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"Optical Appearance" of a Rolling Ring

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The positions of the points on a rolling ring are calculated at retarded points of time, giving the "optical appearance" of a rolling ring.

1. INTRODUCTION

The appearance of a rotating disk, as perceived by a corotating observer, has recently been calculated, in accordance with two operational procedures by McFarlane (1981). The first procedure gives the "geometrical appearance" of the disk, as defined by the positions of the disk points at a fixed point of time in the reference frame, S, of an inertial observer instantaneously at rest relative to a disk-point (Grøn, 1975). The second procedure gives the "optical appearance" corresponding to an extrapolation of the direction of motion of the photons reaching the eye. All light sources (the points of the disk) are at rest, as seen by the rotating observer, but the picture of the disk is distorted due to the curved null geodesics in the rotating reference frame, R.

A calculation of the corresponding optical appearance of the disk points, as perceived by the inertial observer S, has never been published. As far as I have found the extensive literature on the "visual appearance" of objects with relativistic velocities (Penrose, 1959; Terrell, 1959; Weinstein, 1960; Boas, 1961; Gamow, 1961; Ranninger, 1961; Sherwin, 1961; Atwater, 1962; Yngstrøm, 1962; Scott and Viner, 1965; Schröter, 1966; Guess, 1967; McGill, 1968; Bhandari, 1970, 1978; Lang, 1970; Scott and van Driel, 1970; Ferguson, 1971; Gamba, 1971; Mathews and Lakshmanan, 1972; Lai, 1975; Hollenbach, 1976; Grøn, 1978; Hickey, 1979; McKinley, 1979, 1980; Daherty, 1980; Gibbs, 1980) has been concerned only with nonrotating bodies without internal motions, except for one case: the visual appearance of expanding objects (Rees, 1966).

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In the present paper I generalize the earlier works on relativistic visual appearance to include rotating bodies, and also complete the study of McFarlane on the appearance of the relativistically rotating disk.

Since the disk may be considered as an assembly of concentric rings, all essential points of the calculation and of the results are included by the consideration of a rotating ring, which is given below.

I use McFarlane's terminology "optical appearance" rather than the more usual expression "visual appearance" since the result of the calculation gives the coordinates of the retarded events that give rise to a picture, but not the projection onto the film corresponding to the picture itself.

2. THE "OPTICAL APPEARANCE" OF A ROLLING RING

As observed in the instantaneous rest frame S of a point in the ring it is rolling. The angular velocity is ω, and the radius of the ring has a velocity $v = R\omega$, say, in the positive x direction. A camera is placed at the origin of S. At the moment that the ring touches the origin, the camera takes a picture of it. There are n equally spaced radiating points on the ring. The shape of the ring and the distribution of points on it will now be found at the retarded points of time when the photons giving rise to the picture left the points.

The form of the ring is most easily found by assuming that the ring is sliding, with zero rotational motion, instead of rolling. From the Lorentz contraction follows that, as measured by simultaneity in S, the ring has elliptical shape with major half-axis R and minor half-axis $\gamma^{-1}R$, $\gamma = (1-\beta^2)^{-1/2}$, $\beta = v/c$, as shown in Figure 1. P is one of the radiating points. Q is its position when it emitted a photon reaching the camera at the event A when the ring passes the origin of S. Let this moment be $t = 0$. From Figure 1 is seen that

$$v^2t^2 = c^2t^2 + s^2 - 2sct\cos\phi, \qquad s^2 = x^2 + y^2 \tag{1}$$

where t is the time used by the photon from Q to A. Also

$$\cos\phi = \cos\alpha_1\cos\alpha_2 + \sin\alpha_1\sin\alpha_2 = (y/s)(y/ct) + (x/s)(x - vt)/ct \tag{2}$$

giving

$$sct\cos\phi = s^2 - xvt \tag{3}$$

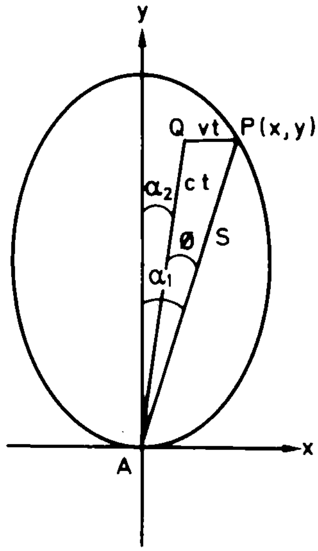

Fig. 1. A sliding ring as observed in S.

Inserted into equation (1) this gives

$$(c^2 - v^2)t^2 + 2xvt - x^2 - y^2 = 0 \tag{4}$$

with positive solution

$$ct = \gamma^2\left[(x^2 + \gamma^{-2}y^2)^{1/2} - \beta x\right] \tag{5}$$

The equation of the ellipse is

$$(\gamma x^E)^2 + (y - R)^2 = R^2 \tag{6}$$

which may be written

$$\gamma x_1^E = -(2Ry - y^2)^{1/2}, \qquad \gamma x_2^E = (2Ry - y^2)^{1/2} \tag{7}$$

Substitution into equation (5) gives

$$ct_1 = \gamma\left[(2Ry)^{1/2} + \beta(2Ry - y^2)^{1/2}\right]$$

$$ct_2 = \gamma\left[(2Ry)^{1/2} - \beta(2Ry - y^2)^{1/2}\right] \tag{8}$$

The equation for the retarded shape of the ring is

$$x_1^B = x_1^E - vt_1, \qquad x_2^B = x_2^E - vt_2 \tag{9}$$

Equations (7), (8), and (9) give

$$x_1^B = -\gamma\left[(2Ry - y^2)^{1/2} + \beta(2Ry)^{1/2}\right]$$

$$x_2^B = \gamma\left[(2Ry - y^2)^{1/2} - \beta(2Ry)^{1/2}\right] \tag{10}$$

The retarded shape of the ring is shown in Figure 4C.

The distribution of the points along the ring will now be calculated. The inertial rest frame of the axis is S_R. By definition the events P_i constituting "the ring as *observed* in S" are simultaneous in S, happening, say, at $t = 0$. The corresponding points of time as referred to S_R, are

$$t_R^P = -\gamma(v/c^2)x^P \tag{11}$$

The spatial coordinates of these events are (Figure 2)

$$x_R^P = R\cos\theta, \qquad y_R^P = R(1+\sin\theta), \qquad \theta = \theta_i + \omega t_R^P \tag{12}$$

Transforming to S gives

$$\gamma x^P = R\cos\theta, \qquad y^P = R(1+\sin\theta) \tag{13}$$

which is consistent with equation (6).

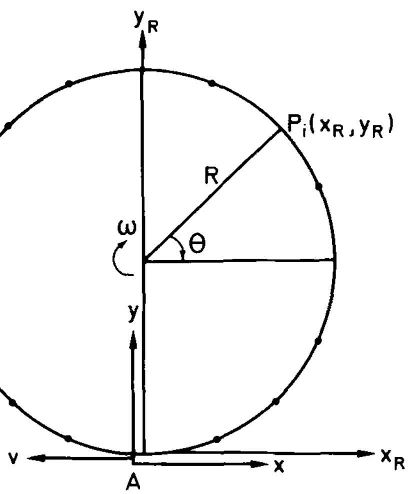

Fig. 2. The ring as observed in S_R.

Substituting this into equation (11) leads to

$$\omega t_R^P = -\beta^2 \cos(\theta_i + \omega t_R^P) \tag{14}$$

The transcendental equation (14) is solved numerically. The coordinates of the events P_i as referred to S are now given by equation (13). The result is found in Figure 4B, which gives the distribution of points on the ring, observed by simultaneity in S. The points cover the ellipse given by equation (6), giving the usual Lorentz contracted observed object.

To every event P_i there is associated an event Q_i such that the photons emitted at every Q_i reach the camera at $t = 0$. The coordinates of Q_i as referred to S_R are

$$x_R^Q = R\cos(\theta - \omega t_R^Q), \qquad y_R^Q = R\left[1 + \sin(\theta - \omega t_R^Q)\right] \tag{15}$$

where $t_R^Q < 0$ and $|t_R^Q|$ is the time taken by a photon from Q_i to A. From Figure (3) follows that

$$(ct_R^Q)^2 = R^2 + R^2 - 2R^2\cos(\pi/2 + \theta - \omega t_R^Q) \tag{16}$$

or

$$\omega t_R^Q = -\beta\{2[1 + \sin(\theta - \omega t_R^Q)]\}^{1/2} \tag{17}$$

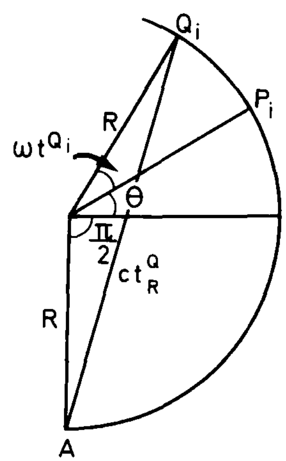

Fig. 3. Calculating the coordinates of R_i, as referred to S_R.

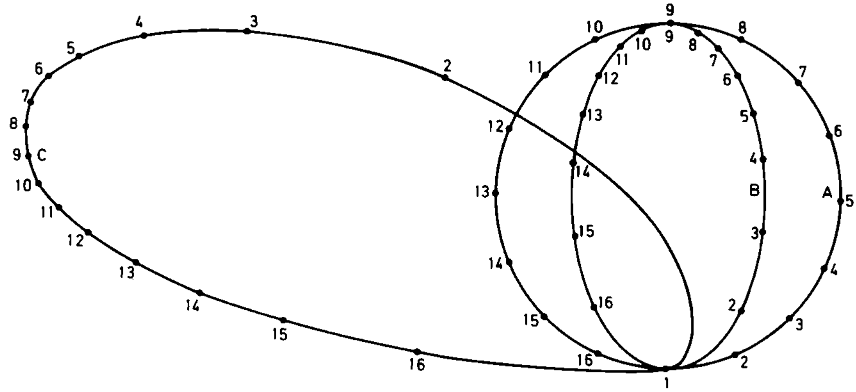

Fig. 4. The ring as observed by simultaneity in $S_R(A)$, by simultaneity in $S(B)$, and at retarded points of time (C).

This transcendental equation for t_R^Q is solved numerically. A transformation to S now gives

$$x^Q = \gamma\left[R\cos\left(\theta - \omega t_R^Q\right) + vt_R^Q\right]$$

$$y^Q = R\left[1 + \sin\left(\theta - \omega t_R^Q\right)\right] \qquad (18)$$

This is the equation for the distribution of the disk points at the retarded points of time.[1] The result is shown in Figure 4C, with $\beta = 0{,}8$.

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[1]A short calculation shows that the distribution of points given by equations (17) and (18) covers the curve described by equation (10).

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