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THE APPLICATION OF THE SOLID-ANGLE THEOREM IN THE SPECIAL THEORY OF RELATIVITY†

G. B. MALYKIN

Nizhnii Novgorod (e-mail: malykin@appl.sci-nnov.ru)

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Ishlinskii's theorem, well known in classical mechanics, asserts that if an axis, selected in a rigid body, having zero projection of the angular velocity onto this axis, described a closed conical surface during the motion of the body, then, after the axis has returned to its initial position the body will have described an angle around it numerically equal to solid angle of the described cone. It is shown that the same relation also exists in the Special Theory of Relativity—the angle of rotation described by a rigid body during motion along a curvilinear trajectory due to the Thomas precession effect, is numerically equal to the solid angle observed in a fixed frame of reference described by an axis connected with the body due to a change in the rotation of the image of the rigid body. The latter phenomenon is due to the Lorentz contraction of the length and the retardation of light radiated by different parts of the body [10–13]. © 2000 Elsevier Science Ltd. All rights reserved.

1. ISHLINSKII'S THEOREM AND ITS APPLICATIONS

More than half a century ago Ishlinskii proved a theorem, which is also called the solid-angle theorem [1–4] (see also [5–7]), which can be formulated as follows [7]. If a certain axis, selected in a rigid body with three degrees of freedom, describes a closed conical surface during the motion of the body, and the projection of the angular velocity of the body onto this axis is zero, then, after the axis has returned to its initial position the body will have rotated around it by an angle numerically equal to the solid angle of the cone described (Fig. 1). Note that this equality is satisfied to within $2\pi N$, where N is an integer [14]. The translational motion of the axis if of no importance here.

We will give a particular example of the occurrence of the Ishlinskii effect, which is sometimes called non-commutativity in classical mechanics. If the axis on which a flywheel is mounted without fiction and is at rest at the initial instant of time describes a certain solid angle in space, then, when the axis returns to its position the flywheel will have rotated through an angle numerically equal to the solid angle described by the axis [2, 3]. It has been suggested [15] that this additional angle of rotation, acquired by the body during spatial evolution, should be called the Ishlinskii angle. It was shown in [15] that the Ishlinskii angle is a manifestation of a geometrical (topological) phase, which, is often called the Berry phase [16] in classical mechanics. If the axis describes the same conical surface, but in the opposite directions, the absolute value of the Ishlinskii angle for opposite directions of displacement will be the same, but the sign will be different.

The value of the Ishlinskii angle is not determined by the initial and final position of the axis in space, but depends on the trajectory which the axis describes during spatial evolution. Consequently, the buildup of the Ishlinskii angle is a non-holonomic phenomenon [17]. This also follows from the fact that, as was shown in [2, 3], the manifestation of the Ishlinskii effect in a mechanical system occurs when there are non-holonomic constraints in this system.

Ishlinskii's theorem finds applications in the gyroscope. In particular, it explains the occurrence of an angular error in a spatial gyrocompass—a gyroframe, inside which there are two coupled mechanical gyroscopes, the axes of which are parallel, and also of gyroscopes with a strong correction. This error is due to the change in the spatial orientation of the vertical axis around which the gyroframe freely rotates or, correspondingly, the external collar of the gyroscope with the strong correction, when the gyrocompass is displaced along the Earth's surface [2, 3]. Note that this effect is intimately related to the so-called translation of a vector in Riemannian geometry [2, 3].

2. THE THOMAS PRECESSION

We will consider the Thomas precession effect and its physical consequences. As was noted above, this effect leads to the rotation of a gyroscope moving along a curvilinear trajectory. In general we mean by a gyroscope a certain rigid body or point mass which is given a certain direction in space.

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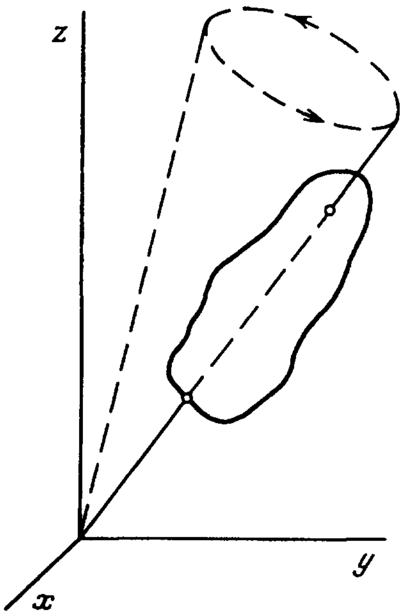

Fig. 1.

An example of this is a rigid body, having spherical symmetry about the centre of gravity, moving in a gravity field along a curvilinear trajectory—a circular or elliptic orbit, or along a parabola, retaining its spatial orientation by virtue of the inertia law. Another example is a point mass with a spin—an electron, neutron, etc.

In a laboratory reference frame the expression for the angular velocity of precession has the form [9]

$$\Omega_T = (1 - 1/\gamma)[\mathbf{v} \times \mathbf{a}]/v^2, \quad \gamma = (1 - v^2/c^2)^{-1/2} \tag{2.1}$$

where **v** and **a** are the velocity and acceleration in the laboratory reference frame and c is the velocity of light. In the special case when the motion is along a circle of radius r with angular velocity $\omega = v/r$

$$\Omega_T = \omega(1 - \sqrt{1 - v^2/c^2}) \tag{2.2}$$

In body axes reference frame the values of the angular velocity of the Thomas precession is γ times greater than in the laboratory reference frame (2.1) and (2.2).

After one revolution in a circle the angle of rotation of the body is

$$\alpha = 2\pi\Omega_T/\omega = 2\pi(1 - \sqrt{1 - v^2/c^2}) \tag{2.3}$$

3. THE OBSERVED ROTATION OF AN OBJECT MOVING RAPIDLY IN A CIRCLE AND THE THOMAS PRECESSION

It was noted in [11, 12] that light quanta arriving simultaneously at an observer were emitted by different points of the object at a different time–points situated further from the observer emitted quanta earlier than closer points. For this reason an effect occurs which compensates the Lorentz contraction, and when the dimensions of the object are much less than the distance to it, the object, and more correctly, its image on the retina of the eye of the observer or on the film of a camera, appears to be undistorted, and only rotated by a certain angle. A large number of papers have been devoted to this problem; it is considered in most detail in [13, 14].

In this case the question of the change in the orientation of an object which moves along a circular trajectory, recorded by a fixed observer, is of interest. This angle may be specified by the form of the object, for example, one of its surfaces if it has the form of a polyhedron, or an axis connected with the object.

In the simple case when the object moves rectilinearly with a velocity **v**, the expression relating the angle Θ', which specifies a certain direction on the object in a reference frame connected with the object, and the angle Θ at which this direction is observed in the laboratory (fixed) reference frame, is determined by the well-known formulae which define the relativistic aberration, and has the form [13, 18]

$$\sin\Theta = \frac{\sqrt{1-v^2/c^2}\,\sin\Theta'}{1+\cos\Theta' v/c} \tag{3.1}$$

A fixed observer sees the object rotated by the aberration angle $\Delta\Theta = \Theta - \Theta$. Suppose the object moves rectilinearly in a plane which is orthogonal to the straight line connecting the object and the observer, while the axis which specifies the direction also lies in this plane, i.e. $\Theta' = \pi/2$. From relation (3.1) we have

$$\Delta\Theta = \Theta - \pi/2, \quad \cos(\Delta\Theta) = \sqrt{1-v^2/c^2}$$

We will now consider the case when the object moves in a circle in the plane considered, and, as previously, $\Theta' = \pi/2$. Here the direction of the axis observed by a fixed observer will be changed, the image of the object will be rotated, and in one rotation of the object the axis describes a cone with an angle at the vertex of $2\Delta\Theta$. The value of the solid angle, contained inside the cone, is numerically equal to the area bounded on a sphere of unit radius by the generatrix of the cone, the vertex of which is situated at the centre of the sphere. Hence we obtain an expression relating the value of the solid angle and the angle at the vertex of the cone

$$\chi = 4\pi\sin^2(\Delta\Theta/2) = 2\pi(1-\cos(\Delta\Theta)) = 2\pi(1-\sqrt{1-v^2/c^2}) \tag{3.2}$$

This phenomenon can be illustrated using the example of the images of a dice (a cube) moving along a circular trajectory, as seen by a fixed observer; the dice during motion retains its orientation in space (Figs 2 and 3). The dice rotates in a plane which is orthogonal to the line connecting the centre of the circle with the point at which the observer is situated (the pupil of the observer's eye), where the distance from the observer to the plane is much greater than the diameter of the circle. When the dice is at the upper point of the circle (position 1 in Figs 2 and 3), it is orientated in such a way that the face turned to the observer, which shows a "six", ahead of the motion is a "four", behind a "three", upwards a "five", downwards a "two", and on the side opposite to the observer's position, a "one". The direction of motion of the dice at each point is shown by the arrow. If the dice velocity $v \ll c$, a fixed observer in all positions of the dice sees the side showing a "six", and the orientation of the image during the motion does not change (Fig. 2). When $v \sim c$ at different points of the circle the observer sees a dice turned at different angles to him (Fig. 3). Hence, the observer sees that after one rotation of the dice, an axis connected with it (for example, the edge of the cube) describes a certain solid angle. The representation shown in Fig. 3 corresponds to the case $\gamma = 2$.

Comparing expressions (2.3) and (3.2) we obtain that $\alpha \equiv \chi$, i.e. the angle of rotation of a body due to the Thomas precession is equal to the Ishlinskii angle which the body acquires when it moves in a circle if the actual change in its angle of orientation is equal to the change in the angle of rotation, observed in the laboratory reference frame, of a body moving relativistically along a curvilinear trajectory. Hence, the Thomas precession can be interpreted as a consequence of the formal application of Ishlinskii's theorem to the solid angle corresponding to the change in the observed rotation of the image of the body when it moves along a curvilinear trajectory with respect to a fixed observer.

We emphasize that in this case the question is not the actual solid angle which is described by an axis connected with the body, but the observed solid angle corresponding to a change in the rotation of the image of the body when it moves along a curvilinear trajectory.

Note that the Thomas precession arises not because the body is observed to rotate by a certain angle in the laboratory reference frame, but because this angle changes during the motion of the body along a curvilinear trajectory, which also leads to the fact that an axis, selected in the body, describes a solid angle.

4. DISCUSSION OF THE RESULTS. CONCLUSIONS

We will compare the physical reasons that give rise to the effects considered.

The Ishlinskii effect is due to the fact that the kinematics of a rigid body—a system of point masses—in classical mechanics does not reduce to the kinematics of a point mass. The kinematic equations of a rigid body, written in any form, have a much more complex structure than the kinematic equations

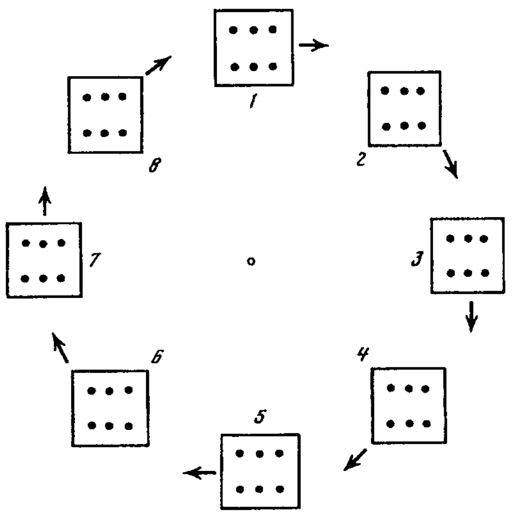

Fig. 2.

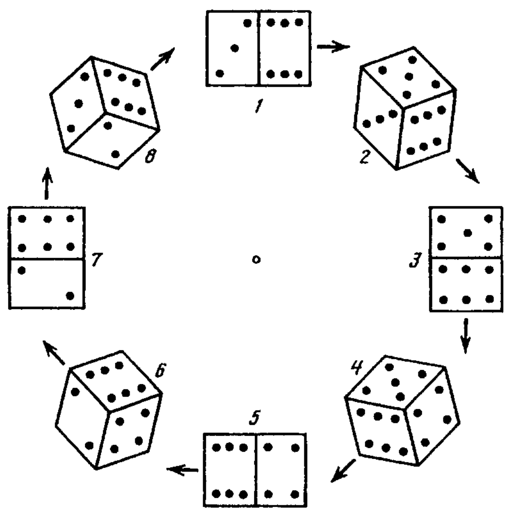

Fig. 3.

of a point mass. If the projection of the velocity of a point onto any axis is zero, there is no change in the corresponding coordinate. The situation is not the same with a rigid body. If the angular velocity of the body in a projection onto any axis is zero, the body does not remain fixed with respect to this axis [7].

The Thomas precession is explained by the relativity idea of the curvilinear translational motion of a system of point masses. If in one inertial reference frame K the velocities of all points of the body at the instant of time t are the same, in another inertial reference frame K' at the instant of time t' in accelerated motion of the body they will be different [9]. The existence of this effect indicates that in the Special Theory of Relativity no curvilinear translational motion of a rigid body exists.

Thus, both these effects, both in classical mechanics and in the Special Theory of Relativity, are caused by the specific feature of the curvilinear motion of a rigid body, of a system of point masses.

We can here draw an analogy between the two effects considered. These are the classical and relativistic (also called the quadratic) Doppler effects. They are due to different causes—the departure of an object with a certain velocity from a fixed observer and, correspondingly, the relativistic time retardation of a moving object, which radiates a wave (for example, an electromagnetic wave), relative to a fixed observer. However, the consequences in both cases are the same—the observer records a reduction in the radiation frequency. Note that when $\upsilon \ll c$ the Thomas precession, since it is also a relativistic Doppler effect, depends quadratically on the velocity (see expression (2.2)). Hence, by analogy with the relativistic Doppler effect, the Thomas precession can be regarded as a relativistic Ishlinskii effect.

The main results of this paper can be formulated as follows:

1. We have shown that Ishlinskii's theorem can be applied in the Special Theory of Relativity. As a consequence, there is a physical analogy between the two different kinematic effects—the Ishlinskii effect in classical mechanics and the Thomas precession in the Special Theory of Relativity. The latter effect can be regarded as the relativistic Ishlinskii effect.

2. The reasons for the change in the spatial orientation of a rigid body for both effects are different: for the first it is consequence of the actual rotation of the body (an axis connected with the body) when it undergoes conical motion, while for the second it is consequence of the change in the rotation of the body, observed in the laboratory reference frame (an axis, connected with the body) when it is undergoing curvilinear motion. However, the consequences of the action of both the first and second effects on the body are the same: after the axis connected with the body returns to its initial position the body is rotated by an angle that is numerically equal to the solid angle described by the axis.

3. Both of the effects considered, both in the Special Theory of Relativity and in classical mechanics, are due to the specific features of the curvilinear motion of a rigid body as a system of point masses.

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