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Special-Relativistic Resolution of Ehrenfest's Paradox: Comments on Some Recent Statements by T. E. Phipps, Jr.

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It is shown how a consistent kinematic resolution of Ehrenfest's paradox may be given in accordance with the special theory of relativity. Some statements by T. E. Phipps, Jr., connected with these matters, are commented upon. Problems connected with the relation between stress and strain are solved by a manifestly covariant formulation of Hooke's law.

1. INTRODUCTION

T. E. Phipps, Jr. has recently(1) commented on Ehrenfest's paradox.(2) His statements are interesting, although, in my opinion, somewhat misleading. The importance of the topic treated—is the special theory of relativity able to describe accelerated motion of extended bodies in a logically consistent way?—makes it necessary to give the matters a renewed discussion.

In this paper I will comment on some statements by Phipps, trying to clarify how the questions he raises are treated according to special relativity. A consistent special-relativistic resolution of Ehrenfest's paradox will be formulated including a manifestly covariant formulation of Hooke's law of elasticity.

It has been stressed by Phipps that the logical order of the development of physics is: first get kinematics right, and then go on to dynamics.(3) Kinematics is defined as the science of pure motion, *considered apart from causes*. Now Phipps writes: "So defined, a kinematics of extended structures

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that is (1) logically consistent and (2) of physical pedigree—i.e., capable of preserving the physical connectedness of stress and strain—does not currently exist."

Below I will show that the kinematics of special relativity is both logically consistent and of physical pedigree, in the sense of Phipps.

2. IS THE LORENTZ CONTRACTION UNIVERSAL?

Central in the discussion of Phipps is the relativistic concept "Lorentz contraction." Phipps writes: "With the advent of the Ehrenfest paradox it became apparent on logical grounds that the Lorentz contraction could not occur universally." This is shown by the following consideration: Where a circular disk of solid material is set into rotation about its center (considered at rest in an inertial frame S) the Lorentz transformation, applied locally to each portion of the disk, requires that (a) lengths transverse to the relative motion (as measured in S) transform invariantly, so the disk radius at all times retains a constant length in S, and (b) lengths parallel to relative motion contract, so that the disk rim should begin to contract in S upon the onset of rotation. If the material of the disk is ideally uniform, there is in principle no weakest place where its structural integrity can first fail in consequence of such alleged azimuthal shrinkage. Hence a Lorentz contraction of the rime cannot occur in S—ergo the Lorentz contraction is not universal."

In the special theory of relativity the term "Lorentz contraction" has the following meaning. Let an object be at rest in the inertial frame S'. The frame S moves with velocity $\mathbf{v}$ relative to S'. If the distance in the $\mathbf{v}$ direction between two points on the object, as measured in S', is l', then the distance between the two points, as measured by simultaneity in S, is $l = l'(1 - v^2/c^2)^{1/2}$. That $l < l'$ is referred to as the Lorentz contraction.

The definitional part here is the specification that the length of a body is measured by simultaneity in the reference frame of the *observer*. Then it follows *universally*, as a consequence of the postulates of special relativity, that l is Lorentz contracted relatively to l'. In this connection an important observation was already made by Planck in 1910. He writes[4]: "Der Satz, dass das Volumen eines mit der Geschwindigkeit q bewegten Körpers einem ruhenden Beobachter im Verhältnis $(c^2 - q^2)^{1/2}$: c kleiner erscheint als einem mit der Geschwindigkeit q mitbewegten Beobachter, muss wohl unterschieden werden von dem anderen Satz, dass das Volumen eines Körpers sich im Verhältnis $(c^2 - q^2)^{1/2}$: c verkleinert, wenn er von der Geschwindigkeit 0 auf die Geschwindigkeit q gebracht wird. Ersterer Satz ist eine der Grundforderungen der Relativitätstheorie, letzterer Satz aber is unrichting, wenigstens in dieser Allgemeinheit."[2]

As applied to the situation considered by Phipps the term "Lorentz contraction" concerns the relation between the length l of an azimuthal disk segment, as measured in the rest frame of the axis, and the length l' of the segment as measured by a comoving observer. They are connected by $l = l'(1 - r^2\omega^2/c^2)^{1/2}$, where r is the distance of the segment from the axis and ω is the angular velocity of the disk.

I now turn to the question: in what cases does a body which is put into motion, get a contraction equal to the Lorentz contraction corresponding to its instantaneous velocity?

The answer to this question is: when the acceleration program is such that the rest length of every part of the body remains constant. So in connection with giving motion to a body the Lorentz contraction is *not* universal. The contraction of a body depends upon the acceleration program that it undergoes. Only in the special case of unchanging rest length does the body get a Lorentz contraction as measured in the *initial* rest frame of the body.

3. KINEMATICAL RESOLUTION OF EHRENFEST'S PARADOX[5–8]

In order not to confuse the kinematical discussion with dynamical aspects, I here consider the movement of incoherent dust (no forces between the particles) in a circular pipe.

Ehrenfest formulated his paradox as follows.[2] Let r' be the radius of the rotating disk, as observed in the inertial rest frame S of the axis, and r be the radius of the disk when it is at rest. Then r' must fulfill the following two requirements:

1. The periphery of the disk must be Lorentz contracted: $2\pi r' < 2\pi r$.
2. Since a radial line is moving normally to its direction, it is not Lorentz contracted: $r' = r$.

Assume that the dust consists of n points moving after each other in the pipe along the circular path. One is to realize an acceleration program so that rest distance between all neighboring particles remains constant. Then

[2] The statement, that the volume of a body, with velocity q, as observed by an observer at rest, is less in the ratio $(c^2 - q^2)^{1/2}$: c, than as measured by a comoving observer with velocity q, must be clearly distinguished from another statement, that the volume of a body gets less in the ratio $(c^2 - q^2)^{1/2}$: c, when it is brought from a state of rest to a velocity q. The first statement is one of the fundamental requirements of the theory of relativity, while the latter statement is in general, erroneous.

for all pairs of neighboring particles, the two particles have to be accelerated simultaneously as observed in their instantaneous inertial rest frame S'. A Lorentz transformation from S' to S shows that, as observed in S, the front point is accelerated $dt = \gamma(\omega r^2/c^2)\, d\theta'$ earlier than the rear point. Going around the pipe one finds that the point $n-1$ is to be accelerated at a time $n\,dt$ earlier than the point n, while at the same time it should be accelerated dt later than the point n. So it appears that *due to the relativity of simultaneity the acceleration program that would realize an angular acceleration of the dust, while keeping the rest length between neighboring particles constant, represents kinematically self-contradicting boundary conditions.* This means that the motion corresponding to the condition given in point (1) of Ehrenfest cannot be realized according to the special relativistic kinematics. This is the kinematical resolution of Ehrenfest's paradox.

4. DYNAMICAL CONSIDERATIONS

Phipps[1] has formulated a somewhat broader interpretation of the problem than what usually goes under the name "Ehrenfest's paradox," and which was solved kinematically above. He points out that one wants kinematics to be of "physical pedigree," implying that stress and strain should bear some reasonable relationship to one another.

The problem formulated by Phipps is the following: "We must view as physically anomalous the claim that an observer in S *who measures no azimuthal strain* (i.e., no change in the disk circumference) can measure an azimuthal stress. Yet, if we deny the reality of Herglotz stresses, we have an equally anomalous situation in which the rim riding observer measures azimuthal strain without azimuthal stress. Either way—with or without the Herglotz stress hypothesis—we encounter a nonphysical divorcement of stress from strain; so the situation is physically anomalous."

The proper relativistic way of getting rid of this "anomaly" is to give a manifestly covariant formulation of Hooke's law of elasticity.[8] This may be done as follows.

Let the position four-vector X_μ, $\mu = 1, 2, 3, 4$ describe one end, say the front end, of a body relative to the opposite (back) end of the same body, when the body is stress free. In other words, X_μ gives the equilibrium position of the body's front end. The position four-vector x_μ gives the position of the front end of the body relative to its back end, when the body is stressed.

In the rest frame S' of the body the four-strain is given by

$$L'_\mu = (l'_x, l'_y, l'_z, 0) = (x' - X', y' - Y', z' - Z', 0) \tag{1}$$

Assume S is moving relative to S' with a velocity v in the (negative) x-direction. A Lorentz-transformation from S' to S then gives for the components of the four-strain, as referred to S

$$L_\mu = \left(\gamma l'_x, l'_y, l'_z, \gamma \frac{v}{c} l'_x\right), \qquad \gamma = (1 - v^2/c^2)^{-1/2} \tag{2}$$

Length is defined as the spatial distance between two points on the (moving) object, measured by simultaneity in the rest frame of the observer. From this definition and the Lorentz transformation of coordinates follow

$$l_x = \gamma^{-1} l'_x, \qquad l_y = l'_y, \qquad l_z = l'_z \tag{3}$$

The usual strain-components are identified with l_x, l_y, and l_z. Solving with respect to l'_x, l'_y, and l'_z, and substituting into Eq. (2) gives

$$L_\mu = \left(\gamma^2 l_x, l_y, l_z, \gamma^2 \frac{v}{c} l_x\right) \tag{4}$$

Let F_μ be the Minkowski four-force acting on a body giving it a strain L_μ. In this connection F_μ will be called four-stress. The components of the four-stress are given by

$$F_\mu = \gamma \left(f_x, f_y, f_z, \frac{v}{c} f_x\right) \tag{5}$$

where f_x, f_y, and f_z are the components of the ordinary three-vector stress giving rise to the strain.

Hooke's law may now be given the following covariant formulation:

$$F_\mu = k L_\mu \tag{6}$$

where k is a scalar called the elastic constant. Substitution from Eqs. (3) and (5) into Eq. (6) gives

$$\gamma \left(f_x, f_y, f_z, \frac{v}{c} f_x\right) = k \left(\gamma^2 l_x, l_y, l_z, \gamma^2 \frac{v}{c} l_x\right) \tag{7}$$

or the component equations

$$f_x = k\gamma l_x, \qquad f_y = k\gamma^{-1} l_y, \qquad f_z = k\gamma^{-1} l_z \tag{8}$$

According to the definitions given above the strain and stress components are proportional to the spatial components of spacelike four-vectors. If a four-vector is spacelike in one system, it is so in all systems.

There is no system in which the spatial component of a spacelike four-vector vanishes. The definitions adopted here, therefore, make the existence of both stress and strain Lorentz invariant. In every inertial frame the existence of stress is a accompanied by a nonvanishing strain and vice verse.

We now consider a rotating elastic disk. Let the disk get an isotropic angular acceleration. Then all points of the periphery have identical motion, as observed in S. If the acceleration is given to the disk by several tangential blows on n points around the periphery, then these blows must be given to all points simultaneously, as measured in S. A Lorentz transformation to the rest frame S' of two neighboring points, with angular distance $d\theta$, then shows that the front point is accelerated at a point of time $dt' = \gamma(\omega r^2/c^2)\, d\theta$ earlier than the rear point. Accordingly, the rest length of the element increases. So the periphery is strained, even if its length as observed in S remains constant. The rest length of the periphery is increased by $l' = 2\pi(\gamma - 1)r$, relative to its length when the disk was nonrotating. According to Eq. (3) then strain of the periphery is $l = \gamma^{-1}l' = 2\pi\gamma^{-1}(\gamma - 1)r$. From Eq. (8) follows that a tangential stress $f_x = k\gamma l_x = k2\pi(\gamma - 1)r$ builds up in the disk, as its angular velocity increases. This is just the Herglotz–Dewan–Beran[9–10] stress associated with the transformation of a body from a state of rest to a state of motion, while the length of the body, as measured in the *initial* inertial rest frame S of the body, is kept constant.

It should be noted that this stress has a purely kinematical origin, in the sense that it is a *consequence* of the relativity of simultaneity for the special acceleration program that the points of the disk are assumed to follow. If the Galilean kinematics were correct, an identical program would not imply any azimuthal stresses in the disk.

5. METRIC STANDARDS

Phipps formulates two definitions of a metric standard:

"*Definition A*. A metric standard is any extended material structure that invariably and precisely undergoes the Lorentz contraction in the direction of its motion relative to any inertial system (as measured by Einstein's prescriptions).

Definition B. A metric standard is any extended material structure that in acquiring arbitrary states of relative motion never undergoes any change in its internal energy state."

He then states: "Most relativists would accept Definition B, because they would consider it equivalent to definition A. However, we shall show that it is not equivalent. Instead, if Definition B is taken as the physically fundamental one, it will be shown that the title question can be answered in the negative: A metric standard does not undergo the Lorentz contraction."

In order to demonstrate the shortcoming of Definition A Phipps considers the accelerative transfer of standards. He writes: "If a Lorentz contraction of the structure occurs near the onset of motion, this can result only from initial *differences* in the externally applied distributed forces. Such differences have to be carefully and by free choice *engineered* by the experimenter at rest in S. Not nature, but the experimenter, has to implement the Lorentz contraction. Definition A implies a departure in principle from Newtonian behavior even at the lowest finite speeds (cf. the Herglotz–Noether theorem, which denies a Born rigid body the option to change its state of rotation); so, consistently with that definition, there is no possibility to recover classical rigid-body kinematics as a contained limiting case of Einstein's kinematics. Moreover, any attempt to view Definition A as equivalent to Definition B implies that *differential force applications* to different portions of a structure beget no change in the internal *energy state* of that structure—a claim that appears physically incorrect, and is clearly so in the Newtonian regime near the onset of motion. For such reasons Definition A is rejected here."

Firstly, I will show that Definition A is in conformity with the existence of classical rigid-body kinematics as a contained limiting case of Einstein kinematics.

As to the fact that special relativistic kinematics forbids a Born rigid angular acceleration of a disk, it was shown above that this is due to the relativity of simultaneity. The observable effect connected with this result is the increased rest length, for example, of a thread along the periphery, as its angular velocity increases. This effect is of second order in $(r\omega/c)$, which does not exclude Newtonian rigid-body kinematics as a contained limiting case of Einstein's kinematics.

We now consider the rectilinear acceleration of metric standards. As stated by Phipps, the forces $\mathring{f}_a$ and $\mathring{f}_b$ at the rear end and the front end, respectively, of an accelerated metric standard (Definition A), as measured in an *inertial* frame, are different. I have shown[11] that their difference, as measured in their instantaneous inertial rest frame, is

$$\mathring{f}'_a - \mathring{f}'_b = (gL_0/c^2)\mathring{f}'_b \tag{9}$$

where g is the rest acceleration of the origin and L_0 is the rest length of the rod. Since (at the start of the motion)

$$\tfrac{1}{2}v^2 = gL_0$$

the difference may also be written

$$\mathring{f}'_a - \mathring{f}'_b = \tfrac{1}{2}(v/c)^2 \mathring{f}'_b \tag{10}$$

Like most other relativistic effects this, too, is of second order in (v/c). But because of v's dependence upon L_0 it is a first-order effect in the distance between the points of attack of the forces. This shows that the Newtonian description of extended, accelerated bodies is contained in the relativistic descrition, as the limit for small velocities and for bodies of small extension. By "small extension" is meant linear dimensions small compared to (c^2/g).

Now let us discuss a consequence of regarding Definition A as equivalent to Definition B: that the differential force applications to different portions of a structure (Eq. 9) give no change in the internal energy state of that structure. Phipps is of the opinion that this must be incorrect, "and clearly so in the Newtonian regime near the onset of motion."

In this connection it is important to observe that the forces applied to the accelerated metric standard are different as measured in an *instantaneous*, *inertial* rest frame of the rod. It is shown in Ref. 11 that a transformation to the permanent, accelerated rest frame R of the rod gives the result that the applied forces are position independent in R.

Taking the principle of equivalence into account we immediately see that this result must be correct. By that principle the measurements in R are equivalent to those obtained at the surface of the earth with a rod standing vertically at rest on the surface. The situation is obviously static with no change in the internal energy state of the rod.

The reason for the differential forces, as measured in an *inertial* frame, is that the velocity of the rod changes in such a frame, which implies a changing Lorentz contraction, and correspondingly different accelerations for the different points of the rod. As mentioned earlier, the effect is of second order in (v/c).

Without further arguments I may state that *in special relativity* Definition A is equivalent to Definition B. A metric standard will undergo a Lorentz contraction in accordance with its observed instantaneous velocity, whether Definition A or B is taken as a point of departure.

6. CONCLUSION

Questions connected with the description of accelerated motion of extended bodies have been treated within the frame of special relativity. We have seen that in this theory there exists a purely kinematical resolution of Ehrenfest's paradox. The relativity of simultaneity plays an essential role in this resolution.

In order to obtain a proper connection between stress and strain Hooke's law has been given a manifestly covariant formulation.

The acceleration of metric standards, defined as rods undergoing Lorentz contractions, has been considered. Problems stated by Phipps, that seemed to imply that metric standards so defined were inconsistent with classical rigid-body kinematics as a contained limiting case of Einstein's kinematics, were solved, demonstrating the consistency of conventional theory.

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