Relativistic precession

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Inertial frames of reference and nonrotating frames of reference are discussed. Using geometric algebra as the principal mathematical tool, the Thomas precession of torque-free gyroscopes in special relativity is then derived in an exact fashion in two different ways. First, the equations of Fermi–Walker transport are employed to derive expressions for the precession; second, a torque-free reference frame is compared to one in which the coordinate axes are forced by torques to remain parallel to a fixed direction. The precession of a freely falling gyroscope in a gravitational field concludes the paper.

I. INTRODUCTION

Classical mechanics depends for its existence on our primitive notions of space and time and mass and their definitions and description in the conceptual foundation known as "Newton's laws". A reference frame is also a primitive element of these laws but is really undefined beyond imagining it as a set of three mutually perpendicular axes moving and spinning arbitrarily about relative to other such reference frames.

Of the several principles needed as a foundation for classical and relativistic particle mechanics, that most essential to our present concern is Newton's 1st law, the one in which inertial reference frames are supposedly defined. This law is contentious[1,2] and has had physicists debating its meaning for centuries. A version I would like to suggest is the following:

Consider the statements: (a) An object is being observed from an inertial reference frame; (b) the object has a constant velocity; (c) the object is subject to no net force. Newton's 1st law is then: If any two of the statements (a), (b), or (c) are true, then so is the third. This is circular reasoning of a sort, and very deliberately so, for inertial reference frames cannot be defined on the basis of more primative concepts. As Misner, Thorne, and Wheeler so ably and emphatically put it: "Here and elsewhere in science, as stressed not least by Henri Poincaré, that view is out of date which used to say, 'Define your terms before you proceed.' All the laws and theories of physics ... have this deep and subtle character, that they both define the concepts they use and make statements about these concepts."[3]

Now what about rotation?

The physical concept of a reference frame, or triad of mutually perpendicular unit three-vectors $\{\hat{\mathbf{e}}_i,\ i=1,2,3\}$ (where the caret denotes a unit three-vector) that is not rotating relative to inertial reference frames is mathematically described by the following: The reference frame is not rotating if and only if

$$\frac{d\hat{\mathbf{e}}_i}{dt}=0. \tag{1.1}$$

This is physically expressed by attaching the triad to torque-free gyroscopes. If the triad is rotating then there is an angular velocity $\boldsymbol{\omega}$ such that relative to an inertial reference frame

$$\frac{d\hat{\mathbf{e}}_i}{dt}=\boldsymbol{\omega}\times\hat{\mathbf{e}}_i. \tag{1.2}$$

This, of course, is not a separate law, but part of the geometry, part of the kinematics, like velocity, and acceleration. Torque-free gyroscopes are essential to this physical description (replacing the formerly essential, but vague, "fixed stars"). We define their behavior to be simple on the basis of our experience and use them in our fundamental concepts.

In space-time a reference frame, or tetrad of mutually orthogonal unit four-vectors $\{\gamma_\mu,\ \mu=0,1,2,3\}$, where $\gamma_\mu\cdot\gamma_\nu=\eta_{\mu\nu}=\mathrm{diag}(1,-1,-1,-1)$, with $c\equiv1$, is inertial (that is, nonrotating and in free fall) if and only if

$$\frac{d\gamma_\mu}{d\tau}=0, \tag{1.3}$$

where τ is the reference frame's proper time, with the spatial axes again imagined to be attached to torque-free gyroscopes. We are thus led to agree with Brehme:[2] "A frame is inertial if and only if the geometry of its space-time is Minkowskian."

A tetrad $\{\kappa_\mu\}$ arbitrarily moving in space-time has the four velocity

$$U=\kappa_0=U^\mu\gamma_\mu, \tag{1.4}$$

with $U\cdot U=1$, and if the spatial part of the tetrad $\{\kappa_\mu\}$ is not rotating, that is, if it is attached to torque-free gyroscopes, then it is said to be Fermi–Walker transported[4] along its worldline which is expressed by

$$\frac{d\kappa_\mu}{d\tau}=\kappa_\mu\cdot UA-\kappa_\mu\cdot AU=\kappa_\mu\cdot(U\wedge A), \tag{1.5}$$

where $A=dU/d\tau$, and in its instantaneous rest frame we have $U=(1,0)$ and $A=(0,\mathbf{a})$ where $\mathbf{a}$ is the proper three acceleration, and where the product "$\wedge$" is defined in the next section. We have, as required,

$$\frac{d\kappa_0}{d\tau}=A \tag{1.6}$$

and

$$\frac{d\kappa_i}{d\tau}=0 \tag{1.7}$$

the latter expressed in its instantaneous rest frame, showing no rotation there.

In this paper I will use these fundamental and simple concepts to investigate the subtle effects that ensue when one wishes to keep pointing in a "fixed" direction while otherwise moving about arbitrarily. We define a fixed direction to

be that given by a torque-free gyroscope's spin axis. It turns out that such a direction wanders away from that defined in a reference frame relative to which the gyroscope is in general accelerated. "Thomas precession" is the name given to the phenomenon in special relativity, a relativistic effect known for about 70 years.[5] And even a gyroscope freely falling in a gravitational field described by general relativity moves away from pointing constantly in a direction fixed by a coordinate system attached to a gravitating object. Further, angular momentum is the source of a gravitational field which augments this precessional motion to an extent that may now be on the verge of being measurable.

The approach that will be used to obtain the well-known results of relativistic Thomas precessional motion will rely on methods employed first by Rastall[6] and Hestenes,[7] and will be exact in that a manifestly covariant four-vector method that relies on Hestene's geometric algebra[8–11] will be employed that is far more elegant, simple and powerful than the awkward index methods frequently employed in relativity, especially general relativity where the (approximate) derivation herein presented is made as simple as possible. Four-vector equations are above all clear, without ambiguity and make it possible to express results in familiar and necessary three-vector form in any frame of reference. The mathematical formalism employed will first be described briefly and necessarily incompletely: There should be no need to repeat the many able and easily accessible accounts given in this journal[11] and elsewhere.[12,13]

II. SPACE, SPACE-TIME, AND ITS COVARIANT DESCRIPTION

Given a vector space of three or four dimensions an associative "geometric" product is defined between vectors a and b by

$$\begin{aligned} ab &= \tfrac{1}{2}(ab+ba)+\tfrac{1}{2}(ab-ba) \\ &\equiv a\cdot b + a\wedge b \\ &= b\cdot a - b\wedge a, \end{aligned} \tag{2.1}$$

which defines the usual scalar product and the "wedge" product $a\wedge b$. Given a basis $\{\hat{\mathbf{e}}_i\}$ of three dimensional space, "multivectors" of only the following types can be constructed:

$$\begin{aligned} &\text{scalar—}1 \\ &\text{vector—}\hat{\mathbf{e}}_i \\ &\text{bivector or pseudovector—}\hat{\mathbf{e}}_i\wedge\hat{\mathbf{e}}_j=\mathbf{i}\hat{\mathbf{e}}_k(i\neq j\neq k). \\ &\text{pseudoscalar—}\hat{\mathbf{e}}_1\wedge\hat{\mathbf{e}}_2\wedge\hat{\mathbf{e}}_3\equiv\mathbf{i}(\mathbf{ii}=-1), \end{aligned} \tag{2.2}$$

where $\mathbf{i}$ commutes with all other elements, and, given a basis $\{\gamma_\mu\}$ of four dimensional space-time, the following multivectors exist:

$$\begin{aligned} &\text{scalar—}1 \\ &\text{vector—}\gamma_\mu \\ &\text{bivector—}\gamma_\mu\wedge\gamma_\nu(\mu\neq\nu) \\ &\text{trivector or pseudovector—}\gamma_\mu\wedge\gamma_\nu\wedge\gamma_\lambda \\ &\quad=\gamma_5\gamma_\sigma(\mu\neq\nu\neq\lambda\neq\sigma) \\ &\text{pseudoscalar—}\gamma_0\wedge\gamma_1\wedge\gamma_2\wedge\gamma_3\equiv\gamma_5(\gamma_5\gamma_5=-1), \end{aligned} \tag{2.3}$$

where γ_5 anti-commutes with vectors. With the standard metrics

$$\hat{\mathbf{e}}_i\cdot\hat{\mathbf{e}}_j=\delta_{ij} \tag{2.4}$$

and

$$\gamma_\mu\cdot\gamma_\nu=\eta_{\mu\nu} \tag{2.5}$$

it follows that

$$\mathbf{i}=\gamma_5 \tag{2.6}$$

and

$$\hat{\mathbf{e}}_i=\gamma_i\wedge\gamma_0=\gamma_i\gamma_0. \tag{2.7}$$

That is, vectors (and pseudovectors) in three-space are bivectors in space-time.

It is seen that

$$\frac{d\hat{\boldsymbol{\kappa}}_i}{d\tau}=0=\frac{d\kappa_i\kappa_0}{d\tau}=\left(\frac{d\kappa_i}{d\tau}\right)\wedge\kappa_0, \tag{2.8}$$

where $\{\hat{\boldsymbol{\kappa}}_i\}$ are the unit three-vectors that are the basis for three-vectors in the instantaneous (inertial) rest frame of $\{\kappa_\mu\}$. Here Hestenes' "wedge" product for four-vectors[8]

$$A\wedge B\equiv\tfrac{1}{2}(AB-BA) \tag{2.9}$$

and an identical definition for three-vectors, with[14]

$$\begin{aligned} &\mathbf{a}\times\mathbf{b}=-\mathbf{i}\mathbf{a}\wedge\mathbf{b} \\ &\mathbf{i}=\gamma_5=\hat{\mathbf{e}}_1\hat{\mathbf{e}}_2\hat{\mathbf{e}}_3=\gamma_0\gamma_1\gamma_2\gamma_3, \end{aligned} \tag{2.10}$$

which have been essential in relating the four-vectors of a tetrad to the three-vectors of its instantaneous rest frame.

Equation (2.7) is the key to reducing expressions written covariantly in four-vector symbolism to three-vector expressions in any given reference frame. For example, if $P=mU$ is an object's four-momentum, then its ordinary three-momentum in the reference frame $\{\gamma_\mu\}$ is simply

$$\mathbf{p}=P\wedge\gamma_0, \tag{2.11}$$

which, when applied to a spacelike four-vector S $(S\cdot S=-1)$ fixed in a rotating reference frame, gives

$$\left(\frac{dS}{d\tau}\right)\wedge\gamma_0=\frac{ds}{d\tau}=\boldsymbol{\omega}\times\mathbf{s}, \tag{2.12}$$

so the angular velocity $\boldsymbol{\omega}$ is automatically given. This result is our prime calculational tool in what follows.

Any mapping of an orthonormal tetrad $\{\gamma_\mu\}$ into another $\{\gamma'_\mu\}$ is called a Lorentz transformation. In general the reference frame $\{\gamma'\}$ will be moving and rotating relative to $\{\gamma\}$. A restricted Lorentz transformation[15] is one in which the spatial axes of $\{\gamma'\}$ remain parallel to those of $\{\gamma\}$, and this is described by

$$\gamma'_\mu=L\gamma_\mu L^{-1}, \tag{2.13}$$

where[6,12]

$$L=(1+\gamma'_0\gamma_0)/[2(1+\gamma'_0\cdot\gamma_0)]^{1/2}, \tag{2.14}$$

with L^{-1} being given by a similar expression, but with γ and γ' exchanged. To see that the spatial axes remain parallel we orient the 1-axis in the direction of the boost and calculate

$$\gamma'_1=(1+\gamma'_0\gamma_0)\gamma_1(1+\gamma_0\gamma'_0)/(\ldots)=(a\gamma_0+b\gamma_1)/(\ldots) \tag{2.15}$$

and

$$\gamma'_{2,3}=\gamma_{2,3} \tag{2.16}$$

where a and b are scalars, so there is no rotation of the spatial axes in the boost. As the boost changes in time and direction, however, there will be a spatial rotation: this is Thomas precession.

III. SPIN AND THOMAS PRECESSION

In a reference frame $\{\kappa_\mu\}$, itself arbitrarily moving with respect to an inertial reference frame $\{\gamma_\mu\}$, let there be an object spinning in the fixed direction $\hat{\mathbf{s}}=s^i\hat{\boldsymbol{\kappa}}_i$, so the components s^i are constant. In covariant four-vector notation we express this spin four-vector as $S=s^i\kappa_i=S^\mu\gamma_\mu$ with

$$S\cdot S=-1, \quad S\cdot U=0, \tag{3.1}$$

where $U=\kappa_0=U^\mu\gamma_\mu$ is the spinning object's four velocity.

In general terms the spin vector satisfies

$$\frac{dS}{d\tau}=S\cdot\Omega=\frac{1}{2}(S\Omega-\Omega S), \tag{3.2}$$

where Ω is a bivector.[16] Since a three-vector or a pseudo-three-vector in ordinary space is a bivector in space-time, as shown in (2.7), we can write

$$\Omega=\gamma_5\boldsymbol{\omega} \tag{3.3}$$

and we obtain, on projecting (3.2) onto the three-space in its instantaneous rest frame

$$\frac{dS}{d\tau}\wedge\kappa_0=\frac{1}{2}[(S\cdot\Omega)\kappa_0-\kappa_0(S\cdot\Omega)]$$

$$=-\frac{1}{2}\gamma_5(\boldsymbol{\omega}\hat{\mathbf{s}}-\hat{\mathbf{s}}\boldsymbol{\omega})=\boldsymbol{\omega}\times\hat{\mathbf{s}}, \tag{3.4}$$

which we now know describes how a unit spin three-vector changes with proper time.

In an arbitrary inertial reference frame where the spin vector is $S=(S_0,\mathbf{S})$, and satisfies, if it is Fermi–Walker transported along its worldline as in (1.5), that is, if it is a torque-free gyroscope,

$$\frac{dS}{d\tau}=-US\cdot A, \tag{3.5}$$

where $U=(\gamma,\gamma\mathbf{v})$ is the gyroscope's four-velocity, $\gamma=(1-v^2)^{-1/2}$ (which must not be confused with four-vectors γ_μ) and where $A=dU/d\tau$. The three-vector part of this is

$$\gamma\frac{d\mathbf{S}}{dt}=-\gamma\mathbf{v}(-\hat{\mathbf{s}}_0\cdot\hat{\mathbf{a}}_0), \tag{3.6}$$

where $S\cdot U$ has been evaluated in the instantaneous rest frame of the arbitrarily moving gyroscope. If we solve for $\hat{\mathbf{S}}=\mathbf{S}/|\mathbf{S}|$ instead we find

$$\frac{d\hat{\mathbf{S}}}{dt}=\left(\frac{\hat{\mathbf{s}}_0\cdot\hat{\mathbf{a}}_0}{|\mathbf{S}|}\right)(\hat{\mathbf{S}}\times\mathbf{v})\times\hat{\mathbf{S}}, \tag{3.7}$$

which is exact but which looks nothing at all like the usual expression given for Thomas precession.

Now consider a circular orbit of radius r in the $x-y$ plane at constant angular speed $\Omega=v/r$, where the position vector would be given by

$$\mathbf{r}=r(\hat{\mathbf{x}}\cos\Omega t+\hat{\mathbf{y}}\sin\Omega t). \tag{3.8}$$

Note that there is no precession if $\hat{\mathbf{S}}=\hat{\mathbf{z}}$, that is, if $\hat{\mathbf{S}}$ is perpendicular to $\mathbf{v}$ and $\mathbf{a}$; therefore without loss of generality we can set $\hat{\mathbf{S}}=\hat{\mathbf{x}}$, and we find, after time averaging over one orbit (where $\langle\cos^2\rangle=\langle\sin^2\rangle=1/2$, $\langle\cos\sin\rangle=0$),

$$\frac{d\hat{\mathbf{S}}}{dt}\approx\left(-\frac{1}{2}\mathbf{v}\times\mathbf{a}\right)\times\hat{\mathbf{S}} \tag{3.9}$$

thus giving the well-known approximate result that the spin vector's direction precesses backward against the direction of motion at the Thomas precession rate of

$$\boldsymbol{\omega}_T\approx-\tfrac{1}{2}\mathbf{v}\times\mathbf{a}. \tag{3.10}$$

It is important to remember that this is the low speed first approximation to the precession. For related discussions of Thomas precession see Refs. 17–21. For especially clear and simple derivations of the approximate result see Refs. 22, 23.

IV. THOMAS PRECESSION II

A more elegant and physically appealing description of this precessional motion is to be considered now, a method based on one first used by Rastall.[6]

Consider two reference frames moving arbitrarily together: the orthonormal tetrad $\{\pi_\mu\}$ has its spatial axes attached to torque-free gyroscopes, but the tetrad $\{\kappa_\mu\}$ is moving in such a way that it is always connected to the inertial frame $\{\gamma_\mu\}$ by a restricted Lorentz transformation of the type considered earlier. That is, the spatial axes of $\{\kappa_\mu\}$ are always parallel to those of $\{\gamma_\mu\}$. A unit torque-free spin vector fixed in $\{\pi_\mu\}$ is to be compared with one fixed in $\{\kappa_\mu\}$; the former is Fermi–Walker transported while the latter is not. The latter spin vector is given by $J=j^i\kappa_i$, with j^i=const. Now, since

$$\kappa_\mu=L\gamma_\mu L^{-1}, \tag{4.1}$$

we have

$$\frac{dJ}{d\tau}=\frac{j^i\,d\kappa_i}{d\tau}=j^i(\dot{L}L^{-1}\kappa_i-\kappa_i\dot{L}L^{-1}), \tag{4.2}$$

where

$$L=(1+\kappa_0\gamma_0)/[2(1+\kappa_0\cdot\gamma_0)]^{1/2}. \tag{4.3}$$

We find

$$\dot{L}L^{-1}=A\wedge(\gamma_0+\kappa_0)/[2(1+\gamma_0\cdot\kappa_0)], \tag{4.4}$$

where

$$A=\frac{dU}{d\tau}=\frac{d\kappa_0}{d\tau} \tag{4.5}$$

is the four-acceleration. We now project this changing spin four-vector into the three-space of its instantaneous rest frame to find the precession. The following exact result is obtained:

$$\dot{J}\wedge\kappa_0=-[J\cdot A\,\gamma_0\wedge\kappa_0-J\cdot\gamma_0A\wedge\kappa_0]/(1+\gamma). \tag{4.6}$$

In the instantaneous rest frame we have

$$J\cdot A=-\mathbf{j}_0\cdot\mathbf{a}_0 \tag{4.7}$$

and

$$A\wedge\kappa_0=\mathbf{a}_0. \tag{4.8}$$

What follows now is a somewhat subtle consideration: All three-vectors must be in the same three-space, namely that of

the instantaneous rest frame, or that of $\{\kappa_\mu\}$. Let the three-velocity components of the moving tetrad $\{\kappa_\mu\}$ relative to the inertial reference frame $\{\gamma_\mu\}$ be v^i, then $-\mathbf{v}=-v^i\hat{\boldsymbol{\kappa}}_i$ is the velocity of the frame $\{\gamma_\mu\}$ relative to $\{\kappa_\mu\}$. Therefore

$$\gamma_0\wedge\kappa_0=-\gamma\mathbf{v} \tag{4.9}$$

and

$$J\cdot\gamma_0=-\hat{\mathbf{j}}\cdot(-\gamma\mathbf{v})=\gamma\hat{\mathbf{j}}\cdot\mathbf{v} \tag{4.10}$$

and we finally have

$$\dot{J}\wedge\kappa_0=[\gamma/(1+\gamma)](\mathbf{v}\times\mathbf{a}_0)|\times\hat{\mathbf{j}}_0. \tag{4.11}$$

This is the precession of the gyroscope fixed in the reference frame $\{\kappa_\mu\}$ relative to the torque-free gyroscope fixed in $\{\pi_\mu\}$. Therefore the torque-free gyroscope will appear to precess relative to fixed-direction coordinate axes at the Thomas precession rate

$$\begin{aligned}\boldsymbol{\omega}_T&=-[\gamma/(1+\gamma)]\mathbf{v}\times\mathbf{a}_0\\&\approx-1/2\mathbf{v}\times\mathbf{a}\quad\text{for } v\ll 1.\end{aligned} \tag{4.12}$$

It is crucial to realize here that $\mathbf{a}_0$ is the proper acceleration and that $-\mathbf{v}$ is the velocity of an inertial reference frame relative to which the moving gyroscope is to be observed, and both are three vectors in the gyroscope's instantaneous rest frame.

V. PRECESSION IN A GRAVITATIONAL FIELD

The method of the preceeding section is especially well suited for describing a spinning torque-free gyroscope in free fall in a gravitational field described by general relativity. The four-velocity U of an object in free fall satisfies the simplest of descriptions:

$$\frac{dU}{d\tau}=0, \tag{5.1}$$

where τ is the object's proper time, so that Fermi–Walker transport of the four-vector of an object that is in free fall and not rotating with respect to spatial axes satisfies a similar equation. Therefore a spin four-vector S of a freely falling torque free gyroscope satisfies

$$\frac{dS}{d\tau}=0 \tag{5.2}$$

and

$$S\cdot U=0 \tag{5.3}$$

and we can also assume that the gyroscope's spin satisfies, without loss of generality,

$$S\cdot S=-1. \tag{5.4}$$

As in the previous section let $\{\pi_\mu\}$ be a freely falling reference frame with the spatial axes attached to torque-free gyroscopes. Then

$$\frac{d\pi_\mu}{d\tau}=0 \tag{5.5}$$

as $\{\pi_\mu\}$ is a (local) inertial reference frame. We may set

$$S=S^i\pi_i \tag{5.6}$$

with S^i=constant. Now let $\{\kappa_\mu\}$ be a reference frame falling with $\{\pi_\mu\}$, but one that always has its spatial axes parallel to those of a reference frame $\{\gamma_\mu\}$ that is fixed in a gravitational field. Note that all three tetrads remain orthonormal at all times. We have

$$\frac{d\kappa_0}{d\tau}=0 \tag{5.7}$$

because $\kappa_0=\pi_0$ is the four velocity of the freely falling gyroscope. Since the spatial axes of $\{\kappa_\mu\}$ and $\{\gamma_\mu\}$ remain parallel at all times, they are related by a restricted Lorentz transformation:

$$\kappa_\mu=L\gamma_\mu L^{-1}, \tag{5.8}$$

where

$$L=(1+\kappa_0\gamma_0)/[2(1+\kappa_0\cdot\gamma_0)]^{1/2}. \tag{5.9}$$

Let $J=J^i\kappa_i$, J^i=constant, be a spacelike four-vector ($J\cdot J=-1$) fixed in $\{\kappa_\mu\}$. We have

$$\frac{dJ}{d\tau}=J^i\frac{d\kappa_i}{d\tau} \tag{5.10}$$

and

$$\dot{\kappa}_\mu=\dot{L}L^{-1}\kappa_\mu-\kappa_\mu\dot{L}L^{-1}+L\dot{\gamma}_\mu L^{-1}. \tag{5.11}$$

If we substitute Eq. (5.9) into (5.11) we find (5.7) follows immediately for $\mu=0$.

The gravitational field must now be introduced. A set of coordinates x^μ fixed in the gravitational field generates a coordinate basis $\{e_\mu\}$ that satisfies the formal definition

$$e_\mu=\partial_\mu x, \tag{5.12}$$

where x is a space-time event, and where $\partial_\mu=\partial/\partial x^\mu$, with

$$e_\mu\cdot e_\nu=g_{\mu\nu} \tag{5.13}$$

and

$$ds^2=g_{\mu\nu}\,dx^\mu\,dx^\nu \tag{5.14}$$

as usual. Because of (5.12) many authors[24,25] today simply identify e_μ with the directional derivative ∂_μ. Along a path parametrized by the proper time τ we have[26]

$$\dot{e}_\mu=\Gamma^\lambda_{\mu\nu}\dot{x}^\nu e_\lambda, \tag{5.15}$$

where Γ are the usual Christoffel symbols and where $U=\kappa_0=(dx^\mu/d\tau)e_\mu$ is the four-velocity.

A rotating mass has the gravitational field[27]

$$ds^2=f^2\,dt^2-g^2\,d\mathbf{r}^2+h_i\,dx^i\,dt, \tag{5.16}$$

where, in the first nonzero approximation,

$$\begin{aligned}f^2&\approx(1+2\phi),\\g^2&\approx(1-2\phi),\end{aligned} \tag{5.17}$$

where $\phi=-M/r$ is the gravitational potential (in units with $G\equiv 1$), and if the mass is rotating about the $+x^3=z$ axis, we have

$$h_1=-4yH/r^3,\quad h_2=+4xH/r^3,\quad h_3=0, \tag{5.18}$$

where H is the rotating mass's angular momentum.

The coordinate basis $\{e_\mu\}$ and the orthonormal basis $\{\gamma_\mu\}$ are readily related:

$$\begin{aligned}e_0&\approx f\gamma_0\\e_i&\approx g\gamma_i+\tfrac{1}{2}h_i\gamma_0\end{aligned} \tag{5.19}$$

to lowest order, and

$$\gamma_0 \approx f^{-1} e_0,$$
$$\gamma_i \approx g^{-1} e_i - \tfrac{1}{2} h_i e_0, \tag{5.20}$$

where f, $g \approx 1$, where undifferentiated, and $|\partial f, \partial g| \ll 1$, $|h_i| \ll 1$. To evaluate $d\gamma_\mu / d\tau$ we need to calculate the Christoffel symbols. Using the Lagrangian

$$\mathscr{L} = f^2 \dot{t}^2 - g^2 (\dot{x}^i)^2 + h_i \dot{t} \dot{x}^i \tag{5.21}$$

and the equations

$$\frac{d}{d\tau} \frac{\partial \mathscr{L}}{\partial \dot{x}^\mu} - \frac{\partial \mathscr{L}}{\partial x^\mu} = 0 \tag{5.22}$$

all of the nonzero Christoffel symbols are readily found from the geodesic equation:

$$\begin{aligned} &\ddot{x}^\lambda + \Gamma^\lambda_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = 0, \\ &\Gamma^0_{ij} = \tfrac{1}{4}(\partial_i h_j + \partial_j h_i), \\ &\Gamma^0_{0i} = \partial_i f = \Gamma^i_{00}, \\ &\Gamma^i_{0j} = -\tfrac{1}{4}(\partial_j h_i - \partial_i h_j), \\ &\Gamma^i_{jk} = \partial_j g\, \delta_{ik} + \partial_k g\, \delta_{ij} - \partial_i g\, \delta_{jk}. \end{aligned} \tag{5.23}$$

With these we find

$$\dot{\gamma}_0 = \partial_i f \gamma_i - \tfrac{1}{4}(\partial_j h_i - \partial_i h_j) \dot{x}^j \gamma_i \tag{5.24}$$

and

$$\dot{\gamma}_i = \gamma_0 [\cdots] - \tfrac{1}{4}(\partial_i h_j - \partial_j h_i) \gamma_j + \partial_i g \dot{x}^j \gamma_j - \partial_j g \dot{x}^i \gamma_j. \tag{5.25}$$

From this last result we have

$$L \dot{\gamma}_i L^{-1} \wedge \kappa_0 = -\tfrac{1}{4}(\partial_i h_j - \partial_j h_i) \hat{\boldsymbol{\kappa}}_j - (\mathbf{v} \times \nabla g) \times \hat{\boldsymbol{\kappa}}_i \tag{5.26}$$

since

$$L \gamma_0 L^{-1} \wedge \kappa_0 = \kappa_0 \wedge \kappa_0 = 0. \tag{5.27}$$

What remains is very similar to the last section. The gyroscope fixed in $\{\kappa_\mu\}$ precesses according to

$$\frac{dJ}{d\tau} \wedge \kappa_0 = \frac{J^i d\kappa_i}{d\tau} \wedge \kappa_0, \tag{5.28}$$

which is, finally,

$$\frac{dJ}{d\tau} \wedge \kappa_0 = \left[\mathbf{v} \times \left(\frac{1}{2} \nabla f - \nabla g \right) - \frac{1}{4} (\nabla \times \mathbf{h}) \right] \times \hat{\mathbf{j}}. \tag{5.29}$$

This, recall, is the gyroscope fixed in $\{\kappa_\mu\}$, whose spatial axes are constrained by appropriate torques to remain parallel to those of $\{\gamma_\mu\}$, a reference frame fixed in the gravitational field. Therefore the freely falling, torque-free gyroscope appears to precess in the opposite direction, with the angular velocity $\boldsymbol{\omega}$:

$$\boldsymbol{\omega} = -\tfrac{3}{2} \mathbf{v} \times \nabla \phi + \tfrac{1}{4} \nabla \times \mathbf{h}. \tag{5.30}$$

Note that the spatial metric g contributes twice as much to the precession rate as the time component f. Note also that the source of gravity in the second term is angular momentum.

For an orbit in the equatorial plane the precessional angular velocity becomes

$$\boldsymbol{\omega} = -\frac{3}{2} \frac{GM}{r^2 c^2} \mathbf{v} \times \hat{\mathbf{r}} - \frac{GH}{r^3 c^2} \hat{\mathbf{z}}, \tag{5.31}$$

where all dimensional constants have been restored, and where, again, H is the Earth's angular momentum, the result first obtained by Schiff.[28,29] In a low earth orbit, for example, the first term is two orders of magnitude greater than the second, and indicates a precession rate of about 10 arcseconds per year. The direction of precession is in the direction of the orbit, so is opposite to Thomas precession (which does not exist here as there is no proper acceleration or nongravitational forces).

It will be most interesting to see if the $600 million Gravity Probe B orbiting gyroscope,[30] scheduled to be sent aloft in a few years, will be able to accurately measure these tiny angles, thereby putting general relativity to its most severe test to date.

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