The rigidly rotating disk of dust and its black hole limit

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Abstract

The exact global solution of the Einstein equations [Neugebauer & Meinel, Phys. Rev. Lett. 75 (1995) 3046] describing a rigidly rotating, self–gravitating disk is discussed. The underlying matter model is a perfect fluid in the limit of vanishing pressure. The solution represents the general–relativistic analogue of the classical Maclaurin disk. It was derived by applying solution techniques from soliton theory to the axisymmetric, stationary vacuum Einstein equations. In contrast to the Newtonian solution, there exists an upper limit for the total mass of the disk – if the angular momentum is fixed. At this limit, a transition to a rotating black hole, i.e., to the Kerr solution occurs. Another limiting procedure leads to an interesting cosmological solution. These results prove conjectures formulated by Bardeen and Wagoner more than twenty–five years ago.

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1 Introduction: Rotating bodies in general relativity

The problem of self–gravitating rotating bodies is a *global* one: One has to solve simultaneously the interior equations (with matter) as well as the exterior (vacuum) equations. The shape of the surface cannot be prescibed arbitrarily. A famous solution within Newton's theory of gravitation is the Maclaurin spheroid of a rigidly rotating perfect fluid with constant mass density (cf. [1], [2]). Within Einstein's theory of gravitation the problem is even more complicated, mainly because of the influence of the body's rotation onto the gravitational field (related to the so–called gravitomagnetic potential). Fortunately, for the exterior equations a powerful solution technique (the 'inverse scattering method' known from soliton theory) exists in the case of axial symmetry and stationarity ([15] – [18]). It does not apply, unfortunately, to the interior equations. However, there is an interesting limiting case: For infinitesimally thin disks the interior problem 'shrinks' to *boundary conditions* for the exterior solution, and the solution technique mentioned can be utilized for solving the global problem.

A rigidly rotating disk of *dust* is the universal limit of rigidly rotating perfect fluid configurations as $p/\epsilon \to 0$ (p denotes the pressure and ϵ the mass–energy density), cf. the disk limit of the Maclaurin sequence. This disk is interesting for two reasons: On the one hand, it represents, in a sense, the simplest model of a self–gravitating rotating body (with no interaction except gravitation). On the other hand, it may serve as a crude model for astrophysical disks, for example galaxies (with the stars considered as dust grains). Of course, normal galaxies are sufficiently well described by Newton's theory of gravitation. Nevertheless, the relativistic model might be interesting, e.g. in the context of quasars. An approximate solution was presented by Bardeen and Wagoner [3], [4].

The rigorous solution of the problem of the rigidly rotating disk of dust ([5] – [7]) seems to provide the first example of an exactly solvable rotating–body problem within general relativity, apart from the Kerr solution describing a rotating black hole.

The lecture is organized as follows: In Section 2, a brief sketch of the method for solving boundary value problems of the axisymmetric, stationary vacuum Einstein equations is given. In section 3 the boundary value problem related to the dust disk and its solution are discussed. The black hole limit of the solution is investigated in some detail.

I would like to emphasize that most of the material of this lecture is based on the joint work with Gernot Neugebauer and Andreas Kleinwächter ([5] – [12]).

2 Solution of boundary value problems to the axisymmetric, stationary vacuum Einstein equations

2.1 The linear system

The axisymmetric, stationary vacuum Einstein equations (equivalent to the so–called Ernst equation, see [13], [14]) are the *integrability condition* of a related *linear system* ([15] – [18]). Neugebauer's form [19] of the linear system reads

$$\Phi,_z = \left\{ \begin{pmatrix} N & 0 \\ 0 & M \end{pmatrix} + \lambda \begin{pmatrix} 0 & N \\ M & 0 \end{pmatrix} \right\} \Phi, \tag{1}$$

$$\Phi,_{\bar z} = \left\{ \begin{pmatrix} \bar M & 0 \\ 0 & \bar N \end{pmatrix} + \frac{1}{\lambda} \begin{pmatrix} 0 & \bar M \\ \bar N & 0 \end{pmatrix} \right\} \Phi. \tag{2}$$

$\Phi(z, \bar z, \lambda)$ is a 2×2 – matrix function depending on

$$z = \rho + i\zeta, \quad \bar z = \rho - i\zeta \tag{3}$$

and

$$\lambda = \sqrt{\frac{K - i\bar z}{K + iz}}, \tag{4}$$

where K is an additional (complex) parameter, called the 'spectral parameter', which does not depend on the coordinates ρ and ζ. (ρ and ζ are cylindrical coordinates, ρ is the distance to the symmetry – [ζ –] axis.) A bar denotes complex conjugation. The scalar functions M and N do not depend on λ:

$$M = M(z, \bar z), \quad N = N(z, \bar z). \tag{5}$$

The integrability condition

$$\Phi,_{z\bar z} = \Phi,_{\bar z z} \tag{6}$$

leads to a <u>first order system</u>[1] of nonlinear partial differential equations for M, N, $\bar M$ and $\bar N$ which is equivalent to

$$M = \frac{f,_z}{f + \bar f}, \quad N = \frac{\bar f,_z}{f + \bar f} \tag{7}$$

[1] To obtain this system, one has to use the relations

$$\lambda,_z = \frac{\lambda}{4\rho}(\lambda^2 - 1), \quad \lambda,_{\bar z} = \frac{1}{4\rho\lambda}(\lambda^2 - 1)$$

following from (4). Comparing the coefficients of different powers of λ in the resulting equations (they must be valid for all K!) one obtains

$$N,_{\bar z} = N(\bar M - \bar N) - \frac{1}{4\rho}(N + \bar M), \quad M,_{\bar z} = M(\bar N - \bar M) - \frac{1}{4\rho}(M + \bar N)$$

and the complex conjugate relations.

and the Ernst equation

$$(\Re f)(f,_{\rho\rho} + f,_{\zeta\zeta} + \frac{1}{\rho} f,_{\rho}) = f,_{\rho}^{2} + f,_{\zeta}^{2} \tag{8}$$

for the complex function $f(\rho, \zeta)$, called the Ernst potential. (The relation to the metric can be found in section 3.1.)

The existence of such a linear system with a spectral parameter allows for the construction of exact solutions of the corresponding nonlinear partial differential equation (here: the Ernst equation), e.g. by means of Bäcklund transformations [17], [18]. These solutions contain an arbitrary number of free parameters. More importantly, it is even possible to construct solutions containing *free functions*. In this way, one can solve, in principle, initial and/or boundary value problems. This method was discovered by Gardner, Greene, Kruskal and Miura in 1967 [20] as a method for solving the Cauchy problem of the Korteweg–de Vries (KdV) equation. The term 'inverse scattering method' comes from the fact that one step of the solution procedure consists in solving an inverse scattering problem for the one–dimensional stationary Schrödinger equation which plays the role of one part of the linear system related to the KdV equation.

The general idea behind is the discussion of the matrix function Φ as a function of the complex spectral parameter K or, in our case, as a function of λ. (ρ and ζ play the role of parameters in this context.) It is possible to obtain solutions containing free functions via the solution of related Riemann–Hilbert problems in the complex λ–plane. This leads to linear integral equations, cf. [21].

2.2 The Riemann–Hilbert technique

A quite general solution Φ of the linear system (1), (2) can be obtained by solving a matrix Riemann–Hilbert problem: This problem consists in finding a $\Phi(\lambda)$ that is holomorphic for all values of λ in the complex plane except those which lie on some closed curves Γ and Γ' defined by (4) and

$$K \in \Gamma_K, \tag{9}$$

with Γ_K being a closed curve in the complex K–plane which is symmetric with respect to the real axis. (There exist two curves in the λ–plane since $\lambda(K)$ is double–valued; $\lambda \in \Gamma \Leftrightarrow -\lambda \in \Gamma'$.) On Γ and Γ' the following jump relations shall be satisfied:

$$\Phi_i = \Phi_e \, C(K) \quad \text{on } \Gamma, \quad \Phi_i = \Phi_e \, C'(K) \quad \text{on } \Gamma', \tag{10}$$

where Φ_i and Φ_e denote the values of Φ that appear by approaching the contour from inside and outside, respectively. It is assumed that the jump matrices $C(K)$ and $C'(K)$ do not depend on z and $\bar{z}$. As a consequence, the expressions $\Phi,_z \Phi^{-1}$ and $\Phi,_{\bar{z}} \Phi^{-1}$ *do not jump* on Γ and Γ'. Moreover, together with some additional assumptions in case of zeros of the determinant $\det\Phi(\lambda)$, one can show that $\Phi_{,z}\,\Phi^{-1}$ and $\Phi_{,\bar{z}}\,\Phi^{-1}$ are holomorphic functions of λ everywhere except at the points $\lambda=\infty$ and $\lambda=0$, respectively. There, in agreement with (1) and (2), simple poles occur – provided

$$0 \notin \Gamma, \quad \infty \notin \Gamma. \tag{11}$$

Some constraints on the jump matrices C, C' and a suitable normalization condition ensure that $\Phi_{,z}\,\Phi^{-1}$ and $\Phi_{,\bar{z}}\,\Phi^{-1}$ have exactly the structure as in (1), (2), and one can read off the M's and N's or calculate the Ernst potential $f(\rho,\zeta)$. The solution of a matrix Riemann–Hilbert problem can be found via a system of *linear* integral equations. In this way a solution of the Ernst equation is obtained which depends on free functions (some elements of the jump matrices which can be choosen arbitralily). Normally, this solution is regular for all values of ρ and ζ satisfying the condition (11), i.e., a curve Σ in the ρ–ζ–plane has to be excluded [cf. (4)]:

$$\Sigma: \quad \rho=|\Im K|, \quad \zeta=\Re K, \quad K\in\Gamma_K. \tag{12}$$

This defines the surface of a body of revolution. One can try to solve boundary value problems, e.g. of the Dirichlet type, with boundary data given on Σ. The inverse scattering method would then consist of the following steps:

1. Determination of the jump matrices $C(K)$ and $C'(K)$ from the boundary data. [The contour Γ_K follows from the surface Σ according to (12)].

2. Solution of the Riemann–Hilbert problem.

3. Calculation of $f(\rho,\zeta)$ from $\Phi(z,\bar{z},\lambda)$.

The first step is the most difficult one. It has to be solved by considering the linear system (1), (2) along the boundary Σ. (In the case of the application of the method to the solution of the Cauchy problem of the KdV equation the first step is simpler and consists in solving a 'direct' scattering problem: One has to determine the 'scattering data' for a given potential.)

The second step consists in the solution of a system of linear integral equations. (In the KdV case it corresponds to the inverse scattering problem: One has to reconstruct a potential from scattering data. This leads to the famous Gelfand–Levitan–Marchenko integral equation.)

The third step is trivial and provides us with the desired solution of the boundary value problem.

In the next section we discuss the problem of the rigidly rotating disk of dust, which is the first example of a successful application of the procedure outlined.

3 The rigidly rotating disk of dust

3.1 The boundary value problem

The disk of dust is characterized by the following energy–momentum tensor:

$$T^{ik} = \epsilon u^i u^k, \quad u^i = e^{-V}(\xi^i + \Omega\eta^i), \tag{13}$$

with the mass–energy density ϵ and the four–velocity u^i. ξ^i and η^i are the Killing vectors corresponding to stationarity and axisymmetry, respectively. The (positive) scalar $\exp(-V)$ follows from $u^i u_i = -1$, and Ω is the angular velocity as measured by an observer at infinity. Rigid rotation means

$$\Omega = constant. \tag{14}$$

The line element can be written in the Weyl–Lewis–Papapetrou form

$$ds^2 = e^{-2U}[e^{2k}(d\rho^2 + d\zeta^2) + \rho^2 d\varphi^2] - e^{2U}(dt + a\, d\varphi)^2, \tag{15}$$

$$0 \le \rho < \infty, \quad -\infty < \zeta < \infty, \tag{16}$$

where $\exp(2U)$, $\exp(2k)$ and a depend on ρ and ζ only. The Killing vectors, in these coordinates, are given by $\xi^i = \delta^i_t$, $\eta^i = \delta^i_\varphi$. (Note, that we use units where the velocity of light c as well as Newton's gravitational constant G are equal to 1.) The disk is defined by $\zeta = 0$, $\rho \le \rho_0$; ρ_0 being the (coordinate) radius of the

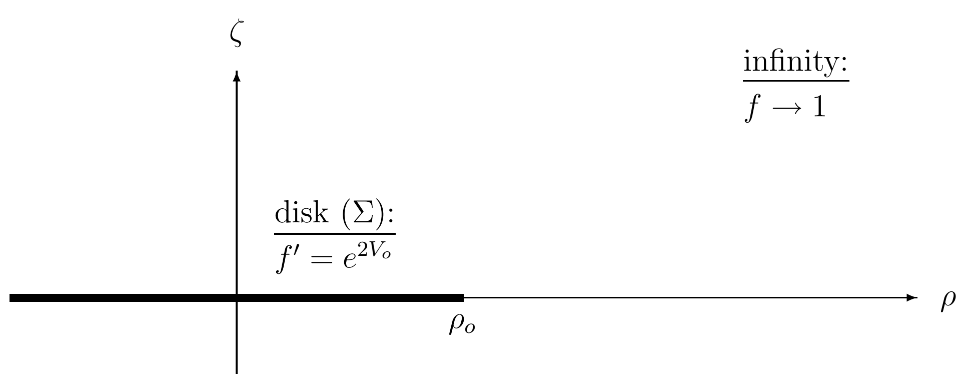

Figure 1: The boundary value problem.

disk. The mass density $\epsilon(\rho, \zeta)$ can be written formally as

$$\epsilon = \delta(\zeta) e^{U-k} \sigma_p(\rho), \tag{17}$$

where $\delta(\zeta)$ is Dirac's delta function and $\sigma_p(\rho)$ is the (proper) surface mass–density. The boundary conditions can easily be obtained by integrating the Einstein equations across the disk, taking into account that the metric coefficients are continuous, but their normal derivatives may jump. Together with the symmetry of the problem with respect to the plane $\zeta = 0$ one obtains

$$e^{2U'} = e^{2V_0} = constant, \quad a'_{,\zeta} = 0, \quad k'_{,\zeta} = 0 \quad \text{as} \quad \zeta = 0, \quad \rho \leq \rho_0. \tag{18}$$

The primed quantities refer to the corotating frame of reference defined by

$$\rho' = \rho, \quad \zeta' = \zeta, \quad \varphi' = \varphi - \Omega t, \quad t' = t. \tag{19}$$

The constant V_0 is related to the relative redshift z_0 measured by an observer at infinity for photons coming from the center of the disk:

$$z_0 = e^{-V_0} - 1. \tag{20}$$

[Note that the V in (13) turns out to be constant and equal to V_0.] The surface mass–density can be calculated (after one has solved the global problem) by

$$\sigma_p = \frac{1}{2\pi} e^{U-k} \, U'_{,\zeta} \bigg|_{\zeta=0^+} . \tag{21}$$

To formulate the global problem as a boundary value problem for the Ernst equation, we need the connection of the metric functions with the complex Ernst potential $f(\rho,\zeta)$:

$$f = e^{2U} + ib, \tag{22}$$

i.e., the real part of the Ernst potential is just equal to $\exp(2U)$. The imaginary part b is related to the gravitomagnetic potential a:

$$a_{,\rho} = \rho e^{-4U} b_{,\zeta} \, , \quad a_{,\zeta} = -\rho e^{-4U} b_{,\rho} \, . \tag{23}$$

Hence $a(\rho,\zeta)$ can be calculated via a line integral from the Ernst potential and its derivatives. This integral is path–independent. [The integrability condition is satisfied as a consequence of (8).] Similarly, the metric function $k(\rho,\zeta)$ follows from

$$k_{,\rho} = \rho[U_{,\rho}^2 - U_{,\zeta}^2 + \frac{1}{4} e^{-4U} (b_{,\rho}^2 - b_{,\zeta}^2)], \quad k_{,\zeta} = 2\rho(U_{,\rho} U_{,\zeta} + \frac{1}{4} e^{-4U} b_{,\rho} b_{,\zeta}). \tag{24}$$

Note that, for regularity reasons, $a = 0$ and $k = 0$ on the symmetry axis ($\rho = 0$), and we obtain, for example,

$$a = \int_0^\rho \tilde{\rho} e^{-4U} b_{,\zeta} \, d\tilde{\rho}, \tag{25}$$

$$k = \int_0^{\rho} \tilde{\rho}[U,_{\tilde{\rho}}^2 - U,_{\zeta}^2 + \frac{1}{4}e^{-4U}(b,_{\tilde{\rho}}^2 - b,_{\zeta}^2)]d\tilde{\rho}. \tag{26}$$

[In the integrands, one has $U = U(\tilde{\rho}, \zeta)$ and $b = b(\tilde{\rho}, \zeta)$.]

Now, from (18), (22) and (23) we conclude that the boundary condition for the 'corotating' Ernst potential f' is simply given by [5]

$$f' = e^{2V_0} = constant \quad \text{as} \quad \zeta = 0, \quad \rho \leq \rho_0. \tag{27}$$

It should be noted that we have combined the result $b',_{\rho} = 0$ and the freedom of adding an imaginary constant to the Ernst potential, to set $b' = 0$ in the disk. The simple condition (27) for f' in the corotating system corresponds to a quite complicated, nonlocal boundary condition[2] for f in the original system. On the other hand, the condition of asymptotic flatness is much simpler for f:

$$f \to 1 \quad \text{as} \quad \rho^2 + \zeta^2 \to \infty. \tag{28}$$

This condition ensures that $U \to 0$, $k \to 0$ and $a \to 0$ at infinity, i.e., the line element (15) becomes Minkowskian (in cylindrical coordinates).

The global problem one has to solve, is to find a (or: *the*) solution of the Ernst equation (8) which satisfies (27) and (28), and which is *regular everywhere outside the disk*, cf. Figure 1.

The solution depends on two parameters only. One can choose, e.g., V_0 and ρ_0, or V_0 and Ω. A relation $\Omega = \Omega(V_0, \rho_0)$, see next section, follows from the regularity condition at the rim of the disk.

3.2 The solution

The solution was obtained by applying the method outlined in Section 2. The curve Γ_K in the complex K–plane is a part of the imaginary axis (from $-i\rho_0$ to $+i\rho_0$) corresponding to the surface Σ, i.e. the disk, cf. (12) and Figure 1.

The first step of the solution procedure (the determination of the jump matrices) lead to the 'small' integral equation [5] which could be solved in terms of elliptic functions [6]. The 'big' integral equation [5] corresponding to the second step (the solution of the matrix Riemann–Hilbert problem) could be solved in

[2] The coordinate transformation (19) preserves the form of the line element (15). One obtains

$$e^{2U'} = e^{2U}[(1+\Omega a)^2 - \Omega^2\rho^2 e^{-4U}], \quad (1-\Omega a')e^{2U'} = (1+\Omega a)e^{2U}, \quad e^{2k'-2U'} = e^{2k-2U}.$$

To get b', one has to integrate the (primed) relations (23). That means, the boundary condition for f is given by complicated, nonlinear and nonlocal relations corresponding to $f' = e^{2V_0}$. Note that these boundary conditions are equivalent to those applied by Bardeen and Wagoner [3], [4]. V_0 is identical with the parameter ν_c of Bardeen and Wagoner.

terms of hyperelliptic functions and lead to the following result for the Ernst potential [7]:

$$f = \exp\left\{\int_{K_1}^{K_a} \frac{K^2 dK}{Z} + \int_{K_2}^{K_b} \frac{K^2 dK}{Z} - v_2\right\}, \tag{29}$$

with

$$Z = \sqrt{(K + iz)(K - i\bar{z})(K^2 - K_1^2)(K^2 - K_2^2)}, \tag{30}$$

$$K_1 = \rho_0 \sqrt{\frac{i - \mu}{\mu}} \quad (\Re K_1 < 0), \quad K_2 = -\bar{K}_1. \tag{31}$$

The real (positive) parameter μ is given by

$$\mu = 2\Omega^2 \rho_0^2 e^{-2V_0}. \tag{32}$$

The upper integration limits K_a and K_b in (29) have to be calculated from

$$\int_{K_1}^{K_a} \frac{dK}{Z} + \int_{K_2}^{K_b} \frac{dK}{Z} = v_0, \quad \int_{K_1}^{K_a} \frac{K dK}{Z} + \int_{K_2}^{K_b} \frac{K dK}{Z} = v_1, \tag{33}$$

where the functions v_0, v_1 and v_2 in (33) and (29) are given by

$$v_0 = \int_{-i\rho_0}^{+i\rho_0} \frac{H}{Z_1} dK, \quad v_1 = \int_{-i\rho_0}^{+i\rho_0} \frac{H}{Z_1} K dK, \quad v_2 = \int_{-i\rho_0}^{+i\rho_0} \frac{H}{Z_1} K^2 dK, \tag{34}$$

$$H = \frac{\mu \ln\left[\sqrt{1 + \mu^2(1 + K^2/\rho_0^2)^2} + \mu(1 + K^2/\rho_0^2)\right]}{\pi i \rho_0^2 \sqrt{1 + \mu^2(1 + K^2/\rho_0^2)^2}} \quad (\Re H = 0), \tag{35}$$

$$Z_1 = \sqrt{(K + iz)(K - i\bar{z})} \quad (\Re Z_1 < 0 \quad \text{for } \rho, \zeta \text{ outside the disk}). \tag{36}$$

In (34) one has to integrate along the imaginary axis. The integrations from K_1 to K_a and K_2 to K_b in (29) and (33) have to be performed along the same paths in the two–sheeted Riemann surface associated with $Z(K)$. The problem of finding K_a and K_b from (33) is a special case of Jacobi's inversion problem. It generalizes the definition of elliptic functions and can be solved in terms of hyperelliptic theta functions ([22], [23], see also [24] – [26]). Using a formula for Abelian integrals of the third kind derived by Riemann (see [24]) it is also possible to express the Ernst potential f directly in terms of theta functions [12]. On the symmetry axis ($\rho = 0$) and in the plane of the disk ($\zeta = 0$) all integrals in (29) and (33) are reduced to elliptic ones [6].

The solution (29) satisfies the boundary conditions (27) and (28), has a positive surface mass– (particle number–) density (vanishing at the rim of the disk), and it is regular everywhere outside the disk – provided

$$0 < \mu < \mu_0 = 4.62966184\ldots \tag{37}$$

(for $\mu > \mu_0$ one or more singular rings appear in the plane $\zeta = 0$, outside the disk). The interesting behaviour for $\mu \to \mu_0$ will be discussed in the next section.

Note that the solution in the form (29) – (36) depends on the parameters ρ_0 and μ only. Since $\exp(2U') = \exp(2U)$ on the symmetry axis ($\rho = 0$), one can calculate the parameter V_0 [cf. (18)] from $\Re f(\rho = 0, \zeta = 0^+)$. The result is [6]:

$$V_0 = -\frac{1}{2}\sinh^{-1}\left\{\mu + \frac{1+\mu^2}{\wp[I(\mu); \frac{4}{3}\mu^2 - 4, \frac{8}{3}\mu(1+\mu^2/9)] - \frac{2}{3}\mu}\right\}, \tag{38}$$

$$I(\mu) = \frac{1}{\pi}\int\limits_0^\mu \frac{\ln(x+\sqrt{1+x^2})dx}{\sqrt{(1+x^2)(\mu - x)}} \tag{39}$$

($\wp$ is the Weierstraß function[3]), i.e., V_0 depends on μ alone. The range $0 < \mu < \mu_0$ corresponds to $0 > V_0 > -\infty$. In this range, the relation (38) can be inverted uniquely to give $\mu(V_0)$. [μ_0 is the first zero of the denominator in (38).] Then, from the definition (32) one obtains the relation $\Omega(V_0, \rho_0)$. (Without loss of generality, we assumed $\Omega > 0$; the solution for negative Ω is simply given by $\bar{f}$.) Alternatively, one can use V_0 and Ω as the primary parameters, with $\rho_0 = \rho_0(V_0, \Omega)$.

3.3 Discussion: ¿From the Newtonian limit to the black hole limit

For $0 < \mu < \mu_0$, the solution (29) can be expanded in terms of $\mu^{1/2}$:

$$f = 1 + \sum_{n=1}^{\infty} f_n\, \mu^{(n+1)/2} = 1 + f_1\, \mu + \mathcal{O}(\mu^{3/2}). \tag{40}$$

The f_n ($n = 1, 2, \ldots, \infty$) are elementary functions of ρ and ζ. (This series corresponds to the Bardeen–Wagoner expansion[4].) The Newtonian limit ($\mu \ll 1$) is represented by f_1:

$$f_1 = -\frac{1}{\pi}\left\{\frac{4}{3}\cot^{-1}\xi + [\xi - (\xi^2 + \frac{1}{3})\cot^{-1}\xi](1 - 3\eta^2)\right\}, \tag{41}$$

with elliptic coordinates ξ and η:

$$\rho = \rho_0\sqrt{1+\xi^2}\sqrt{1-\eta^2}, \quad \zeta = \rho_0\xi\eta \quad (0 \le \xi < \infty,\ -1 \le \eta \le 1) \tag{42}$$

[3] The Weierstraß function $\wp(x; g_2, g_3)$ is defined by

$$\int\limits_{\wp(x;g_2,g_3)}^{\infty} \frac{dt}{\sqrt{4t^3 - g_2 t - g_3}} = x.$$

[4] The expansion parameter γ used by Bardeen and Wagoner [3], [4] is related to μ by $\gamma = 1 - e^{V_0(\mu)} = \mu/2 + \mathcal{O}(\mu^2)$.

($\xi = 0$ is the disk). The Newtonian potential U_N is obtained from

$$g_{tt} = -\Re f = -(1 + f_1\,\mu) + \mathcal{O}(\mu^{3/2}) = -(1 + \frac{2U_N}{c^2}) + \mathcal{O}(\frac{1}{c^3}), \qquad (43)$$

i.e.,

$$U_N = \Omega^2 \rho_0^2\, f_1. \qquad (44)$$

This is exactly the Maclaurin solution. [Note that we have reintroduced the velocity of light c into Eq. (43). From (32) and (38) one obtains $\mu = 2\Omega^2\rho_0^2/c^2 + \mathcal{O}(\mu^2)$.]

With increasing μ, characteristic deviations from the Newtonian solution occur. This concerns, e.g., the radial distribution of the surface mass–density [4], [6]. The gravitomagnetic potential leads to dragging effects and, for $\mu > 1.68849\ldots$, even to the formation of an ergoregion [4], [11]. Some illustrations can also be found in [12].

However, the most striking difference to the Newtonian case is the following [4], [5]:

For given angular momentum the mass of the disk is bounded by

$$\frac{M^2}{J} < 1, \qquad (45)$$

where M and J denote the total (gravitational) mass and the (ζ–component of the) total angular momentum, respectively ($c = G = 1$ again). The equality $M^2/J = 1$ is reached in the limit $\mu \to \mu_0$, the **black hole limit**.

For $\mu \to \mu_0$, one has $V_0 \to -\infty$, cf. (38). As a consequence of (32), for nonvanishing Ω, this results in

$$\rho_0 \to 0. \qquad (46)$$

For $\rho^2 + \zeta^2 \neq 0$, we obtain from (29) in the limit $\mu \to \mu_0$

$$f = \frac{2\Omega r - 1 - i\cos\vartheta}{2\Omega r + 1 - i\cos\vartheta} \qquad (r > 0), \qquad (47)$$

$$\rho = r\sin\vartheta, \quad \zeta = r\cos\vartheta \quad (0 \le \vartheta \le \pi). \qquad (48)$$

This is exactly the ($r > 0$ part of the) <u>extreme Kerr solution</u>[5] with

$$M = \frac{1}{2\Omega}\,, \quad J = \frac{1}{4\Omega^2}\,, \qquad (49)$$

[5] To derive (47) from (29), let us first rewrite (29) and (33) in the equivalent form

$$f = \exp\left\{\int_{K_b}^{K_a} \frac{K^2 dK}{Z} - \tilde{v}_2\right\}, \quad \int_{K_b}^{K_a} \frac{dK}{Z} = \tilde{v}_0, \quad \int_{K_b}^{K_a} \frac{K dK}{Z} = \tilde{v}_1,$$

with

$$\tilde{v}_n = v_n - \int_{K_1}^{K_2} \frac{K^n dK}{Z} \quad (n = 0, 1, 2).$$

i.e.,

$$\frac{M^2}{J} = 1. \tag{50}$$

(Remember that we assumed $\Omega > 0$.) In the coordinates used, the horizon of the extreme Kerr black hole is just given by (the excluded) $r = 0$. Ω plays the role of the 'angular velocity of the horizon'. In the corotating system (19), one obtains

$$f' = -\Omega^2 r^2 \left[\frac{2(1 + i\cos\vartheta)^2}{2\Omega r + 1 - i\cos\vartheta} + \sin^2\vartheta \right]. \tag{51}$$

It can be seen, that the boundary condition (27) with $V_0 \to -\infty$ is indeed satisfied on the horizon ($r = 0$).

A completely different limit of the space–time, for $\mu \to \mu_0$, is obtained for finite values of r/ρ_0 (corresponding just to the previously excluded $r = 0$). Therefore, we consider a coordinate transformation [4]

$$\tilde{r} = r\, e^{-V_0}, \quad \tilde{\varphi} = \varphi - \Omega t, \quad \tilde{\vartheta} = \vartheta, \quad \tilde{t} = t\, e^{V_0}. \tag{52}$$

(Note that finite r/ρ_0 correspond to finite $\tilde{r}$ in the limit.) For $\mu < \mu_0$, this is nothing but the transformation to the corotating system (19) combined with a rescaling of r and t. The transformed Ernst potential $\tilde{f}$ is related to f' according to $\tilde{f} = f' \exp(-2V_0)$, i.e.,

$$\frac{\tilde{f}}{\tilde{r}^2} = \frac{f'}{r^2} \quad \text{as} \quad \mu < \mu_0. \tag{53}$$

However, for $\mu \to \mu_0$, the solutions f' (finite r) and $\tilde{f}$ (finite $\tilde{r}$) separate from each other. (A similar phenomenon has been observed by Breitenlohner *et al.* for some limit solutions of the static, spherically symmetric Einstein–Yang–Mills–Higgs equations [27].) For finite r, the extreme Kerr solution (51) arises, while finite $\tilde{r}$ lead to a solution which still describes a disk. This solution (which can be expressed in terms of theta functions) is regular everywhere outside the disk, but it is *not asymptotically flat*, i.e., it can be considered as a cosmological solution. The space–time structure of both solutions (f' and $\tilde{f}$) coincides at $r = 0$ (the horizon) and $\tilde{r} \to \infty$ (spatial infinity). The relation (53) survives in the form

$$\lim_{\tilde{r}\to\infty} \frac{\tilde{f}}{\tilde{r}^2} = \lim_{r\to 0} \frac{f'}{r^2} \quad \text{as} \quad \mu \to \mu_0. \tag{54}$$

(K_b is now on the other sheet of the Riemann surface.) In the limit $\mu \to \mu_0$ one obtains for $r > 0$, using (38),

$$\tilde{v}_0 = \frac{2\Omega}{r} - \frac{\pi i \cos\vartheta}{2r^2}, \quad \tilde{v}_1 = -\frac{\pi i}{2r}, \quad \tilde{v}_2 = 0$$

(modulo periods). In the above integrals from K_b to K_a, because of $K_1 \to 0$, $K_2 \to 0$ [cf. (31)], Z can be replaced by $Z = K^2\sqrt{(K+iz)(K-i\bar{z})}$. Hence, all integrals become elementary and the result (47) can easily (and uniquely) be obtained.

Accordingly, for $\mu \to \mu_0$ and $\tilde{r} \to \infty$,

$$\tilde{f} \to \tilde{f}_{as} = -\Omega^2 \tilde{r}^2 \left[\frac{2(1 + i\cos\tilde{\vartheta})^2}{1 - i\cos\tilde{\vartheta}} + \sin^2\tilde{\vartheta} \right]. \tag{55}$$

Note that $\tilde{f}_{as}$ belongs to the family of solutions to the Ernst equation of the type $f = r^k Y_k(\cos\vartheta)$ presented by Ernst [28]. The corresponding *asymptotic* line element is given by the following exact solution of the vacuum Einstein equations:

$$ds^2 = e^{-2\tilde{U}}[e^{2\tilde{k}}(d\tilde{r}^2 + \tilde{r}^2 d\tilde{\vartheta}^2) + \tilde{r}^2 \sin^2\tilde{\vartheta}\, d\tilde{\varphi}^2] - e^{2\tilde{U}}(d\tilde{t} + \tilde{a}\, d\tilde{\varphi})^2 \quad , \tag{56}$$

$$e^{2\tilde{U}} = \Omega^2 \tilde{r}^2 \cdot \frac{\cos^4\tilde{\vartheta} + 6\cos^2\tilde{\vartheta} - 3}{\cos^2\tilde{\vartheta} + 1} \quad , \tag{57}$$

$$\tilde{a} = \frac{2}{\Omega^2 \tilde{r}} \cdot \frac{\cos^2\tilde{\vartheta} - 1}{\cos^4\tilde{\vartheta} + 6\cos^2\tilde{\vartheta} - 3} \quad , \tag{58}$$

$$e^{2\tilde{k}} = \frac{1}{4}(\cos^4\tilde{\vartheta} + 6\cos^2\tilde{\vartheta} - 3)\,. \tag{59}$$

These analytical results prove the conjectures formulated by Bardeen and Wagoner [4] on the basis of their numerical results.

Let me conclude with a quotation from Bardeen and Wagoner ([4], page 411): *'The picture we have developed of the extreme relativistic limit of a rotating disk is that it becomes buried in the horizon of the $J = M^2$ Kerr metric, surrounded by its own infinite, non asymptotically flat universe. As approached from the $r > 0$ Kerr region, the disk represents a "singularity" in the horizon, since the whole range $0 \leq \tilde{r} < \infty$, over which there exist considerable changes in the local geometry, corresponds to an infinitesimal range of affine parameter for a typical photon which reaches the horizon from the outside.'*

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