(\mathbf{v}+\mathbf{E}_0). \tag{61}$$

Writing out the terms in the product, we find

$$\Omega_b=\dot{\mathsf{B}}\cdot\mathsf{B}^T=\frac{(\mathbf{v}+\mathbf{E}_0)\wedge\mathbf{a}}{\gamma+1}, \tag{62}$$

after cancellations.

To express Ω_b in the form (45), first write

$$\Omega_b = \mathbf{v}\wedge\mathbf{a} + \frac{(\mathbf{E}_0 - \gamma\mathbf{v})\wedge\mathbf{a}}{\gamma+1}, \tag{63}$$

in the second term of which both $\mathbf{a}$ and $\mathbf{E}_0 - \gamma\mathbf{v}$ are orthogonal to $\mathbf{v}$. The next step requires a simple formula for the dual (17) of such an antisymmetric dyadic.

If $\mathbf{x}$ and $\mathbf{y}$ are orthogonal to $\mathbf{v}$, we can use a rest frame basis (with $\mathbf{e}_0 = \mathbf{v}$) in (17) to write

$$(\mathbf{x}\wedge\mathbf{y})^* = \mathbf{e}_\mu \otimes \mathbf{e}_\nu \epsilon^{\mu\nu\lambda\rho} x_\lambda y_\rho, \tag{64}$$

where $x_\lambda = \mathbf{e}_\lambda \cdot \mathbf{x}$, $y_\rho = \mathbf{e}_\rho \cdot \mathbf{y}$ with $x_0 = y_0 = 0$. Then, because $\epsilon^{\mu\nu\lambda\rho}$ is completely antisymmetric,

$$\begin{aligned}(\mathbf{x}\wedge\mathbf{y})^* &= \mathbf{e}_0 \otimes \mathbf{e}_n \epsilon^{0nlr} x_l y_r + \mathbf{e}_m \otimes \mathbf{e}_0 \epsilon^{m0lr} x_l y_r \\ &= \mathbf{v}\wedge\boldsymbol{\epsilon}(\mathbf{v},\mathbf{x},\mathbf{y})\end{aligned} \tag{65}$$

in terms of the definition (18). Also, since $** = -1$, taking the dual gives

$$\mathbf{x}\wedge\mathbf{y} = -(\mathbf{v}\wedge\boldsymbol{\epsilon}(\mathbf{v},\mathbf{x},\mathbf{y}))^*. \tag{66}$$

Using (66), (63) becomes

$$\Omega_b = \mathbf{v}\wedge\mathbf{a} + \frac{(\mathbf{v}\wedge\boldsymbol{\epsilon}(\mathbf{v},\mathbf{a},\mathbf{E}_0 - \gamma\mathbf{v}))^*}{\gamma+1}, \tag{67}$$

or

$$\Omega_b = \mathbf{v}\wedge\mathbf{a} + (\mathbf{v}\wedge\boldsymbol{\omega}_b)^*, \tag{68}$$

with

$$\boldsymbol{\omega}_b = \frac{\boldsymbol{\epsilon}(\mathbf{v},\mathbf{a},\mathbf{E}_0)}{\gamma+1}. \tag{69}$$

The vector $\boldsymbol{\omega}_b$ is orthogonal both to $\mathbf{v}$ and to $\mathbf{E}_0$; it is spatial both in the lab and in the rest frame. Using (29) and (30), it can be written in terms of relative velocity and relative acceleration with respect to the lab:

$$\boldsymbol{\omega}_b = \frac{\gamma^3}{\gamma+1}\boldsymbol{\epsilon}(\mathbf{E}_0,\mathbf{V},\mathbf{A}). \tag{70}$$

The right-hand side is more familiar as a cross product of spatial vectors in the lab, $\gamma^3\mathbf{V}\times\mathbf{A}/(\gamma+1)$. This form gives the standard angular velocity of Thomas precession of a lab representative of the F–W frame [see (99)].

VII. RELATIVE ANGULAR VELOCITY

Any vector in $\mathscr{V}$ can be expanded in any basis, in an inertial basis $\mathbf{E}_\mu$ or a moving basis $\mathbf{e}_\mu(\tau)$. For some τ-dependent vector $\mathbf{W}(\tau)$ typical expansions are

$$\mathbf{W}(\tau) = W_E^\mu(\tau)\mathbf{E}_\mu = W_e^\mu(\tau)\mathbf{e}_\mu(\tau), \tag{71}$$

where the subscripts on the components refer to the relevant basis

$$W_E^\mu(\tau) = \eta^{\mu\nu}\mathbf{E}_\nu \cdot \mathbf{W}(\tau), \quad W_e^\mu(\tau) = \eta^{\mu\nu}\mathbf{e}_\nu \cdot \mathbf{W}(\tau). \tag{72}$$

The ("absolute") derivative of $\mathbf{W}(\tau)$ with respect to τ, which we have been using up till now without comment, is defined by

$$\frac{d\mathbf{W}(\tau)}{d\tau} = \lim_{\delta\to 0}\frac{\mathbf{W}(\tau+\delta) - \mathbf{W}(\tau)}{\delta}. \tag{73}$$

In terms of the expansion in an inertial basis,

$$\frac{d\mathbf{W}(\tau)}{d\tau} = \frac{dW_E^\mu(\tau)}{d\tau}\mathbf{E}_\mu, \tag{74}$$

but using a moving basis,

$$\frac{d\mathbf{W}(\tau)}{d\tau} = \frac{dW_e^\mu(\tau)}{d\tau}\mathbf{e}_\mu + W_e^\mu\frac{d\mathbf{e}_\mu}{d\tau} = \frac{dW_e^\mu(\tau)}{d\tau}\mathbf{e}_\mu + \Omega_e \cdot \mathbf{W}, \tag{75}$$

in the notation of (36).

It is useful to have available a new concept, a relative derivative of vectors with respect to a moving basis. The idea is the same as is used with rotating frames of reference in three-dimensional dynamics—see, for example, Woodhouse.[10] For the basis $\mathbf{e}_\mu$, the relative derivative is defined by

$$\frac{d^e\mathbf{W}}{d\tau} = \frac{dW_e^\mu}{d\tau}\mathbf{e}_\mu; \tag{76}$$

only the *components* in the $\mathbf{e}_\mu$-decomposition are differentiated. A superscript on the d indicates the new derivative. Then, from (75),

$$\frac{d\mathbf{W}}{d\tau} = \frac{d^e\mathbf{W}}{d\tau} + \Omega_e \cdot \mathbf{W}. \tag{77}$$

For the derivative of a scalar product, $\mathbf{A}\cdot\mathbf{B} = A_e^\mu \eta_{\mu\nu} B_e^\nu$, we have

$$\frac{d}{d\tau}(\mathbf{A}\cdot\mathbf{B}) = \frac{d\mathbf{A}}{d\tau}\cdot\mathbf{B} + \mathbf{A}\cdot\frac{d\mathbf{B}}{d\tau} = \frac{d^e\mathbf{A}}{d\tau}\cdot\mathbf{B} + \mathbf{A}\cdot\frac{d^e\mathbf{B}}{d\tau}, \tag{78}$$

the last form because of the antisymmetry of Ω_e.

If the relative derivative of a dyadic $\mathsf{T} = \mathbf{e}_\mu \otimes \mathbf{e}_\nu \mathsf{T}_e^{\mu\nu}$ is defined similarly,

$$\frac{d^e\mathsf{T}}{d\tau} = \mathbf{e}_\mu \otimes \mathbf{e}_\nu \frac{d\mathsf{T}_e^{\mu\nu}}{d\tau}, \tag{79}$$

then

$$\frac{d\mathsf{T}}{d\tau} = \frac{d^e\mathsf{T}}{d\tau} + \Omega_e \cdot \mathsf{T} - \mathsf{T}\cdot\Omega_e. \tag{80}$$

From these definitions, if $\mathbf{A} = \mathsf{T}\cdot\mathbf{B}$, then not only $\dot{\mathbf{A}} = \dot{\mathsf{T}}\cdot\mathbf{B} + \mathsf{T}\cdot\dot{\mathbf{B}}$, but also

$$\frac{d^e\mathbf{A}}{d\tau} = \frac{d^e\mathsf{T}}{d\tau}\cdot\mathbf{B} + \mathsf{T}\cdot\frac{d^e\mathbf{B}}{d\tau}. \tag{81}$$

With two moving bases to consider, $\mathbf{e}_\mu$ and $\mathbf{g}_\mu$, say, then from (77) for each basis,

$$\frac{d^g\mathbf{W}}{d\tau} = \frac{d^e\mathbf{W}}{d\tau} + (\Omega_e - \Omega_g)\cdot\mathbf{W}. \tag{82}$$

We call

$$\Omega_{e|g} \equiv \Omega_e - \Omega_g \tag{83}$$

the relative angular velocity of the $\mathbf{e}_\mu$ basis with respect to the $\mathbf{g}_\mu$ basis. It appears in the generalisation of (36)

$$\frac{d^g\mathbf{e}_\mu}{d\tau} = \Omega_{e|g}\cdot\mathbf{e}_\mu, \tag{84}$$

which is just (82) applied to $\mathbf{W} = \mathbf{e}_\mu$.

The relative angular velocity of the $\mathbf{g}_\mu$ basis with respect to the $\mathbf{e}_\mu$ basis has the opposite sign:

$$\Omega_{g|e} = \Omega_g - \Omega_e = -\Omega_{e|g}. \tag{85}$$

Note also that

$$\Omega_e = \Omega_g + \Omega_{e|g} \tag{86}$$

and

$$\Omega_{e|g} = \Omega_{e|h} + \Omega_{h|g}. \tag{87}$$

When both moving frames are rest frames, the relative angular velocity takes a simple form

$$\Omega_{e|g} = (\mathbf{v}\wedge\boldsymbol{\omega}_e)^* - (\mathbf{v}\wedge\boldsymbol{\omega}_g)^* = (\mathbf{v}\wedge(\boldsymbol{\omega}_e - \boldsymbol{\omega}_g))^* \tag{88}$$

since the $\mathbf{v}\wedge\mathbf{a}$ terms of (45) cancel. The relative motion of the two rest frames is then described by the rest-spatial vector

$$\boldsymbol{\omega}_{e|g} \equiv \boldsymbol{\omega}_e - \boldsymbol{\omega}_g. \tag{89}$$

As an immediate application of (88), the relative angular velocity of the boosted frame $\mathbf{b}_\mu$ with respect to the F–W frame $\mathbf{f}_\mu$ is, from (68) and (46),

$$\Omega_{b|f} = (\mathbf{v}\wedge\boldsymbol{\omega}_b)^* \tag{90}$$

with

$$\boldsymbol{\omega}_b = \boldsymbol{\omega}_{b|f} = \boldsymbol{\epsilon}(\mathbf{v},\mathbf{a},\mathbf{E}_0)/(\gamma+1). \tag{91}$$

But since the $\mathbf{b}_\mu$ frame is completely known, it is perhaps more valuable to emphasise the motion of the F–W frame with respect to it, given by the vector $\boldsymbol{\omega}_{f|b} = -\boldsymbol{\omega}_b$.

VIII. UNBOOSTED FERMI–WALKER FRAME

It can be helpful to introduce a rotating frame in the lab which serves as a representative for the F–W rest frame. The advantages of the new frame depend on some simple properties.

First the definition. In parallel with $\mathbf{E}_\mu = \mathsf{B}^T\cdot\mathbf{b}_\mu$, which could be said to "unboost" the boosted frame back to the lab frame, we define here a frame which is "unboosted" back to the lab from the Fermi–Walker frame

$$\mathbf{F}_\mu(\tau) \equiv \mathsf{B}^T\cdot\mathbf{f}_\mu(\tau). \tag{92}$$

Because $\mathbf{f}_0 = \mathbf{b}_0 = \mathbf{v}$, the time basis vector of the new frame is the same as the lab's $\mathbf{F}_0 = \mathbf{E}_0$, and this is why the new frame can be said to be rotating in the lab. Its rate of change can be specified completely by a lab-spatial angular velocity vector.

From $\Omega_b = \dot{\mathsf{B}}\cdot\mathsf{B}^T$ and the antisymmetry of Ω_b, we get

$$\dot{\mathsf{B}}^T = -\mathsf{B}^T\cdot\Omega_b, \tag{93}$$

so that, differentiating Eq. (92) and using Eq. (90) with an identity proved below [Eq. (97)],

$$\begin{aligned}\dot{\mathbf{F}}_\mu &= \mathsf{B}^T\cdot(\Omega_f - \Omega_b)\cdot\mathbf{f}_\mu = -\mathsf{B}^T\cdot(\mathbf{v}\wedge\boldsymbol{\omega}_b)^*\cdot\mathsf{B}\cdot\mathbf{F}_\mu \\ &= -(\mathbf{E}_0\wedge\boldsymbol{\omega}_b)^*\cdot\mathbf{F}_\mu.\end{aligned} \tag{94}$$

The last step in (94) required a simple identity. For any two vectors $\mathbf{x}$ and $\mathbf{y}$, the dyadic $(\mathbf{x}\wedge\mathbf{y})^*$, written with the $\mathbf{b}_\mu$-basis, is

$$(\mathbf{x}\wedge\mathbf{y})^* = \mathbf{b}_\mu\otimes\mathbf{b}_\nu\epsilon^{\mu\nu\lambda\rho}(\mathbf{x}\cdot\mathbf{b}_\lambda)(\mathbf{y}\cdot\mathbf{b}_\rho). \tag{95}$$

Then,

$$\mathsf{B}^T\cdot(\mathbf{x}\wedge\mathbf{y})^*\cdot\mathsf{B} = \mathbf{E}_\mu\otimes\mathbf{E}_\nu\epsilon^{\mu\nu\lambda\rho}(\mathbf{x}\cdot\mathbf{b}_\lambda)(\mathbf{y}\cdot\mathbf{b}_\rho). \tag{96}$$

But, $\mathbf{x}\cdot\mathbf{b}_\lambda = \mathbf{x}\cdot\mathsf{B}\cdot\mathbf{E}_\lambda = (\mathsf{B}^T\cdot\mathbf{x})\cdot\mathbf{E}_\lambda$ is the $\mathbf{E}_\lambda$-component of $\mathsf{B}^T\cdot\mathbf{x}$. So,

$$\mathsf{B}^T\cdot(\mathbf{x}\wedge\mathbf{y})^*\cdot\mathsf{B} = ((\mathsf{B}^T\cdot\mathbf{x})\wedge(\mathsf{B}^T\cdot\mathbf{y}))^*. \tag{97}$$

In addition, for (94), $\mathsf{B}^T\cdot\mathbf{v} = \mathbf{E}_0$, $\mathsf{B}^T\cdot\boldsymbol{\omega}_b = \boldsymbol{\omega}_b$, the last because $\boldsymbol{\omega}_b$ is orthogonal to both $\mathbf{v}$ and $\mathbf{E}_0$.

According to (94) and (70) the $\mathbf{F}_\mu$-frame rotates in the lab with angular velocity (measured with respect to τ)

$$-\boldsymbol{\omega}_b = \frac{\gamma^3}{\gamma+1}\,\boldsymbol{\epsilon}(\mathbf{E}_0,\mathbf{A},\mathbf{V}). \tag{98}$$

This angular velocity, reckoned with respect to lab time instead of proper time, is the standard rate of Thomas precession

$$\boldsymbol{\omega}_T = \frac{\gamma^2}{\gamma+1}\,\mathbf{A}\times\mathbf{V}. \tag{99}$$

It is the rate of rotation, with respect to the lab, of a frame that serves as the lab representative of the *nonrotating* F–W rest frame. The frame $\mathbf{F}_\mu$ is a representative of the F–W frame by means of the unboosting operation, a process which connects partially parallel frames (two spatial directions unchanged). This is the best that can be achieved; parallelism is impossible (three spatial directions cannot remain invariant if an orthonormal basis changes). Hence, although the F–W frame does not rotate, the $\mathbf{F}_\mu$ frame does.

The problem of finding the $\mathbf{F}_\mu$-frame is the problem of integrating Eq. (94). Because $\mathbf{F}_0 = \mathbf{E}_0$, it is a three-dimensional problem; in terms of lab time T,

$$\frac{d\mathbf{F}_m}{dT} = \boldsymbol{\omega}_T\times\mathbf{F}_m. \tag{100}$$

When this is solved, the F–W frame is found by reversing Eq. (92):

$$\mathbf{f}_\mu = \mathsf{B}\cdot\mathbf{F}_\mu. \tag{101}$$

The second advantage provided by the new frame is best explained in the context of the example in which it is most used, classical spin.[1,3]

The intrinsic spin of an elementary point particle is modelled as a vector $\mathbf{S}\in\mathscr{V}$ which is orthogonal to the particle's spacetime velocity:

$$\mathbf{S}\cdot\mathbf{v} = 0. \tag{102}$$

Therefore, $\mathbf{S}$ is spatial in any rest frame. There are several relevant expansions

$$\mathbf{S} = S_E^\mu\mathbf{E}_\mu = S_b^m\mathbf{b}_m = S_f^m\mathbf{f}_m. \tag{103}$$

The lab frame $\mathbf{E}_\mu$ is relevant just because it is the most accessible frame; it is, after all, the lab. But the components S_E^μ are not independent because of (102). The F–W frame, $\mathbf{f}_\mu$, in which there is no redundant timelike component, is important because the equations of motion take by far their simplest form in it. (The equations will be discussed in Sec. IX.) The boost frame has no timelike component, and it is calculable (by $\mathbf{b}_\mu = \mathsf{B}\cdot\mathbf{E}_\mu$), but the equations are not in their conceptually simplest form (the Thomas precession intervenes).

A way of combining the advantages of the different frames is to introduce a lab frame representative for the spin vector $\mathbf{S}$, defined, like (92), by unboosting the spin

$$\boldsymbol{\sigma}(\tau) = \mathsf{B}^T\cdot\mathbf{S}(\tau). \tag{104}$$

(The notation $\boldsymbol{\sigma}$ is chosen because of the similarity with the quantum mechanical spin vector.) Using (103) we get two important decompositions for $\boldsymbol{\sigma}$,

$$\sigma(\tau)=S_b^m(\tau)\mathbf{E}_m=S_f^m(\tau)\mathbf{F}_m(\tau). \tag{105}$$

Exactly the same components appear in the decomposition of σ with respect to the inertial lab basis $\mathbf{E}_m$ as appear in the decomposition of $\mathbf{S}$ with respect to the boost basis $\mathbf{b}_m$. Similarly, the $\mathbf{F}_\mu$-components of σ are the $\mathbf{f}_\mu$-components of $\mathbf{S}$.

From (105), we see that the derivative of σ is

$$\frac{d\sigma}{d\tau}=\frac{dS_b^m}{d\tau}\mathbf{E}_m=\mathsf{B}^T\cdot\frac{d^b\mathbf{S}}{d\tau}, \tag{106}$$

the lab representative of the derivative of $\mathbf{S}$ with respect to the $\mathbf{b}_\mu$-basis. A rather complicated derivative in the rest frame is made simple in terms of the representative in the lab. Again from (105),

$$\frac{d\sigma}{d\tau}=\dot{S}_f^m\mathbf{F}_m+S_f^m\dot{\mathbf{F}}_m=\mathsf{B}^T\cdot\frac{d^f\mathbf{S}}{d\tau}+\epsilon(\mathbf{E}_0,-\omega_b,\sigma). \tag{107}$$

The right-hand sides of the last two equations are just the results of applying B^T to each side of

$$\frac{d^b\mathbf{S}}{d\tau}=\frac{d^f\mathbf{S}}{d\tau}+(\Omega_f-\Omega_b)\cdot\mathbf{S}. \tag{108}$$

Such equations are very awkward indeed to write in a formalism in which vectors are regarded as collections of components (printed, usually, as single bold face symbols, with no indication of the relevant basis). In such a formalism a τ-derivative is automatically the derivative with respect to the relevant basis. This is an advantage which is recovered [as in (106)] by using the idea of lab representative.

IX. SPIN EQUATIONS FOR A MAGNETIC DIPOLE IN AN EXTERNAL FIELD

We consider in this section various forms of the relativistic spin equations for a point particle with mass m, charge e, spin $\mathbf{S}$ and magnetic dipole moment

$$\mu=\kappa\mathbf{S},\quad \kappa=\frac{ge}{2m}. \tag{109}$$

The equations were developed by Thomas[1] and Frenkel[11] immediately after the idea of spin was proposed. An excellent textbook treatment is provided by Jackson.[3]

Only the effect of an external electromagnetic field F will be taken into account here. The field can be decomposed into lab frame electric and magnetic fields

$$\mathsf{F}=\mathbf{E}_0\wedge\mathbf{E}_L-(\mathbf{E}_0\wedge\mathbf{B}_L)^*, \tag{110}$$

where $\mathbf{E}_0\cdot\mathbf{E}_L=\mathbf{E}_0\cdot\mathbf{B}_L=0$, or into *rest* frame electric and magnetic fields

$$\mathsf{F}=\mathbf{v}\wedge\mathbf{E}-(\mathbf{v}\wedge\mathbf{B})^*, \tag{111}$$

where $\mathbf{v}\cdot\mathbf{E}=\mathbf{v}\cdot\mathbf{B}=0$. (The notations nearly clash, but not quite: basis vectors $\mathbf{E}_0$, $\mathbf{E}_m$, $\mathbf{E}_\mu$, lab electric field $\mathbf{E}_L$, rest frame electric field $\mathbf{E}$.) The rest frame electric field depends only on $\mathbf{v}$, not on the specific rest basis.

The nonrelativistic equations for the spin of a particle at rest in a lab magnetic field are (using the notations of a relativistic rest frame as if it were a nonrelativistic inertial frame)

$$\frac{d\mathbf{S}}{d\tau}=\mu\times\mathbf{B}_L. \tag{112}$$

The relativistic generalisation, for a particle in arbitrary motion, is most succinctly put in terms of the derivative of $\mathbf{S}$ with respect to the "nonrotating" F–W rest frame,

$$\frac{d^f\mathbf{S}}{d\tau}=\epsilon(\mathbf{v},\mu,\mathbf{B}). \tag{113}$$

The right-hand side is the vector cross product in the rest frame of μ and the rest frame magnetic field. If the particle is stationary in the lab, $\mathbf{v}=\mathbf{E}_0$, and (113) reduces to the nonrelativistic form (112).

Although (113) is conceptually the simplest equation at the present level, it does presuppose a knowledge of the F–W frame. To remove from the equation its dependence on the special F–W frame we may use

$$\frac{d\mathbf{S}}{d\tau}=\frac{d^f\mathbf{S}}{d\tau}+\Omega_f\cdot\mathbf{S}=\frac{d^f\mathbf{S}}{d\tau}+(\mathbf{v}\wedge\mathbf{a})\cdot\mathbf{S}, \tag{114}$$

to get, in view of $\mathbf{v}\cdot\mathbf{S}=0$,

$$\frac{d\mathbf{S}}{d\tau}-\mathbf{v}(\mathbf{a}\cdot\mathbf{S})=\epsilon(\mathbf{v},\mu,\mathbf{B}). \tag{115}$$

Here, the τ-derivative is absolute (geometrical) and all vectors are, of course, basis-independent. It is worth noting that there is no sign of a Thomas precession either in (113) or in (115): it comes in only when the boost basis is used or the lab representative σ for the spin.

The right-hand side of (115) may be expressed in terms of the electromagnetic field F instead of the rest frame magnetic field $\mathbf{B}$. To do that, note in (111) that $\mathbf{E}=\mathsf{F}\cdot\mathbf{v}$, so

$$\begin{aligned}\mathsf{F}\cdot\mu&=(\mathbf{v}\wedge(\mathsf{F}\cdot\mathbf{v}))\cdot\mu-(\mathbf{v}\wedge\mathbf{B})^*\cdot\mu\\&=\mathbf{v}(\mu\cdot\mathsf{F}\cdot\mathbf{v})+\epsilon(\mathbf{v},\mu,\mathbf{B}).\end{aligned} \tag{116}$$

Therefore,

$$\frac{d\mathbf{S}}{d\tau}-\mathbf{v}(\mathbf{a}\cdot\mathbf{S})=\mathsf{F}\cdot\mu+\mathbf{v}(\mathbf{v}\cdot\mathsf{F}\cdot\mu). \tag{117}$$

Approximating $m\mathbf{a}$ with the Lorentz force $e\mathsf{F}\cdot\mathbf{v}$ gives the famous Bargmann–Michel–Telegdi (BMT) version[12] of the equation:

$$\frac{d\mathbf{S}}{d\tau}=\frac{e}{m}\left[\frac{g}{2}\mathsf{F}\cdot\mathbf{S}+\left(\frac{g}{2}-1\right)\mathbf{v}(\mathbf{v}\cdot\mathsf{F}\cdot\mathbf{S})\right]. \tag{118}$$

Equation (118) is used to get information about the rate of transfer of longitudinal polarization to transverse polarisation; this procedure avoids having to find redundant components of $\mathbf{S}$ in a decomposition in an inertial basis. Equation (113), although elegant, requires a knowledge of the F–W frame, which is not in general easily available.

To get an equation for the spin's lab representative σ, first rewrite (113) in terms of a derivative with respect to the boost frame [as in (82)]

$$\frac{d^b\mathbf{S}}{d\tau}=\epsilon(\mathbf{v},\mu,\mathbf{B})-(\mathbf{v}\wedge\omega_b)^*\cdot\mathbf{S}=\kappa\epsilon\left(\mathbf{v},\mathbf{S},\mathbf{B}+\frac{1}{\kappa}\omega_b\right). \tag{119}$$

We see here the effect of using a frame which is rotating (with respect to the F–W frame).

Because the boosted frame is completely known, we can unboost (119), as in (104) and (106), to get an equation for the lab representative $\sigma=\mathsf{B}^T\cdot\mathbf{S}$,

$$\frac{d\boldsymbol{\sigma}}{d\tau}=\kappa \mathsf{B}^T\cdot\boldsymbol{\epsilon}\left(\mathbf{v},\mathbf{S},\mathbf{B}+\frac{1}{\kappa}\,\boldsymbol{\omega}_b\right). \tag{120}$$

The right-hand side can be dealt with the way (97) was derived:

$$\frac{d\boldsymbol{\sigma}}{d\tau}=\kappa\boldsymbol{\epsilon}\left(\mathbf{E}_0,\boldsymbol{\sigma},\mathsf{B}^T\cdot\mathbf{B}+\frac{1}{\kappa}\,\boldsymbol{\omega}_b\right). \tag{121}$$

Only one problem now remains: unboosting the rest frame magnetic field **B**, and re-expressing it in terms of lab fields.

From Eqs. (110) and (111), we can find the rest frame **B** in terms of lab fields $\mathbf{E}_L$ and $\mathbf{B}_L$,

$$\mathbf{B}=\mathsf{F}^*\cdot\mathbf{v}=(\mathbf{E}_0\wedge\mathbf{E}_L)^*\cdot\mathbf{v}+(\mathbf{E}_0\wedge\mathbf{B}_L)\cdot\mathbf{v}. \tag{122}$$

Recalling the formula (59) for B, and $\mathbf{b}_0=\mathbf{v}=\gamma(\mathbf{E}_0+\mathbf{V})$,

$$\mathsf{B}^T\cdot\mathbf{B}=\left(\eta+\frac{\mathbf{v}+\mathbf{E}_0}{\gamma+1}\,\mathbf{E}_0\right)\cdot\mathbf{B}=\mathbf{B}-\frac{\mathbf{v}+\mathbf{E}_0}{\gamma+1}\,\gamma\mathbf{V}\cdot\mathbf{B}_L$$
$$=-\gamma\boldsymbol{\epsilon}(\mathbf{E}_0,\mathbf{V},\mathbf{E}_L)+\gamma\mathbf{B}_L-\frac{\gamma^2}{\gamma+1}\,\mathbf{V}(\mathbf{V}\cdot\mathbf{B}_L). \tag{123}$$

Inserting this in (121), and using the cross product notation for lab-spatial vectors, we get

$$\frac{d\boldsymbol{\sigma}}{d\tau}=\gamma\kappa\boldsymbol{\sigma}\times\left[\mathbf{B}_L-\mathbf{V}\times\mathbf{E}_L-\frac{\gamma}{\gamma+1}\,\mathbf{V}(\mathbf{V}\cdot\mathbf{B}_L)+\frac{1}{\kappa}\,\frac{\gamma^2}{\gamma+1}\,\mathbf{V}\times\mathbf{A}\right]. \tag{124}$$

The equation for the spin representative $\boldsymbol{\sigma}$ has in (124) reached almost its most practical form. All that remains is to convert the proper time derivative $\dot{\boldsymbol{\sigma}}$ to a lab time derivative $\gamma d\boldsymbol{\sigma}/dT$ and to express the relative acceleration **A** in terms of lab fields $\mathbf{E}_L$ and $\mathbf{B}_L$. Assuming, as for the BMT Eq. (118), that the translational motion of the particle is given sufficiently accurately by the Lorentz force, then $m\mathbf{a}=e\mathsf{F}\cdot\mathbf{v}$, or in terms of lab frame vectors,

$$m(\dot\gamma[\mathbf{E}_0+\mathbf{V}]+\gamma^2\mathbf{A})=e[\mathbf{E}_0\wedge\mathbf{E}_L-(\mathbf{E}_0\wedge\mathbf{B}_L)^*]\cdot\gamma(\mathbf{E}_0+\mathbf{V}), \tag{125}$$

from which, since $\dot\gamma=-\mathbf{a}\cdot\mathbf{E}_0=-e\mathbf{E}_0\cdot\mathsf{F}\cdot\mathbf{v}/m$,

$$m\gamma\mathbf{A}=e(\mathbf{E}_L-\mathbf{V}(\mathbf{V}\cdot\mathbf{E}_L)+\boldsymbol{\epsilon}(\mathbf{E}_0,\mathbf{V},\mathbf{B}_L)). \tag{126}$$

When this is substituted in (124), we get

$$\frac{d\boldsymbol{\sigma}}{dT}=\frac{e}{m}\,\boldsymbol{\sigma}\times\left\{\left(\frac{g}{2}-1+\frac{1}{\gamma}\right)\mathbf{B}_L-\left(\frac{g}{2}-\frac{\gamma}{\gamma+1}\right)\mathbf{V}\times\mathbf{E}_L-\frac{\gamma}{\gamma+1}\left(\frac{g}{2}-1\right)\mathbf{V}(\mathbf{V}\cdot\mathbf{B}_L)\right\}, \tag{127}$$

which is Thomas's equation.[1]

To compare with the practical Eq. (127) for $\boldsymbol{\sigma}$, the simplest theoretical equation for this vector is, from (107) and (113),

$$\frac{d^F\boldsymbol{\sigma}}{d\tau}=\mathsf{B}^T\cdot\boldsymbol{\epsilon}(\mathbf{v},\boldsymbol{\mu},\mathbf{B})=\kappa\boldsymbol{\epsilon}(\mathbf{E}_0,\boldsymbol{\sigma},\mathsf{B}^T\cdot\mathbf{B}). \tag{128}$$

The unboosted F–W frame plays the role in the lab of the F–W frame itself in the rest space.

However convenient for calculations (127) may be, it must be remembered that its solution is not the spin vector **S**, but its representative $\boldsymbol{\sigma}$. The former can be generated from the latter by the inverse of (104)

$$\mathbf{S}=\mathsf{B}\cdot\boldsymbol{\sigma} \tag{129}$$

either in terms of boost frame components

$$\mathbf{S}=\mathsf{B}\cdot(\sigma^m\mathbf{E}_m)=\sigma^m\mathbf{b}_m, \tag{130}$$

or by using the explicit form (59) for B to get

$$\mathbf{S}=\boldsymbol{\sigma}+\frac{\gamma^2\mathbf{V}(\boldsymbol{\sigma}\cdot\mathbf{V})}{\gamma+1}+\mathbf{E}_0(\gamma\boldsymbol{\sigma}\cdot\mathbf{V}). \tag{131}$$

The spin equations of the present section have all been various versions of the theoretically simple equation (113), the equation for the case of an external electromagnetic field. If radiation reaction is taken into account the theory is much more complicated. An equation for this case was derived[13] as a generalization of (115):

$$\frac{d\mathbf{S}}{d\tau}-\mathbf{v}(\mathbf{a}\cdot\mathbf{S})=\boldsymbol{\epsilon}(\mathbf{v},\boldsymbol{\mu},[\mathbf{B}+\mathbf{b}]), \tag{132}$$

where

$$\mathbf{b}=\frac{2}{3}\,\frac{d^3\boldsymbol{\mu}}{d\tau^3}-\frac{2}{3}\,\mathbf{a}\cdot\mathbf{a}\,\frac{d\boldsymbol{\mu}}{d\tau}-\frac{2}{3}\,\mathbf{a}\cdot\boldsymbol{\mu}\,\frac{d\mathbf{a}}{d\tau}-2\left(\mathbf{a}\cdot\frac{d\boldsymbol{\mu}}{d\tau}\right)\mathbf{a}-\frac{4}{3}\left(\frac{d\mathbf{a}}{d\tau}\cdot\boldsymbol{\mu}\right)\mathbf{a}. \tag{133}$$

Equivalent equations in yet more complicated notations (as measured by their lengths) have been derived by Bhabha[14] and by van Weert.[15] If the proper time derivatives are evaluated in terms of derivatives with respect to the F–W frame the equation simplifies considerably. With $\mathbf{S}=S_f^m\mathbf{f}_m$ and $\boldsymbol{\mu}=\mu_f^m\mathbf{f}_m$, one gets

$$\frac{d^f\mathbf{S}}{d\tau}=\frac{dS_f^m}{d\tau}\,\mathbf{f}_m=\boldsymbol{\epsilon}\left(\mathbf{v},\boldsymbol{\mu},\left[\mathbf{B}+\frac{2}{3}\,\frac{d^3\mu_f^m}{d\tau^3}\,\mathbf{f}_m-\frac{2}{3}\,(\mathbf{a}\cdot\mathbf{a})\,\frac{d\mu_f^m}{d\tau}\,\mathbf{f}_m\right]\right). \tag{134}$$

The simplification highlights the significance of the F–W frame.

X. SPINNING ELECTRON IN A UNIFORM MAGNETIC FIELD

The spinning electron in a uniform magnetic field is an attractive example in which to study various systems of basis vectors. The example is completely solvable and, being a sanitised version of several $g-2$ experiments, it is physically relevant. (More realistic experimental details are described in experimental review papers).[12,16,17]

In the absence of an electric field in the lab, the electromagnetic field dyadic F for a constant and uniform lab magnetic field, directed along the lab z axis, is

$$\mathsf{F}=-(\mathbf{E}_0\wedge\mathbf{B}_L)^*=-B_L(\mathbf{E}_0\wedge\mathbf{E}_3)^*=B_L\mathbf{E}_1\wedge\mathbf{E}_2. \tag{135}$$

The translational equation of motion for an electron (mass m, charge $e<0$) in such a field, assuming that the effects of radiation reaction, spin, and quantum mechanics may all be neglected, is once again taken to be

$$m\mathbf{a}=e\mathsf{F}\cdot\mathbf{v}. \tag{136}$$

Decomposing with respect to the lab's time axis, $\mathbf{v}=\gamma(\mathbf{E}_0+\mathbf{V})$, and (136) becomes

$$m\dot{\gamma}(\mathbf{E}_0+\mathbf{V})+m\gamma^2\mathbf{A}=e\gamma B_L\boldsymbol{\epsilon}(\mathbf{E}_0,\mathbf{V},\mathbf{E}_3). \tag{137}$$

A solution of (137) requires $\dot{\gamma}=0$ and

$$\mathbf{A}=-\frac{eB_L}{\gamma m}\boldsymbol{\epsilon}(\mathbf{E}_0,\mathbf{E}_3,\mathbf{V}). \tag{138}$$

The simplest solution of (138) is given by uniform motion on a circle in a plane parallel to the xy plane of the lab. Take the center of the circle as spatial origin. As a displacement from the spacetime origin the electron's worldline is

$$\mathbf{z}=T\mathbf{E}_0+\mathbf{R}(T), \quad \mathbf{V}=d\mathbf{R}/dT. \tag{139}$$

For uniform anticlockwise circular motion about $\mathbf{E}_3$ with lab angular velocity Ω (the scalar Ω is lab angle with respect to lab time),

$$\mathbf{V}=\Omega\boldsymbol{\epsilon}(\mathbf{E}_0,\mathbf{E}_3,\mathbf{R}). \tag{140}$$

This gives a solution of (136) if

$$\Omega=-\frac{eB_L}{m\gamma}\ (>0). \tag{141}$$

From (139),

$$\mathbf{v}=\frac{d\mathbf{z}}{d\tau}=\frac{dT}{d\tau}(\mathbf{E}_0+\mathbf{V}), \tag{142}$$

and $\mathbf{v}\cdot\mathbf{v}=-1$ if

$$\frac{dT}{d\tau}\equiv\gamma=(1-V^2)^{-1/2}, \quad V^2=\Omega^2R^2. \tag{143}$$

Then,

$$\mathbf{a}=\gamma^2\mathbf{A}=\gamma^2\Omega\boldsymbol{\epsilon}(\mathbf{E}_0,\mathbf{E}_3,\mathbf{V}). \tag{144}$$

The worldline is completely solvable in this case because of the simple external field and the neglect, in (136), of a number of (thankfully small) dynamical complications.

With the electron traveling uniformly on a lab circle at angular velocity Ω, it is useful to introduce a co-rotating basis $\mathbf{R}_\mu$:

$$\begin{aligned}
&\mathbf{R}_1=\cos\Omega T\mathbf{E}_1+\sin\Omega T\mathbf{E}_2\\
&\mathbf{R}_2=-\sin\Omega T\mathbf{E}_1+\cos\Omega T\mathbf{E}_2\\
&\mathbf{R}_3=\mathbf{E}_3, \quad \mathbf{R}_0=\mathbf{E}_0.
\end{aligned} \tag{145}$$

Assuming the electron is on the positive x axis at $T=0$, then for all T it remains on the $\mathbf{R}_1$ axis and

$$\mathbf{v}=\gamma(\mathbf{R}_0+V\mathbf{R}_2) \tag{146}$$

$$\mathbf{a}=-\Omega V\gamma^2\mathbf{R}_1. \tag{147}$$

The dyadic $\mathsf{R}=\eta^{\mu\nu}\mathbf{R}_\mu\otimes\mathbf{E}_\nu$ which generates $\mathbf{R}_\mu=\mathsf{R}\cdot\mathbf{E}_\mu$ satisfies

$$\Omega_R\equiv\dot{\mathsf{R}}\cdot\mathsf{R}^T=-\gamma\Omega\mathbf{R}_1\wedge\mathbf{R}_2=-\gamma\Omega\mathbf{E}_1\wedge\mathbf{E}_2 \tag{148}$$

and

$$\dot{\mathbf{R}}_\mu=\Omega_R\cdot\mathbf{R}_\mu. \tag{149}$$

Note the minus sign associated with the anticlockwise rotation about $\mathbf{E}_3$, and the factor γ arising from the τ derivative.

The unboosted Fermi-Walker frame rotates in the opposite sense to the $\mathbf{R}_\mu$ frame. By (94) and (70),

$$\begin{aligned}
\dot{\mathbf{F}}_\mu&=\frac{\gamma^3}{\gamma+1}(\mathbf{E}_0\wedge\boldsymbol{\epsilon}(\mathbf{E}_0,\mathbf{A},\mathbf{V}))^*\cdot\mathbf{F}_\mu\\
&=-\Omega\gamma(\gamma-1)(\mathbf{E}_0\wedge\mathbf{E}_3)^*\cdot\mathbf{F}_\mu\\
&=\gamma\Omega(\gamma-1)(\mathbf{E}_1\wedge\mathbf{E}_2)\cdot\mathbf{F}_\mu.
\end{aligned} \tag{150}$$

This represents a clockwise rotation about $\mathbf{E}_3$ at an angular velocity, with respect to lab time, of $\Omega(\gamma-1)$. It should be recalled that this frame is the lab representative of the "nonrotating" F–W frame. If B_L were zero, but the electron was forced to pursue the same orbit without putting a torque on the spin, the representative of the nonrotating spin would be at rest in the $\mathbf{F}_\mu$ frame. The rotation ceases in the nonrelativistic limit, as $\gamma\rightarrow1$.

Turning to the electron's rest frames, the boosted basis generated from $\mathbf{E}_\mu$ by

$$\mathbf{b}_\mu=\mathsf{B}\cdot\mathbf{E}_\mu=\left[\eta-2\mathbf{v}\otimes\mathbf{E}_0+\frac{(\mathbf{v}+\mathbf{E}_0)\otimes(\mathbf{v}+\mathbf{E}_0)}{\gamma+1}\right]\cdot\mathbf{E}_\mu \tag{151}$$

is the one we have been working with in earlier sections, but an even more useful one is that reached by a boost from $\mathbf{R}_\mu$:

$$\mathbf{r}_\mu=\mathsf{B}\cdot\mathbf{R}_\mu=\mathsf{B}\cdot\mathsf{R}\cdot\mathbf{E}_\mu. \tag{152}$$

Explicitly, it is given by

$$\begin{aligned}
&\mathbf{r}_0=\mathbf{v}, \quad \mathbf{r}_1=\mathbf{R}_1, \quad \mathbf{r}_3=\mathbf{R}_3\\
&\mathbf{r}_2=\left[\mathbf{R}_2+\frac{\gamma V}{\gamma+1}(\mathbf{v}+\mathbf{R}_0)\right]=\gamma(\mathbf{R}_2+V\mathbf{R}_0).
\end{aligned} \tag{153}$$

Because the two rest frame bases, $\mathbf{b}_\mu$ and $\mathbf{r}_\mu$, are derived by the same linear boost from the lab frames $\mathbf{E}_\mu$ and $\mathbf{R}_\mu$, the former pair are related to each other by the same rotation as the latter pair. Hence, $\mathbf{r}_0=\mathbf{b}_0=\mathbf{v}$, $\mathbf{r}_3=\mathbf{b}_3$, and (with $T=\gamma\tau$)

$$\begin{aligned}
&\mathbf{r}_1=\mathbf{b}_1\cos\gamma\Omega\tau+\mathbf{b}_2\sin\gamma\Omega\tau\\
&\mathbf{r}_2=-\mathbf{b}_1\sin\gamma\Omega\tau+\mathbf{b}_2\cos\gamma\Omega\tau.
\end{aligned} \tag{154}$$

The explicit expressions for the boost basis $\mathbf{b}_\mu$ in terms of the inertial basis $\mathbf{E}_\mu$ will not be needed but are included for the sake of completeness. The two frames must coincide in the nonrelativistic limit, when $V\rightarrow0$, $\gamma\rightarrow1$.

$$\begin{aligned}
\mathbf{b}_0&=\gamma\mathbf{E}_0+\gamma V(-\mathbf{E}_1\sin\gamma\Omega\tau+\mathbf{E}_2\cos\gamma\Omega\tau),\\
\mathbf{b}_1&=\mathbf{E}_1(1+[\gamma-1]\sin^2\gamma\Omega\tau)\\
&\quad-\mathbf{E}_2(\gamma-1)\sin\gamma\Omega\tau\cos\gamma\Omega\tau-V\gamma\mathbf{E}_0\sin\gamma\Omega\tau,\\
\mathbf{b}_2&=\mathbf{E}_2(1+[\gamma-1]\cos^2\gamma\Omega\tau)\\
&\quad-\mathbf{E}_1(\gamma-1)\sin\gamma\Omega\tau\cos\gamma\Omega\tau+V\gamma\mathbf{E}_0\cos\gamma\Omega\tau.
\end{aligned}$$

The τ derivative of $\mathbf{b}_\mu$ is given by $\dot{\mathbf{b}}_\mu=\Omega_b\cdot\mathbf{b}_\mu$, where, by (63),

$$\begin{aligned}
\Omega_b&=\mathbf{v}\wedge\mathbf{a}+\frac{1}{\gamma+1}(\mathbf{E}_0-\gamma\mathbf{v})\wedge\mathbf{a}\\
&=\mathbf{v}\wedge\mathbf{a}+\frac{1}{\gamma+1}(-\gamma V\mathbf{r}_2)\wedge(-\Omega V\gamma^2\mathbf{r}_1)\\
&=\mathbf{v}\wedge\mathbf{a}-\gamma\Omega(\gamma-1)\mathbf{r}_1\wedge\mathbf{r}_2.
\end{aligned} \tag{155}$$

To get Ω_r we can use (152) and (149):

$$\dot{\mathbf{r}}_\mu=\Omega_r\cdot\mathbf{r}_\mu=\dot{\mathsf{B}}\cdot\mathbf{R}_\mu+\mathsf{B}\cdot\dot{\mathbf{R}}_\mu=\Omega_b\cdot\mathbf{r}_\mu+\mathsf{B}\cdot\Omega_R\cdot\mathsf{B}^T\cdot\mathbf{r}_\mu.$$

Therefore, using (148) and (155),

$$\Omega_r = \Omega_b + \mathsf{B}\cdot(-\gamma\Omega\mathbf{R}_1\wedge\mathbf{R}_2)\cdot\mathsf{B}^T = \mathbf{v}\wedge\mathbf{a} - \gamma^2\Omega\mathbf{r}_1\wedge\mathbf{r}_2. \tag{156}$$

The derivation used the general formula (63) but could equally well have proceeded directly from the explicit definition of $\mathbf{r}_\mu$.

For the idealized physical situation we are considering the $\mathbf{r}_\mu$ frame is the most useful of the constructable rest frames. From it, we can derive the F—W frame by using (84):

$$\frac{d^r\mathbf{f}_\mu}{d\tau} = (\Omega_f - \Omega_r)\cdot\mathbf{f}_\mu = \Omega\gamma^2\mathbf{r}_1\wedge\mathbf{r}_2\cdot\mathbf{f}_\mu. \tag{157}$$

The F—W frame rotates clockwise around $\mathbf{r}_3 = \mathbf{E}_3$ with angular velocity (here a τ rate) $\Omega\gamma^2$ with respect to the $\mathbf{r}_\mu$-frame. There is no need to write the explicit formulae. Alternatively, the F—W frame can be obtained by boosting $\mathbf{F}_\mu$ [which can itself be constructed from (150)].

For the rotation of the (classical) electron spin, the equations of Sec. IX apply. It is perhaps simplest first to write down the equations for the lab representative $\boldsymbol{\sigma}$ of the spin $\mathbf{S}$. For easy comparisons all time derivatives are written with respect to proper time.

In the present case, Thomas' equation, (127), is equivalent to

$$\frac{d\boldsymbol{\sigma}}{d\tau} = \Omega\gamma\left[1+\gamma\left(\frac{g}{2}-1\right)\right]\mathbf{E}_3\times\boldsymbol{\sigma}, \tag{158}$$

representing an anticlockwise rotation of the spin representative about $\mathbf{E}_3$ when the square bracket is positive.

Using (150), the equation for spin motion with respect to the unboosted F—W frame is

$$\frac{d^F\boldsymbol{\sigma}}{d\tau} = \frac{d\boldsymbol{\sigma}}{d\tau} + \Omega\gamma(\gamma-1)\mathbf{E}_3\times\boldsymbol{\sigma} = \Omega\gamma^2\,\frac{g}{2}\,\mathbf{E}_3\times\boldsymbol{\sigma}. \tag{159}$$

This is also an anticlockwise rotation but with respect to a frame which itself is rotating clockwise with respect to the fixed lab frame.

The rotation with respect to the $\mathbf{R}_\mu$ frame is particularly interesting because it involves an angular velocity proportional to $(g/2-1)$, thus effectively proportional to the anomalous magnetic moment. Using (148) and (158),

$$\frac{d^R\boldsymbol{\sigma}}{d\tau} = \frac{d\boldsymbol{\sigma}}{d\tau} - \Omega\gamma\mathbf{E}_3\times\boldsymbol{\sigma} = \Omega\gamma^2\left(\frac{g}{2}-1\right)\mathbf{E}_3\times\boldsymbol{\sigma}. \tag{160}$$

This equation, or its boosted form, is the basis for many experimental measurements of the magnetic moment anomaly for electrons and muons.[16]

The spin vector itself is generated from $\boldsymbol{\sigma}$ by a boost $\mathbf{S} = \mathsf{B}\cdot\boldsymbol{\sigma}$, so corresponding to the different forms

$$\boldsymbol{\sigma} = \sigma_R^m\mathbf{R}_m = \sigma_F^m\mathbf{F}_m = \sigma_E^m\mathbf{E}_m, \tag{161}$$

there are forms for $\mathbf{S}$:

$$\mathbf{S} = \sigma_R^m\mathbf{r}_m = \sigma_F^m\mathbf{f}_m = \sigma_E^m\mathbf{b}_m. \tag{162}$$

Each of the Eqs. (158), (159), and (160) can be boosted to a rest frame equation. For example, from the last,

$$\frac{d^r\mathbf{S}}{d\tau} = \Omega\gamma^2\left(\frac{g}{2}-1\right)\boldsymbol{\epsilon}(\mathbf{v},\mathbf{r}_3,\mathbf{S}), \tag{163}$$

where in fact the rotation axis in the rest frame is still $\mathbf{E}_3 = \mathbf{r}_3$.

XI. FINITE WIGNER ROTATION

The methods of the previous sections can be used to analyze the relations between discrete inertial frames. The need for this occurs in the general addition-of-velocities formula (which is used in relativistic polarization analysis[18]), and also in Wigner's construction of irreducible representations of the Poincaré group (see, for example, Ref. 19).

Suppose a free particle worldline l has $\mathbf{v}$ as its unit future-pointing spacetime velocity vector and that the relative velocity with respect to an inertial frame K (the lab) is $\mathbf{V}$. Then,

$$\mathbf{v} = \gamma(\mathbf{E}_0+\mathbf{V}), \quad \mathbf{E}_0\cdot\mathbf{V} = 0, \quad \gamma = -\mathbf{E}_0\cdot\mathbf{v}. \tag{164}$$

A second worldline l' has spacetime velocity $\mathbf{v}'$ and relative velocity $\mathbf{V}'$ with respect to K,

$$\mathbf{v}' = \gamma'(\mathbf{E}_0+\mathbf{V}'), \quad \mathbf{E}_0\cdot\mathbf{V}' = 0, \quad \gamma' = -\mathbf{E}_0\cdot\mathbf{v}'. \tag{165}$$

Let $\mathsf{B}(\mathbf{E}_0\mapsto\mathbf{v})$ be the boost that carries the lab basis to a basis for a frame K_v, at rest for the worldline l; it is completely fixed by the fact that it carries $\mathbf{E}_0$ to $\mathbf{v}$. Similarly, $\mathsf{B}(\mathbf{E}_0\mapsto\mathbf{v}')$ carries the lab basis to a basis for the rest frame $K_{v'}$ of l'.

To get the relation between K_v and $K_{v'}$ we consider the product of three boosts

$$\mathsf{R} \equiv \mathsf{B}(\mathbf{v}'\mapsto\mathbf{E}_0)\cdot\mathsf{B}(\mathbf{v}\mapsto\mathbf{v}')\cdot\mathsf{B}(\mathbf{E}_0\mapsto\mathbf{v}). \tag{166}$$

The product carries a basis to a basis, and carries $\mathbf{E}_0$ to itself. Therefore, R is a spatial rotation in the lab. The boosts to K_v and to $K_{v'}$, are related by

$$\mathsf{B}(\mathbf{E}_0\mapsto\mathbf{v}')\cdot\mathsf{R} = \mathsf{B}(\mathbf{v}\mapsto\mathbf{v}')\cdot\mathsf{B}(\mathbf{E}_0\mapsto\mathbf{v}). \tag{167}$$

(The finite Wigner rotation R that appears here is just the counterpart of the continuous Thomas precession of the unboosted frame $\mathbf{F}_m$.)

Before calculating explicitly the product of the three boosts, it is helpful to prepare by considering the general form of a rotation dyadic. A rotation is determined by an axis, an angle less than π, and a sense. (Rotations of π can be treated as limits.) The vectors in the plane orthogonal to the axis are rotated into each other. Suppose a unit vector $\mathbf{x}$ in the plane is rotated into $\mathbf{y}$. Then, as may easily be checked, the appropriate rotation dyadic is

$$\mathsf{R}(\mathbf{x}\mapsto\mathbf{y}) = \eta - \mathbf{x}\wedge\mathbf{y} + \frac{(\mathbf{x}\wedge\mathbf{y})\cdot(\mathbf{x}\wedge\mathbf{y})}{1+\mathbf{x}\cdot\mathbf{y}}. \tag{168}$$

(There is a similar formula with the wedge product for a boost.) If the angle between $\mathbf{x}$ and $\mathbf{y}$ is $\phi<\pi$, and $\mathbf{m},\mathbf{n}$ are two *orthogonal* unit vectors in the plane with the same sense as $\mathbf{x},\mathbf{y}$, then

$$\mathsf{R}(\mathbf{x}\mapsto\mathbf{y}) = \eta - \sin\phi\,\mathbf{m}\wedge\mathbf{n} + (1-\cos\phi)(\mathbf{m}\wedge\mathbf{n})\cdot(\mathbf{m}\wedge\mathbf{n}). \tag{169}$$

From (169), we see that the antisymmetric part of the rotation dyadic R determines it unambiguously. This fact is very helpful while interpreting the result of the product of the three boosts (166). After some calculation, one finds

$$\mathsf{R} = \eta - \frac{b}{a}(\gamma'\mathbf{V}')\wedge(\gamma\mathbf{V}) + \frac{1}{a}[(\gamma'\mathbf{V}')\wedge(\gamma\mathbf{V})]\cdot[(\gamma'\mathbf{V}')\wedge(\gamma\mathbf{V})], \tag{170}$$

where $a=(1+\gamma)(1+\gamma')(1+\Gamma)$ and $b=1+\gamma+\gamma'+\Gamma$ with

$$\Gamma \equiv -\mathbf{v}\cdot\mathbf{v}' = \gamma\gamma'(1-\mathbf{V}\cdot\mathbf{V}') = \gamma\gamma'(1-VV'\cos\chi) \tag{171}$$

in terms of the angle $\chi<\pi$ between $\mathbf{V}'$ and $\mathbf{V}$. Comparing second terms and third terms in (169) and (170), we get

$$\sin\phi = \frac{b}{a}(\gamma' V')(\gamma V)\sin\chi$$

and (172)

$$1-\cos\phi = \frac{1}{a}(\gamma' V' \gamma V)^2 \sin^2\chi,$$

from which

$$\cos\phi + 1 = \frac{b^2}{a}. \tag{173}$$

The rotation R is by an angle ϕ from $\mathbf{V}'$ towards $\mathbf{V}$. The angle ϕ is called the Wigner angle.

As an application of the three boosts formula we can reconsider the nonrotating rest frames for an accelerating particle. As in Sec. V, the Fermi–Walker frame $\mathbf{f}_\mu(\tau)$ is defined by differential equations that require the instantaneous angular velocity with respect to the rest space to vanish. The representative $\mathbf{F}_\mu$ in the lab of the Fermi–Walker frame is constructed by "unboosting" (or boosting back to the lab) $\mathbf{F}_\mu = \mathsf{B}(\mathbf{v}\mapsto\mathbf{E}_0)\cdot\mathbf{f}_\mu$. The surprising fact is that the frame $\mathbf{F}_\mu$ rotates in the lab.

If the spacetime velocities at successive proper times τ and $\tau+\delta\tau$ (or, as measured in the lab at the particle, at times T and $T+\delta T$) are $\mathbf{v}$ and $\mathbf{v}'$ as in (164) and (165), then

$$\mathbf{F}_\mu(T) = \mathsf{B}(\mathbf{v}\mapsto\mathbf{E}_0)\cdot\mathbf{f}_\mu(\tau),$$

and

$$\mathbf{F}_\mu(T+\delta T) = \mathsf{B}(\mathbf{v}'\mapsto\mathbf{E}_0)\cdot\mathbf{f}_\mu(\tau+\delta\tau). \tag{174}$$

The requirement that the Fermi–Walker frame should not rotate in the interval τ to $\tau+\delta\tau$ may be expressed

$$\mathbf{f}_\mu(\tau+\delta\tau) = \mathsf{B}(\mathbf{v}\mapsto\mathbf{v}')\cdot\mathbf{f}_\mu(\tau), \tag{175}$$

since an infinitesimal boost is nonrotating. These relations lead immediately to

$$\begin{aligned}\mathbf{F}_\mu(T+\delta T) &= \mathsf{B}(\mathbf{v}'\mapsto\mathbf{E}_0)\cdot\mathsf{B}(\mathbf{v}\mapsto\mathbf{v}')\cdot\mathsf{B}(\mathbf{E}_0\mapsto\mathbf{v})\cdot\mathbf{F}_\mu(T)\\ &= \mathsf{R}\cdot\mathbf{F}_\mu(T).\end{aligned} \tag{176}$$

For the present case $\mathbf{V}'=\mathbf{V}+\delta T\mathbf{A}$, and to zero order $\gamma'=\gamma$, $\Gamma=1$, so the rotation dyadic, to first order, is

$$\mathsf{R} = \eta - \frac{b}{a}(\gamma'\mathbf{V}')\wedge(\gamma\mathbf{V}) = \eta - \frac{\gamma^2}{\gamma+1}\mathbf{A}\wedge\mathbf{V}\,\delta T. \tag{177}$$

Hence,

$$\begin{aligned}\frac{\delta\mathbf{F}_\mu}{\delta T} &= -\frac{\gamma^2}{\gamma+1}(\mathbf{A}\wedge\mathbf{V})\cdot\mathbf{F}_\mu\\ &= \frac{\gamma^2}{\gamma+1}(\mathbf{A}\times\mathbf{V})\times\mathbf{F}_\mu = \boldsymbol{\omega}_T\times\mathbf{F}_\mu,\end{aligned} \tag{178}$$

in agreement with (99).

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THE INTERPRETATION OF QUANTUM MECHANICS

How does it come about then, that great scientists such as Einstein, Schrödinger, and De Broglie are nevertheless dissatisfied with the situation? Of course, all these objections are levelled not against the correctness of the formulae, but against their interpretation. Two closely knitted points of view are to be distinguished: the question of determinism and the question of reality.

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