plications," Adv. Chem. Phys. **52**, 363–505 (1983), and references therein.
[5]D. T. Gillespie, *Markov Processes: An Introduction for Physical Scientists* (Academic, Boston, 1992).
[6]L. Turban, "Anisotropic critical phenomena in parabolic geometries: The directed self-avoiding walk," J. Phys. A **25**, L127–L134 (1992).
[7]F. Iglói, "Directed polymer inside a parabola: Exact solution," Phys. Rev. A **45**, 7024–7029 (1992); F. Iglói, I. Peschel, and L. Turban, "Inhomogeneous systems with unusual critical behaviour," Adv. Phys. **42**, 683–740 (1993).
[8]L. Breiman, "First exit time from the square root boundary," Proc. Fifth Berkeley Symp. Math. Statist. Prob. **2**, 9–16 (1966).
[9]H. E. Daniels, "The minimum of a stationary Markov superimposed on a U-shape trend," J. Appl. Prob. **6**, 399–408 (1969).
[10]K. Uchiyama, "Brownian first exit from sojourn over one sided moving boundary and application," Z. Wahrsch. verw. Gebiete **54**, 75–116 (1980).
[11]P. Salminen, "On the hitting time and the exit time for a Brownian motion to/from a moving boundary," Adv. Appl. Prob. **20**, 411–426 (1988).
[12]See, e.g., A. Messiah, *Quantum Mechanics* (Interscience, New York, 1961).
[13]L. I. Schiff, *Quantum Mechanics* (McGraw–Hill, New York, 1968).
[14]J. Crank, *Free and Moving Boundary Problems* (Clarendon, Oxford, 1984).
[15]H. S. Carslaw and J. C. Jaeger, *Conduction of Heat in Solids* (Oxford U.P., Oxford, 1959), Chap. XI.
[16]R. Ghez, *A Primer of Diffusion Problems* (Wiley, New York, 1988), Chap. IV.
[17]For example, if a function has the asymptotic form $f(x)=x^\alpha \exp(x^\beta)(1+h_1(x)+h_2(x)+\cdots)$ as $x\to\infty$, with $h_i(x)\to 0$ as $x\to\infty$, then the *leading behavior* is $x^\alpha \exp(x^\beta)$ and the *controlling factor* is $\exp(x^\beta)$. See, e.g., p. 80 of Ref. 18.
[18]C. M. Bender and S. A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers* (McGraw–Hill, New York, 1978); M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1965).
[19]F. Leyvraz, (unpublished 1987); D. ben-Avraham, "Computer simulation methods for diffusion-controlled reactions," J. Chem. Phys. **88**, 941–947 (1988); M. E. Fisher and M. P. Gelfand, "The reunions of three dissimilar vicious walkers," J. Stat. Phys. **53**, 175–189 (1988).

Thomas precession and the Liénard–Wiechert field

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The role of the Thomas precession in the dynamic formation of the electric field lines of a moving charged particle is demonstrated. A simple derivation of the Thomas precession formula is given, based only on the Lorentz contraction of moving bodies. A simple and physically appealing construction is developed for determining points on a field line. Field line diagrams are generated and discussed. These diagrams vividly reveal the existence and magnitude of the Thomas precession. © *1996 American Association of Physics Teachers.*

I. INTRODUCTION

The Thomas precession is one of those subtle counterintuitive consequences of special relativity whose existence students often find initially quite difficult to accept. Standard discussions[1–5] often involve fairly advanced concepts and techniques which tend to encourage the belief that the Thomas precession is an esoteric phenomenon whose origin and significance is hard to grasp. It will be shown here that the Thomas precession can be understood very simply in terms of the well-known Lorentz contraction of a moving body. Furthermore, its influence is manifested in and can easily be seen in plots of the electric field lines of a moving charged particle. The first of these insights is not new, but is included for completeness. An alternative simple derivation of the Thomas precession can be found in the appendix of Muller's paper.[5]

This paper extends the work of Purcell,[6] Tsien,[7] and others.[8,9] It provides a simple description of the mechanism by which the field lines are formed and explicitly reveals the role of the Thomas precession in their formation. A straightforward, intuitive derivation of the Thomas precession formula is given in Sec. II. This is followed, in Sec. III, by discussion and further development of a simple picture of the dynamical formation of the electric field lines of a moving charged particle. Section IV is devoted to the display and discussion of field lines for a charged particle undergoing uniform motion in a circle. These results directly illustrate the effect of the Thomas precession in a manner that needs no mathematics to appreciate. The insight achieved in this analysis is summarized in Sec. V.

II. THOMAS PRECESSION FORMULA

Imagine three inertial reference frames, S_0, S_1, and S_2, such that S_1 and S_2 are moving at velocities $\boldsymbol{\beta}$ and $\boldsymbol{\beta}+\delta\boldsymbol{\beta}$, respectively, relative to S_0, where $\boldsymbol{\beta}=\mathbf{v}/c$. We will be specifically interested in the limit $\delta\beta\ll 1$, so that frames S_1 and S_2 are moving nonrelativistically with respect to one another. Imagine a line of posts fixed in S_0 running parallel to $\boldsymbol{\beta}$ and another line running parallel to $\boldsymbol{\beta}+\delta\boldsymbol{\beta}$. With no loss of generality, we may choose the x axis in S_0 parallel to $\boldsymbol{\beta}$ and the y axis such that $\delta\boldsymbol{\beta}$ lies in the xy plane.

Figure 1 shows the situation as seen in each of the three reference frames. In frame S_0, the two lines of posts are at rest and separated by a small angle $\delta\theta$. In frames S_1 and S_2

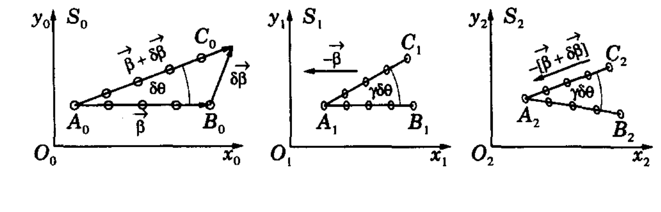

Fig. 1. Objects as seen in three different reference frames arranged to show how the Thomas precession arises. See text for explanation.

the posts are moving in the directions indicated by the reversed velocity arrows. By virtue of the Lorentz contraction, the moving patterns are foreshortened along their respective directions of travel. The angle between the two lines is therefore increased by the relativistic factor $\gamma=1/(1-\beta^2)^{1/2}$. The small difference in γ between frames S_1 and S_2 is not important here because $\delta\theta$ is already a first-order small quantity. The line A_1B_1 is parallel to A_0B_0 and the line A_2C_2 is parallel to A_0C_0. As a result, the lines of posts are tilted in S_2 relative to their orientation in S_1. This rotation is the essence of the Thomas precession, as will now be explained.

First, we must understand that observers in S_0 regard the orientations of the axes in frames S_1 and S_2 as being parallel to their own. A Lorentz boost from S_0 to S_1 will clearly keep the x and y axes parallel. However, the boost from S_0 to S_2 must be handled more carefully. If the new axes are to be regarded as parallel to those in S_0 the angle between the x_2 axis and the reversed velocity vector in S_2 must be the same as the angle between the x_0 axis and the velocity vector in S_0. Thus, in order for S_2 to be oriented parallel to S_0 in this sense, the line A_2C_2 must make an angle $\delta\theta$ with the O_2x_2 axis and the whole pattern of posts will be tilted at an angle $-(\gamma-1)\delta\theta$ in S_2 relative to its orientation in S_1.

Recall that frames S_1 and S_2 are assumed to be moving *nonrelativistically* with respect to one another. Observers in S_1 believe that the moving pattern of posts has the same orientation in both frames. Therefore, a boost without rotation from S_1 into S_2 brings a line that is parallel to A_1B_1 into one that is parallel to line A_2B_2. Viewed from S_0, this boost results in a rotation through angle $\delta\theta_T=-(\gamma-1)\delta\theta$.

The Thomas precession refers to the relativistic precession that arises when a body is continuously boosted parallel to itself through a sequence of instantaneous rest frames. Define the precession angular velocity as

$$\omega_T \equiv \frac{d\theta_T}{dt} = -(\gamma-1)\frac{d\theta}{dt}. \tag{1}$$

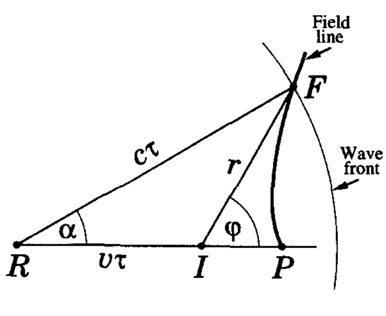

Fig. 2. Geometry used to construct a field line—one-dimensional motion. See text for explanation.

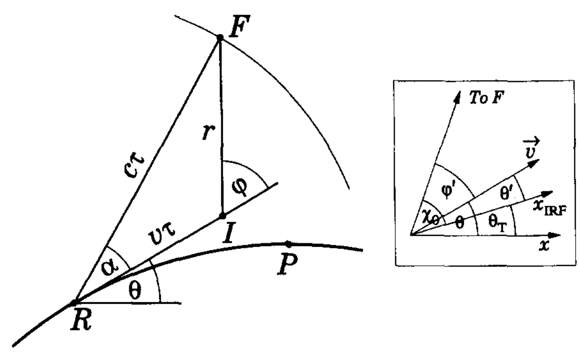

Fig. 3. Geometry used to construct a field line—two-dimensional motion. The inset shows angles as seen in the particle's instantaneous rest frame. See text for explanation.

From the geometry as seen in S_0, we can also easily obtain the following well-known vector expression;

$$\boldsymbol{\omega}_T = -(\gamma-1)\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}}/\beta^2. \tag{2}$$

All quantities in these expressions are as measured in frame S_0.

III. THE LIÉNARD–WIECHERT ELECTRIC FIELD

Purcell[6] has developed an appealing picture of the mechanism by which the electric field lines of a moving charged particle are formed, based on the Lorentz invariance of electric charge. Using simple arguments, he is able to show that the electric field lines of a point charge moving with constant velocity are deflected away from the direction of motion just as though they were embedded in a medium that suffers the standard Lorentz contraction. By invoking the finite speed of propagation of the "information" contained in the field lines, he is also able to treat cases in which the particle undergoes bursts of acceleration. Field lines produced during different phases of constant velocity motion are linked up by invoking symmetry and flux conservation arguments. These arguments can be extended to treat a charge that is being continuously accelerated in one dimension.

To create a picture of the field lines of an accelerated charged particle, imagine that it is continuously shooting out a pattern of field lines that spread out from its instantaneous position at the speed of light. The field lines at some given time, t, can be constructed by joining up the information generated during each earlier instant of the motion. At some earlier time, $t-\tau$, the particle was at its "retarded position," $R(t,\tau)$; see Fig. 2. Corresponding to this retarded position there is an "imputed present position," $I(t,\tau)$, which is the position that the particle would presently occupy if it had traveled at a constant velocity **v** determined by the instantaneous velocity at the retarded time. Since the particle may be undergoing acceleration, $I(t,\tau)$, is not generally the same as the actual present position, $P(t)$.

In order to locate a point, F, on a given field line, we need to determine the angle φ and the distance r from I, or the angle α to combine with the known distance $c\tau$ from R. The angle φ can be found by assuming that the corresponding angle φ' in the particle's instantaneous rest frame (IRF) is constant. We shall call this constant angle, different for each field line, χ_0. Then φ is obtained from the relation

$$\tan\varphi = \gamma \tan\varphi' \tag{3}$$

which results simply from the Lorentz contraction effect. Since the distances RF and RI are already known, the rest of the calculation is just geometry. Alternatively, Tsien[7] has given the convenient formula

$$\tan(\alpha/2) = \left(\frac{1-\beta}{1+\beta}\right)^{1/2} \tan(\varphi'/2) \tag{4}$$

which enables α to be found directly.

The above picture has been developed[6] using flux and symmetry arguments, the latter of which is no longer available when the velocity and acceleration are not collinear. It is not at all certain, *a priori*, whether the above picture can be extended to the more general situation. Although the above construction always yields a point on some field line, it is not immediately clear that each such point lies on the same field line. When they are connected together, the result is therefore not necessarily a field line.

There are two problems to be resolved. First, can this appealing picture of the formation of the electric field lines be extended to the more general situation wherein the velocity and the acceleration of the moving charge are not collinear? Second, if it can be so extended, how does one determine the angle φ' such that successive points lie on a single field line? In the following, we shall restrict our attention to motion confined to a plane and to the electric field lines seen in that plane.

Tsien[7] generates field lines by deriving and analytically integrating an appropriate differential equation for them, hence avoiding the first problem that needs to be resolved here. He suggests that the initial angle of a field line can be fixed relative to the particle's present instantaneous velocity. However, this suggestion leads to the unacceptable consequence that the orientation of the field pattern of a slowly moving charged particle rotates as the direction of the particle's motion changes. Stoner[9] assumes that the above picture is valid. He further assumes that, as it is being generated, a given field line subtends a constant angle with a line in the particle's IRF, which line is held parallel to the stationary observer's x axis.

An alternative possibility, which we propose and demonstrate here, is that a given field line subtends a constant angle with a line that is held fixed in the particle's IRF. If this alternative prescription is correct, the Thomas precession implies that the field line is generated in a frame that is rotating relative to the laboratory observer's frame. *This rotation will be directly observable as a qualitative difference in the pattern of field lines produced.*

Our proposed generalized construction is illustrated in Fig. 3. Points R, I, P, and F have the same significance as before. The main difference is that we have to introduce new angles that were not required before. These are illustrated in the inset to the right. Angle χ_0 is again the constant angle that defines a given field line. Angle θ gives the direction of the velocity vector, $\mathbf{v}$ relative to the laboratory x axis, angle θ' is the corresponding angle in the particle's IRF and θ_T is the angle between the two x axes that results from the Thomas precession. This angle can be computed from ω_T as follows:

$$\theta_T(t) = \int_{t_0}^{t} \omega_T(t')\,dt'. \tag{5}$$

Here, t_0 is an arbitrary constant that fixes a time when the two x axes are parallel to one another.

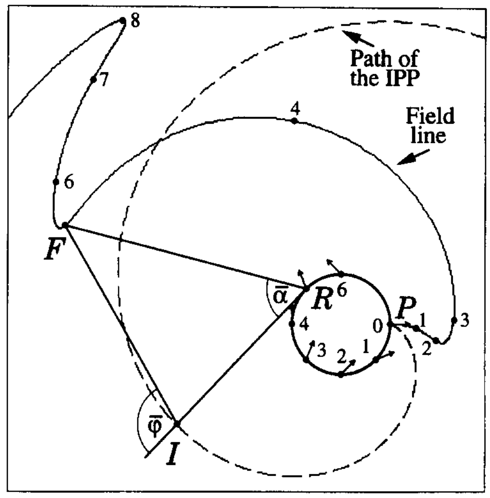

Fig. 4. A field line for a charged particle at point P, showing various construction details. The particle is moving at a constant speed such that $\gamma=1.5$ around the circular track below and to the right of the center of the diagram. See text for explanation.

The desired angle φ' is now given by $\varphi' = \chi_0 - \theta'$ where $\theta' = \theta - \theta_T$. Thus we obtain the simple result

$$\varphi' = \chi_0 + \theta_T - \theta. \tag{6}$$

The rest of the construction is done using Eq. (3) or (4) and standard geometry. As one locates each point on a given field line, all quantities in these equations are to be evaluated at the appropriate retarded time, $t-\tau$.

The above construction embodies a surprisingly simple, intuitive picture of the formation of the electric field lines of a moving charged particle. But we have not yet proven that it is correct. Fortunately, a detailed algebraic analysis is not necessary. Towards the end of his paper, Tsien shows that a formula of this structure will indeed reproduce a field line. What we have added here is a simple physical interpretation of that result with specific emphasis on the role of the Thomas precession in it. An independent algebraic analysis, already completed before Tsien's result was discovered, confirms it.

IV. UNIFORM MOTION IN A CIRCLE

The above results can be nicely illustrated by the field lines of a charged particle that is undergoing uniform circular motion on a track of radius ρ. In this case, $d\theta/dt = v/\rho \equiv \omega$ is a constant. Hence, ω_T is also constant and the integral in Eq. (5) can be immediately evaluated to give

$$\theta_T(t-\tau) = -(\gamma-1)\omega(t-\tau-t_0). \tag{7}$$

In this case, also, all the physics that we are interested in can be seen in each field line. It is therefore advantageous to display just one field line at a time. In the detailed results that follow, we set $c=1$, $\rho=1$, and $t=t_0=0$. These choices ensure that θ_T vanishes at P.

Figure 4 shows a field line computed for $\gamma=1.5$, along with some of the construction details described in Sec. III. The field line itself was computed by direct numerical integration using the Liénard–Wiechert field as given in standard texts.[10,11] The numbered points on the field line were obtained using Eqs. (3) and (4), along with $\chi_0=0$ and appropriate geometry. Of course, we really need to use only one of

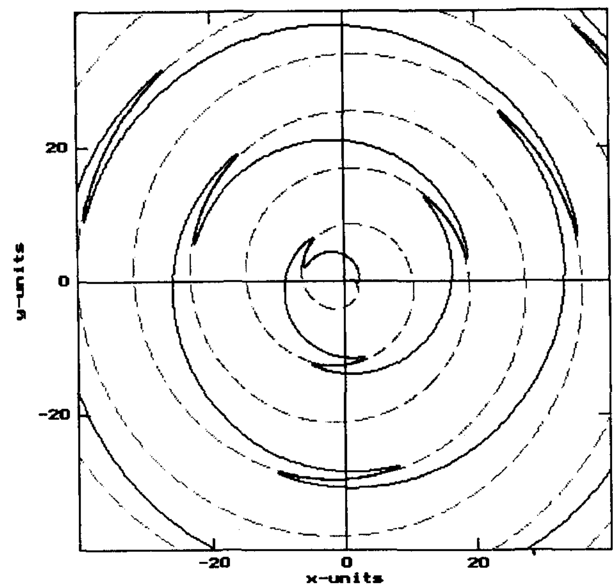

Fig. 5. A broader view of the same field line shown in Fig. 4. The effect of the Thomas precession is readily apparent in this figure. See text for explanation.

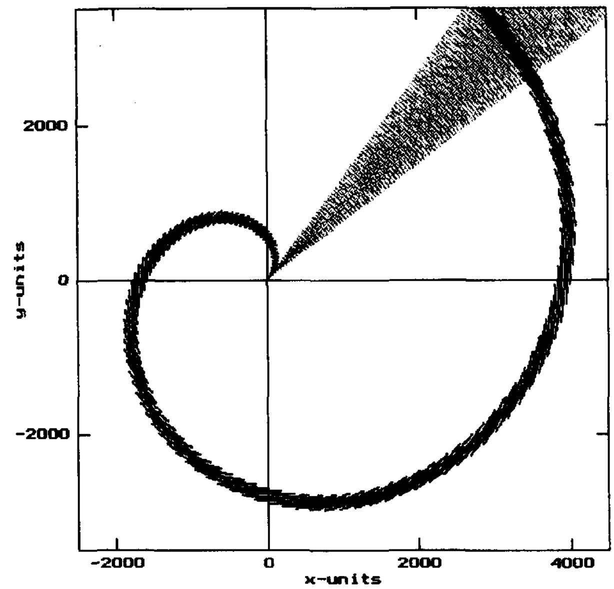

Fig. 7. A field line for a charged particle moving at a speed such that γ=1.01. This field line has χ_0=45°. The thickness of the pattern is caused by the oscillatory nature of the field line. The shaded area is a line formed by the same construction technique as a field line, but omitting the Thomas precession angle.

these equations because each yields the same set of points. The proposed construction is again verified by the fact that the individual numbered points lie on the field line. In subsequent calculations, we have used the construction to create the entire field line and have only occasionally verified it by numerical integration.

The circle below and to the right of the center of the figure represents the track along which the particle is moving in an anticlockwise sense. The numbered positions on the track label (but are not equal to) uniformly increasing values of the retardation, τ. The path followed by the particle's imputed present position (IPP) is shown as a dashed line. A specific example of the construction is also shown, where labels R and F have been substituted for the label 5. Since the particle is, at this retarded moment, moving down and to the left, the barred angle $\bar{\phi}$ and $\bar{\alpha}$ are actually $360°-\phi$ and $360°-\alpha$, respectively.

Each small arrow extending from the labeled points on the particle's track gives the direction of x axis in the particle's IRF at that point. Because of the Thomas precession, these directions rotate in a clockwise sense at half the rate at which the particle's velocity vector is rotating. The center of the kink in the field line (see points 5–8) occurs when the direction of the field line in the particle's IRF is parallel to the particle's velocity vector. Since this particular field line has χ_0=0, its direction in the particle's IRF is always parallel to the arrow. It is now easy to see that a kink will arise centered near position 6 on the track. Another incipient kink is centered near position 1. Because of the Thomas precession, these kinks do not all occur at the same angular position. In fact, the angle traversed by the field line between kinks is $(1-1/\gamma)360°$. When γ=1.5, as we have here, this angle is 120°.

Figure 5 shows a broader view of the same field line. The dashed line now represents the locus of points reached by light signals emitted tangentially forwards from retarded points on the particle's track. Each turn of the dashed spiral represents a full period of the particle's motion along the track. The twisting of the field line around the particle's track is governed by the Thomas precession angular velocity, ω_T. Thus there are $\gamma-1$ turns of the field line during each period of the particle's motion. Kinks in the field line are formed at points where it crosses the dashed line. At these points, independent of the starting angle, χ_0, we have α=0 and the direction of the field line in the particle's IRF is parallel to the particle's velocity vector. The rate of occurrence of these crossing points is governed by $\omega-\omega_T$. Hence there are γ kinks in the field line for each period of the particle's mo-

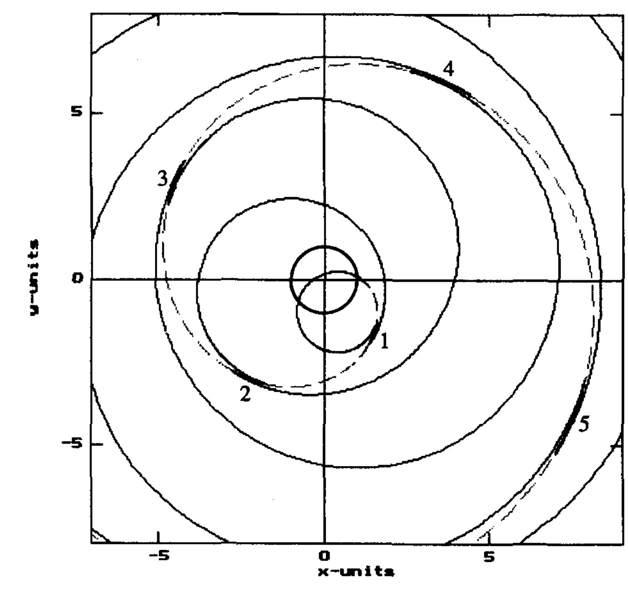

Fig. 6. A field line for a charged particle moving at a speed such that γ=4. See text for explanation.

tion. When $\gamma=1.5$, the field line makes one turn around the track during every two periods and there are three equally spaced kinks for each turn of the field line.

Figure 6 is a similar plot made for $\gamma=4$. Now the field line makes $\gamma-1=3$ turns and $\gamma=4$ kinks during each period of the particle's motion. Note the eccentric nature of the field line's spiral, caused by the particle's motion along its track. These field lines are already quite complex. One can only marvel at the convoluted intricacy of the field pattern when γ reaches up into the thousands!

The opposite extreme occurs when the particle is moving relatively slowly. Figure 7 shows a field line for a particle having $\gamma=1.01$ ($\beta=0.140$). Here the field line requires 100 cycles to complete one turn, producing 101 kinks while doing so. The distance traveled by light during this time is easily found to be 4476 length units. The gray area in Fig. 7 is a line calculated by the same construction method as a field line, but omitting the Thomas precession angle. This line does not form a spiral around the particle's track.

V. CONCLUSION

In this paper we have emphasized the role of the Thomas precession on the dynamic formation of the electric field lines of a moving charged particle. A simple derivation of the Thomas precession has been given, based only on the Lorentz contraction of moving bodies. Extending the work of Purcell[6] and Tsien,[7] a simple and physically appealing construction has been developed for determining points on a field line.

The particle continuously shoots out field line information in fixed directions in its instantaneous rest frame. This information moves away from the particle at the speed of light. At any given instant of time field lines can be formed by combining the information emitted at earlier times. A field line so constructed appears distorted in a fixed reference frame not only because the emission point is continuously changing, but also because the information has to be mapped from the particle's IRF into the fixed laboratory frame. The mapping explicitly involves the Lorentz contraction effect, Eq. (3) or (4) and the Thomas precession, Eqs. (1), (5), and (6).

This work has been illustrated by several diagrams giving the detailed geometry of field lines generated by a changed particle undergoing uniform motion in a circle. These diagrams show in a very graphic way the influence of the Thomas precession on the field lines. Field lines can be formed by integrating the Liénard–Wiechert field equations without any reference to relativistic effects. The existence of the Thomas precession is then directly seen in the twisting of the field lines around the particle's track. The rate of precession can be determined by analyzing the geometry of the field lines. It shows up both in the number of turns made by the field line during each period of the particle's motion and in the number of kinks formed during the same period.

It is not *a priori* obvious that a simple construction, based only a relativistic transformations, will be able to reproduce so elusive a quantity as an electric field line. It is a tribute to the relativistic consistency of electromagnetic theory that such a construction is possible at all.

[1]R. D. Sard, *Relativistic Mechanics* (Benjamin, New York, 1970), pp. 285–290.

[2]J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1975), 2nd ed., Sec. 11.8.

[3]H. Urbantke, "Physical holonomy, Thomas precession and Clifford algebra," Am. J. Phys. **58**, 747–750 (1990).

[4]Abraham A. Ungar, "Thomas precession and its associated grouplike structure," Am. J. Phys. **59**, 824–834 (1991).

[5]Richard A. Muller, "Thomas precession: Where is the torque?," Am. J. Phys. **60**, 313–317 (1992).

[6]Edward M. Purcell, *Electricity and Magnetism* (McGraw-Hill, New York, 1965); Berkeley Physics Course, Vol. 2, Chap. 5.

[7]Roger Y. Tsien, "Pictures of Dynamic Electric Fields," Am. J. Phys. **40**, 46–56 (1972).

[8]John C. Hamilton and Judah L. Schwartz, "Electric Fields of Moving Charges: A series of Four Film Loops," Am. J. Phys. **39**, 1540–1542 (1971).

[9]R. Stoner, *Electricity and Magnetism Simulations* (Wiley, New York, 1995), The Consortium for Upper-Level Physics Software, Chap. 7.

[10]David J. Griffiths, *Introduction to Electrodynamics* (Prentice-Hall, Englewood Cliffs, NJ, 1981), 2nd ed., p. 424.

[11]In Ref. 2, p. 657.

POTHOOKS

I became interested in trying to devise new experimental methods for investigating the physics of elementary particles in 1950, not long after the new ⟨⟨strange particles⟩⟩ had been discovered in cosmic rays. In those days a rather small number of these particles had been observed, and they were still called ⟨⟨*V*-particles⟩⟩ or ⟨⟨pothooks⟩⟩ because of their unusual appearance in the cloud chamber photographs. In fact, I remember that when I left the California Institute of Technology in 1949 after finishing my doctoral research on cosmic radiation under the direction of Professor Carl D. Anderson, there was written at the top of his blackboard the question: ⟨⟨What have we done about the pothooks today?⟩⟩

Donald A. Glaser, "Elementary particles and bubble chambers" (Nobel Lecture, December 12, 1960, reprinted in *Nobel Lectures, Physics*, Vol. 3, 1942–1962, Elsevier Amsterdam, 1964).