Energy considerations in connection with a relativistic rotating ring

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(Received 31 August 1981; accepted for publication 29 January 1982)

The retardation of an elastic rotating ring is considered. Bates has formulated a paradox indicating that as the ring slows down, *unaccounted for* elastic energy appears in the ring. I here show that, due to the relativity of simultaneity, the amounts of work performed at the two ends of a small element of the ring, are unequal, as measured in the instantaneous inertial rest frame of the element. So in addition to the work performed in order to stop the ring, some work is performed which compresses the ring. This work, which vanishes if Galilean kinematics with absolute simultaneity is used, fully accounts for the appearance of elastic energy in the ring.

I. INTRODUCTION

N.F. Bates has formulated the following paradox.[1] A horizontal wheel with radius r rotates with angular velocity ω_0, so that the velocity of the periphery, $r\omega_0$, is comparable with the velocity of light c. The wheel is, for reasons of illustration, considered as an elastic ring, consisting of small springs with elastic constant k and rest length L_0. When the ring rotates the springs are close to each other, end to end, but without stress.

Arrangements are made to slowly draw off the kinetic energy of the wheel to perform work until it has come to a halt. As the wheel slows, the Lorentz contraction of the springs reduces, and the available place appears to each spring to be less than the place it needs to be stressfree. Hooke's law applies here, and the strain will entail elastic potential energy.

Where does the energy come from? There should always be some traceable transfer of energy, some "mechanism" should be involved. In this case the tension of the springs seems to be the result of the changing conditions of space as observed by them.

II. ELASTIC ENERGY ASSOCIATED WITH DEWAN-BERAN STRESS

In the following I identify the "mechanism" responsible for the compression of the springs, and show that there is a work associated with this "mechanism" giving rise to the potential energy in the material when the ring has stopped.

In order to obtain conceptual clarity I assume that the retardation of the ring's rotational motion is obtained by giving it lots of tangential blows at points infinitesimally near to each other. In the limit of infinitely many blows per unit distance and unit time, this simulates stopping the ring by a frictional braking force.

This braking is required to be performed in an axisymmetric way, as observed in the inertial restframe J of the axis. So the braking is performed by groups of simultaneous blows all around the ring, as observed in J.

Now make a Lorentz transformation to the instantaneous inertial rest frame J' of one of the springs, when the angular velocity of the ring is ω. With the axes chosen as illustrated in Figs. 1 and 2, the Lorentz transformation of time from J to J' gives

$$t_2' - t_1' = \gamma[(t_2 - t_1) - (r\omega/c^2)(x_2 - x_1)], \tag{1}$$
$$\gamma = (1 - r^2\omega^2/c^2)^{-1/2}.$$

Since the blows are simultaneous in J, $t_1 = t_2$. The result is

$$t_2' - t_1' = \gamma(r\omega/c^2)(x_1 - x_2). \tag{2}$$

Since $x_2 > x_1$, Eq. (2) gives $t_2' < t_1'$, which means that, as observed in J' blow 2 happens earlier than blow 1. As seen from the direction of the blows, as illustrated in Fig. 2, this results in a compression of the spring. When formulated in these terms it appears that a Dewan–Beran stress is formed in the spring.[2,3]

If we think of the force as a continuous function of time, the role of the relativity of simultaneity in explaining the compression of the springs becomes somewhat more hidden. Instead the compression appears as due to the *constraint* that the endpoints of every spring have a constant separation, during the deceleration of the disk, as observed in J. In the instantaneous inertial frame J' one then observes that the endpoints of the spring are constrained to move so that it is compressed during its deceleration.

Let us now calculate that part of the work performed on a spring, which is associated with its compression, during its deceleration to rest. Assume that the spring has an angular velocity ω, and that a group of blows, simultaneous in J, as described above, changes its angular velocity to $\omega - d\omega$.

From Eq. (2) the time difference between the blows as observed in J' is

$$\Delta t' = t_2' - t_1' = \gamma(r\omega/c^2)L, \tag{3}$$

where L is the initial Lorentz contracted length, $L = \gamma^{-1}L_0$, of the spring. If the velocity change as ob-

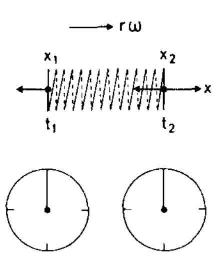

Fig. 1. As observed in J the blows are simultaneous.

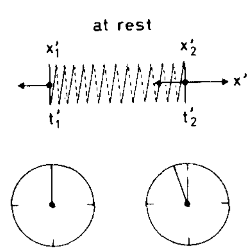

Fig. 2. As observed in J' the blow 2 happens earlier than 1 (greatly exaggerated on the clocks).

served in J is $r\,d\omega$, it follows from the Lorentz transformation that the velocity change in J' is

$$dv' = -\gamma^2 r\,d\omega. \tag{4}$$

During the time interval $\Delta t'$ point 2 moves towards 1 with this velocity, as observed in J'. So the spring gets a compression

$$ds' = dv'\,\Delta t' = -\gamma^3(r^2\omega/c^2)L\,d\omega. \tag{5}$$

Integration gives

$$s' = (\gamma_0 - \gamma)L, \tag{6}$$

where $\gamma_0 = (1 - r^2\omega_0^2/c^2)^{-1/2}$.

The work performed on the ring giving rise to this compression is

$$W'_{\text{pot}} = \int_{\omega_0}^{0} ks'\,ds' = \tfrac{1}{2}ks'^2|_{\omega_0}^{0}, \tag{7}$$

which by use of Eq. (6) gives

$$W'_{\text{pot}} = \tfrac{1}{2}kL^2(\gamma_0 - 1)^2, \tag{8}$$

or as expressed by the initial uncompressed rest length L_0 of the spring:

$$W'_{\text{pot}} = \tfrac{1}{2}kL_0^2(1 - \gamma_0^{-1})^2. \tag{9}$$

This work, as calculated in the successive inertial rest frames of the ring, gives the contribution to the ring's *rest* mass from the relativistic potential energy in it.

Expanding in powers of $(r\omega_0/c)$ and retaining only the first term, gives

$$W'_{\text{pot}} \cong (r\omega_0/2c)^4 2kL_0^2. \tag{10}$$

This shows that the accumulation of potential energy in the ring due to Dewan–Beran stresses is only a fourth-order effect in v/c.

III. CONCLUSION

The increase of potential energy in a rotating ring, consisting of elastic springs, as it is slowed down has appeared as a result of work done by forces compressing the springs, and these forces have been explained as due to the acceleration program and the relativity of simultaneity.

[1] N. F. Bates (unpublished).
[2] E. M. Dewan and M. Beran, Am. J. Phys. **27**, 517 (1959).
[3] E. M. Dewan, Am. J. Phys. **31**, 383 (1963).

Slinky oscillations and the notion of effective mass

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(Received 1 May 1981; accepted for publication 29 January 1982)

Vertical oscillations of a hand-held Slinky spring are investigated. The equation of motion is solved as a Fourier series for general starting conditions. However, it is shown that, for the natural initial condition of dropping from rest, the equation may easily be integrated directly. The resulting motion is periodic with the motion in each period described by a set of second-order polynomials. Finally, the notion of effective mass for the Slinky is investigated.

I. INTRODUCTION

The purpose of this paper is to point out a possible pitfall that may occur when demonstrating simple harmonic motion. Last semester I was lecturing to 260 beginning physics students. When we came to simple harmonic motion, I stood up on the demonstration table, held on to one end of a Slinky spring and let it oscillate vertically. This was to be simple harmonic motion, *but it isn't*. My previous impression had been that the lack of a large mass on the end of the spring could be accounted for by a calculation of the effective mass of the spring. This effective mass would then be used with the spring constant to determine the frequency of the sinusoidal motion of the end of the Slinky. But, in fact, the motion of the end is not even sinusoidal!

Many previous papers[1–7] have investigated the problem of correcting results for the realistic nonzero spring mass. These have studied circular motion,[1,2] general standing waves without gravity,[3] loaded vertical oscillations,[4–6] and oscillations of spiral (nonhelical) springs.[7] However, all of them restrict discussion of oscillations to single-frequency sinusoidal motion.

In Sec. II, we review a simple effective mass calculation for the condition in which a suspended mass is much larger than the spring mass. The equations of motion, boundary conditions, and methods of solution for the Slinky with no suspended mass are presented in Sec. III. The specific initial conditions of dropping the Slinky from rest are applied in Sec. IV, with the surprising result that the motion in each period is described by a set of second-order polynomials in the time and position variables. Finally, in Sec. V, an expression is presented as a candidate for the effective mass of an oscillating spring.

II. EFFECTIVE MASS

Consider a light spring with a large mass M attached to the end. We wish to calculate the effect of the spring mass m on the period T. According to Halliday and Resnick,[8]

$$T = 2\pi[(M + m/3)/k]^{1/2}, \tag{1}$$