

# Relativistic description of a rotating disk\*

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*A consistent relativistic kinematic description of a rotating disk is given. The disk is described from the point of view of an inertial observer momentarily at rest relative to a point on the periphery of the disk, and as observed in a coordinate system  $SS'$  rotating with the disk. The kinematic resolution of Ehrenfest's paradox is stated. Also the following elements of the description are analyzed: (a) the transformation of time from an inertial system, where the axis of the disk is at rest, to  $SS'$ ; (b) the spatial geometry in  $SS'$ ; and (c) the velocity of light in  $SS'$ . The procedure for synchronization of the coordinate clocks on the disk is stated explicitly.*

## I. INTRODUCTION

Some writers talk of the special theory of relativity as the theory which describes phenomena in flat space-time, relative to inertial reference frames, in a Lorentz-covariant way.<sup>1-3</sup> When this point of view is adopted, it is implicitly understood as a consequence of the two postulates of Einstein<sup>4</sup>:

(i) The same laws of electrodynamics and optics will be valid for all frames of reference for which the equations of mechanics hold good.

(ii) Light is always propagated in empty space with a definite velocity  $c$  that is independent of the state of motion of the emitting body.

The first of these demands, in the description of phenomena, a certain covariance between *inertial* systems. The second postulate fixes this as the *Lorentz* covariance.

Such reasoning, however, was sound only before 1910, when the notion of general covariance between arbitrary systems of coordinates was not yet developed.<sup>5</sup> During the next ten years, covariant operations—for example, covariant differentiation—were defined, and it was recognized that it is possible to formulate every classical (non-quantum) theory in tensor form. Accordingly, we have a receipt for formulating the theory in a generally covariant way, independent of coordinates.

This has consequences for the region of applicability of special relativity. By the method of successive inertial systems<sup>6</sup> or by considerations of simplicity and kinematic

consistency (see Sec. V A), one can associate a local coordinate system with an accelerated observer. Now, using the principle of general covariance, which is a mathematical restriction without physical contents, one can transform the laws of phenomena in flat space-time, formulated in an inertial system, to the coordinates of the accelerated observer.<sup>7</sup> In this way special relativity can deal with accelerated coordinate systems.<sup>8-11</sup>

Today one should say that the special theory of relativity is the theory which describes phenomena in flat space-time. Thus the description by accelerated observers, of which a frame of reference rotating with constant angular velocity with respect to an inertial system is one of the simplest examples, is included in special relativity.<sup>12</sup>

A description in terms of arbitrary coordinates implies the use of arbitrary coordinate clocks and arbitrary coordinate measuring rods. A consequence is that the velocity of light is variable in such a description. Thus it is perfectly all right to have a variable velocity of light within the special theory of relativity. However, one should not confuse this coordinate light velocity with the constant light velocity  $c$  measured locally with standard clocks synchronized by the Einstein convention.<sup>13</sup> Such synchronization is generally not possible around a closed path.<sup>14</sup> Thus the light velocity along a circle around the axis in a rotating frame of reference cannot be equal to  $c$ .<sup>15</sup>

In this article a consistent kinematic description of a rotating disk is given within the framework of the special theory of relativity, including the description of the spatial geometry and the velocity of light by the observer rotating with the disk.

## II. THE ROTATING DISK

In a detailed work, Cavalleri<sup>16</sup> gives a dynamical resolution of Ehrenfest's paradox. Sama<sup>17</sup> has recently stated this paradox as given by Ehrenfest,<sup>18</sup> and expressed the opinion that "this paradox per se arises not from an inconsistency in relativity, with which it is not actually connected, but from an ambiguous use of notation." Cavalleri,<sup>16</sup> on the contrary, states in his summary: "It is here shown that Ehrenfest's paradox cannot be solved from a purely kinematical point of view. It follows that the relativistic kinematics for extended bodies is not generally self-consistent." If this statement is correct, it is a very serious matter, because of a fact stated precisely by Phipps<sup>19</sup>: "That dynamics can exist without the foundation of a logically consistent kinematics is absurd; for any structures or motions that can occur for cause can be described apart from causes—and that description is known as kinematics. Kinematics is foundational (logically pre-conditional) to physics. Physics should rest on its foundations, not rescue them. The logical order of development of physics is clear: first get kinematics right, then go on to dynamics."

Thus we want here to proceed beyond the question of notation, and a relativistic kinematic description of a rotating disk is given. The disk is described in three ways: (a) from the point of view of an inertial observer  $S$  at rest relative to the axis of the disk; (b) as measured by

an inertial observer  $S_k$  momentarily at rest relative to a point  $k$  on the periphery of the disk; and (c) as measured in a coordinate system  $SS'$  rotating with the disk. In this way the nature of the seemingly kinematical inconsistencies is made clear, and we see how a careful consideration of the relativity of simultaneity is essential to the kinematic resolution of Ehrenfest's paradox.

The inertial systems associated with  $S$  and  $S_k$  are designated  $SS$  and  $SS_k$ , respectively. The accelerated observer in  $SS'$ , momentarily at rest relative to  $S_k$ , is designated  $S'$ .

### III. DESCRIPTION BY INERTIAL OBSERVERS

This section does not deal with Ehrenfest's paradox. Instead a problem independently raised by Phipps (private communication) concerning the measurements of the inertial observer  $S_k$  is analyzed. It demonstrates clearly the essential role played by the relativity of simultaneity in relativistic descriptions of rotating bodies.

A circular disk with radius  $R$  is rotating with angular velocity  $\omega$ , and the periphery of the disk is marked with  $n$  points, dividing it into  $n$  equal parts, each with length  $2\pi R/n$  as observed by  $S$ .

An inertial observer  $S_k$  momentarily at rest with respect to the  $k$ th element, will measure that it has a rest length  $\gamma 2\pi R/n$ , where  $\gamma = (1 - \beta^2)^{-1/2}$ ,  $v = \omega R$ ,  $\beta = v/c$ . He observes that the center of the disk has velocity  $v$ , and that the opposite element has a velocity  $2v/(1 + \beta^2)$ . Thus, as measured by him, it has a length  $\gamma(2\gamma^2 - 1)2\pi R/n$ . This seems to indicate that  $S_k$  will measure a greater Lorentz contraction for the elements of the circumference of the disk at the opposite side of it than he measures for those on his side of the disk. A tempting conclusion is that he does not measure ellipse shape for the periphery of the disk.

Assume that in the laboratory a ring is drawn on the table along the circumference of the disk.  $S_k$  measures this drawing as an ellipse with axis  $2R$  normal to the direction of velocity and  $2R/\gamma$  along the velocity direction.

According to the preceding statements, one must conclude that  $S_k$  is not measuring the same shape for the circumference of the disk as for the drawing which it just covers. This result is obviously impossible.

The description of  $S_k$ 's measurement of the drawing as an ellipse is a direct consequence of the Lorentz contraction and is doubtless correct. Thus it is the description of how  $S_k$  measures the circumference of the disk that is in error. The correct relativistic description is given below.

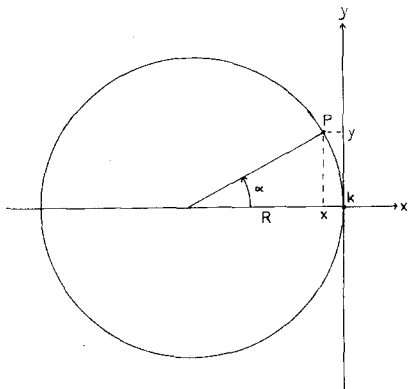


Fig. 1. The disk observed in  $S$ .

Let  $P$  be a point on the periphery of the rotating disk, as shown in Fig. 1. At time  $t = 0$ , the radius vector from the center of the disk to  $P$  makes an angle  $\alpha$  with the  $x$  axis as observed in  $SS$ . This event  $A$  then has the coordinates

$$\begin{aligned} t^A &= 0, & x^A(0) &= -R(1 - \cos \alpha), \\ y^A(0) &= R \sin \alpha. \end{aligned} \quad (1)$$

The inertial observer  $S_k$  moves with velocity  $v = \omega R$  in the  $y$  direction observed by  $S$ . The origin in  $SS_k$  passes the origin in  $SS$  at time  $t = t_k = 0$ , and the axes of  $SS$  and  $SS_k$  are parallel. In  $SS_k$  the event has coordinates

$$t_k^A = -\gamma(v/c^2)R \sin \alpha, \quad x_k^A = x^A, \quad y_k^A = \gamma y^A. \quad (2)$$

The observation of the disk by  $S_k$  is by definition the measurement of the positions of points on the disk, measured simultaneously in  $SS_k$ . Thus we are interested in the position of the point  $P$  at a given point of time in  $SS_k$ , say, at  $t_k = 0$ . By transforming back to  $SS$ , we find that this corresponds to a time

$$t^B = \gamma(v/c^2)y_k^B. \quad (3)$$

Now the spatial coordinates of  $P$  in  $SS$  are

$$x^B = -R(1 - \cos \theta), \quad y^B = R \sin \theta, \quad \theta = \alpha + \omega t^B. \quad (4)$$

Transforming to  $SS_k$  and using (3) now gives

$$x_k^B = x^B, \quad y_k^B = (R/\gamma) \sin \theta. \quad (5)$$

Inserting this in (3) gives

$$\omega t^B = \beta^2 \sin(\alpha + \omega t^B). \quad (6)$$

The coordinates of  $P$  at time  $t_k = 0$  as measured by  $S_k$  are now determined by Eqs. (5) and (6).

Let  $Q$  be a point on the drawn circle with coordinates given by (1) as measured in  $SS$ . Then at  $t_k = 0$  in  $SS_k$  its coordinates are

$$x_k = -R(1 - \cos \alpha), \quad y_k = (R/\gamma) \sin \alpha. \quad (7)$$

Here the angle  $\alpha$  is measured in  $SS$ . Owing to the Lorentz contraction, the connection between  $\alpha$  measured in  $SS$  and the corresponding angle  $\alpha_k$  measured in  $SS_k$ , is

$$\begin{aligned} \sin \alpha &= \sin \alpha_k / (1 - \beta^2 \cos^2 \alpha_k)^{1/2}, \\ \cos \alpha &= \cos \alpha_k / \gamma (1 - \beta^2 \cos^2 \alpha_k)^{1/2}. \end{aligned} \quad (8)$$

Substituting this in Eq. (7) gives

$$x_k = -R[1 - \cos \alpha_k / \gamma (1 - \beta^2 \cos^2 \alpha_k)^{1/2}],$$

$$y_k = R \sin \alpha_k / \gamma (1 - \beta^2 \cos^2 \alpha_k)^{1/2}. \quad (9)$$

This is the description of the drawing by  $S_k$ . It is the equation of an ellipse with semiaxis  $R$  along the  $x_k$  axis and semiaxis  $R/\gamma$  along the  $y_k$  axis.

Using Eqs. (5) and (6) together with (9), we are able to give a comparison of the measurements by  $S_k$  of the circumference of the disk and the drawing around it on the table.

Let  $n = 16$ . Then the points on the periphery of the disk all have an angular difference  $22.5^\circ$  measured in  $SS$ . The broken circle in Fig. 2 and the points on this give the periphery of the disk and the positions of the points at time  $t = 0$  in  $SS$ . Also drawn in Fig. 2 is the ellipse given by Eq. (9): In  $SS_k$  the 16 points are observed on the ellipse with the same abscissas as in  $SS$ . This is not the case with the points on the disk, however, owing to the rotation of the disk. Their coordinates are given by Eqs. (5) and (6). The angle  $\omega t^B$ , given by the transcendental equation (6), was calculated numerically on an electronic computing machine. The coordinates are then calculated from Eq. (5), with the results shown in Fig. 2. The points all have positions, measured at  $t_k = 0$  in  $SS_k$ , which coincided with the drawing on the table. We can see that this had to be the case since Eq. (5) gives, independently of  $t^B$ ,

$$(x_k^B + R)^2 + (\gamma y_k^B)^2 = R^2. \quad (10)$$

This equation describes a circle in the  $x_k, \gamma y_k$  plane, and thus the ellipse in the  $x_k, y_k$  plane. The lines in Fig. 2 are drawn between the same points on the disk, measured at  $t = 0$  in  $SS$  and at  $t_k = 0$  in  $SS_k$ .

The result of this analysis is that the periphery of the disk coincides with the drawing around it, observed both in  $SS$  and in  $SS_k$  for every velocity  $v < c$ .

We now calculate the path of the point  $P$ , observed by  $S_k$ , and compare with the Newtonian result. In  $SS$  the point  $P$  describes a circle. Its coordinates at time  $t$  are

$$x = -[1 - \cos(\omega t)], \quad y = R \sin(\omega t). \quad (11)$$

The transformation to  $SS_k$  is

$$t_k = \gamma[t - (v/c^2)y], \quad x_k = x, \quad y_k = \gamma(y - vt). \quad (12)$$

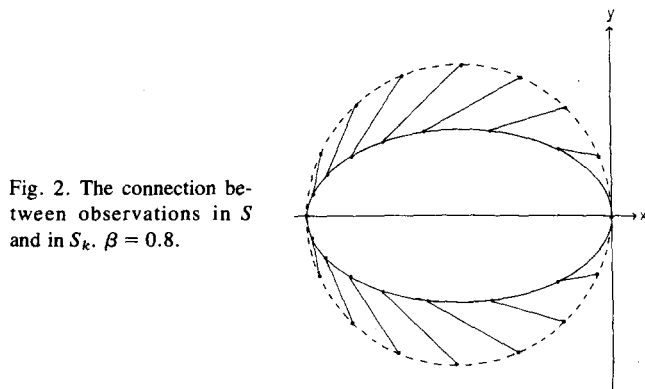


Fig. 2. The connection between observations in  $S$  and in  $S_k$ .  $\beta = 0.8$ .

Thus the coordinates  $x_k$  and  $y_k$  of the point  $P$ , measured by  $S_k$ , expressed by the time  $t_k$ , measured in  $SS_k$ , are

$$\begin{aligned} x_k &= -R(1 - \cos\{\gamma\omega[t_k + (v/c^2)y_k]\}), \\ y_k &= R((1/\gamma)\sin\{\gamma\omega[t_k + (v/c^2)y_k] - \omega t_k\}). \end{aligned} \quad (13)$$

The corresponding nonrelativistic cycloid is

$$x_{k0} = -R[1 - \cos(\omega t)], \quad y_{k0} = R[\sin(\omega t) - \omega t]. \quad (14)$$

The curves given by Eqs. (13) and (14) are drawn in Fig. 3.

Let  $\theta_k$  be the angle between the radius vector from the center of the disk to  $P$  and the  $x$  axis, measured in  $SS_k$ . The function  $f(t_k)$  is defined by

$$f(t_k) = \tan \theta_k, \quad (15)$$

where

$$\begin{aligned} f(t_k) &= [y_k(P) - y_k(\text{center})] / [x_k(P) - x_k(\text{center})] \\ &= (1/\gamma) \tan\{\gamma\omega[t_k + (v/c^2)y_k]\} \end{aligned} \quad (16)$$

by use of Eqs. (13). Thus

$$\tan \theta_k = (1/\gamma) \tan \phi,$$

where

$$\phi = \gamma\omega[t_k + (v/c^2)y_k]. \quad (17)$$

The  $y_k$  component of  $P$ 's velocity is

$$v_{yk} = dy_k/dt_k = -[(1 - \cos \phi)/(1 - \beta^2 \cos \phi)]v \quad (18)$$

by differentiation of the second of Eqs. (13). By using (17), this can be written

$$\begin{aligned} v_{yk} &= -\{[(1 + \gamma^2 \tan^2 \theta_k)^{1/2} \pm 1] \\ &\quad \times [(1 + \gamma^2 \tan^2 \theta_k)^{1/2} \pm \beta^2]^{-1}\}v \end{aligned} \quad (19)$$

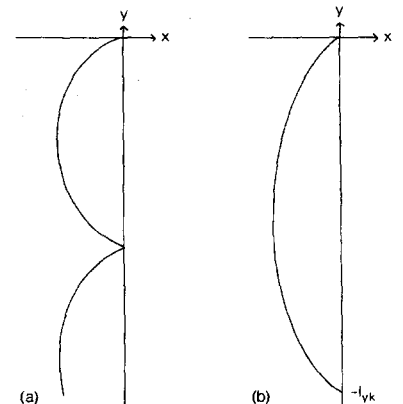


Fig. 3. The path of the point  $k$  as observed from  $S_k$ . (a) Nonrelativistic; (b)  $\beta = 0.8$ .

with  $-$  for  $0 \leq \theta_k < \pi/2$ ,  $3\pi/2 < \theta_k \leq 2\pi$ ,  $+$  for  $\pi/2 < \theta_k < 3\pi/2$ . This velocity is 0 for  $\theta_k = 0$ , and it is  $-2v/(1 + \beta^2)$  for  $\theta_k = \pi$ .

The angular velocity of the radius from the center of the disk to  $P$  is varying with  $\theta_k$ , observed by  $S_k$ . Differentiation of Eqs. (15) and (16) gives

$$\omega_k = \frac{d\theta_k}{dt_k} = \cos^2 \theta_k \frac{df}{dt_k} = \frac{\cos^2 \theta_k}{\cos^2 \phi} \left( 1 + \frac{vv_{yk}}{c^2} \right) \omega. \quad (20)$$

Using Eqs. (17) and (19) in (20), one gets

$$\omega_k = \left\{ (1 + \gamma^2 \tan^2 \theta_k)^{3/2} \cos^2 \theta_k \times \gamma^{-2} [(1 + \gamma^2 \tan^2 \theta_k)^{1/2} \pm \beta^2]^{-1} \right\} \omega \quad (21)$$

with  $-$  for  $0 \leq \theta_k < \pi/2$ ,  $3\pi/2 < \theta_k \leq 2\pi$ ,  $+$  for  $\pi/2 < \theta_k < 3\pi/2$ . The angular velocity has maximal value  $\omega$  for  $\theta_k = 0$  and minimal value  $[(1 - \beta^2)/(1 + \beta^2)]\omega$  for  $\theta_k = \pi$ . Generally, the angular velocity  $\omega_k$  is greater when  $x_k(P) > -R$  than when  $x_k(P) < -R$ . This is the reason that the distance  $l_{yk}$  between the points where  $P$  touches the  $y_k$  axis is greater than  $2\pi R$ , as shown in Fig. 3, in spite of the Lorentz contraction of the disk in the  $y_k$  direction measured by  $S_k$ . The distance  $l_{yk}$  is given by the value of  $t_k$  obtained by setting  $x_k = 0$  in the first of Eqs. (13). This gives

$$\gamma \omega [t_k + (v/c^2)y_k] = 2\pi \quad \text{or} \quad t_k = 2\pi/\gamma \omega - vy_k/c^2. \quad (22)$$

By substitution into the second of Eqs. (13), one obtains

$$l_{yk} = \gamma 2\pi R. \quad (23)$$

#### IV. KINEMATIC RESOLUTION OF EHRENFEST'S PARADOX

We regard the circular rotating disk with  $n$  marks along its periphery, as shown in Fig. 4. It is assumed that  $n$  is so large that the distances between the marks on the periphery are infinitesimally small. With each element between two marks there is associated an inertial observer  $S_k$  such that the element is momentarily at rest as observed by  $S_k$ . We shall investigate whether it is kinematically possible that each mark gets a blow (a sudden acceleration) such that the blows on each pair of marks are observed simultaneously in the inertial system  $SS_k$  associated with the pair.

To investigate this, we associate emission of a light pulse with each blow. With reference to Fig. 4, the hypothetical situation is as follows.  $S_1$  observes the events 1 and 2 simultaneously,  $S_2$  observes the events 2 and 3 simultaneously, and so forth, continuing around the periphery of the disk to  $S_n$  observes the events  $n$  and 1 simultaneously.  $S_1$  moves momentarily with the disk halfway between the points where emissions 1 and 2 happen,  $S_2$  halfway between 2 and 3, and so forth. All the inertial observers  $S_k$  regard themselves as at rest.

Thus, if  $S_1$  should say that the light signals 1 and 2 are emitted simultaneously, he must receive them simultaneously. For that to happen, 1 must be emitted earlier than

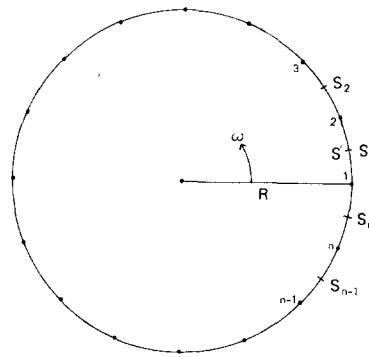


Fig. 4. The rotating disk as observed in  $S$ . Only a few of the points are drawn in the figure.

2, as measured in the laboratory system  $SS$ , since  $S_1$  moves from 1 towards 2.

Assume now that  $S_1$  receives the signals 1 and 2 simultaneously,  $S_2$  receives 2 and 3 simultaneously, and so forth. This implies that, as measured in  $SS$ , emission 1 happens earlier than 2, 2 earlier than 3, and so forth, to  $n-1$  happens earlier than  $n$ . Thus, as measured in  $SS$ , emission 1 happens earlier than  $n$ , but then it is impossible that  $S_n$  receives the signals  $n$  and 1 simultaneously.  $S_n$  then does not observe that the emissions  $n$  and 1 happen simultaneously.

The conclusion is that it is kinematically self-contradicting to assume that the periphery gets blows on all  $n$  points simultaneously as measured in the successive inertial systems  $SS_k$ , where the points  $k$  on the periphery of the disk are momentarily at rest.

There is no corresponding problem if the light signals are emitted simultaneously in the laboratory system  $SS$ . Then all the inertial observers  $S_k$  find that the signal  $k + 1$  was emitted a time  $(v/c^2)2\pi R/n$ ,  $v = \omega R$  earlier than the signal  $k$ .

The reason that there is a problem by simultaneity in the successive rest systems  $SS_k$  and no problem at simultaneity in the laboratory system  $SS$  is that in the former case one observer measures different points of time for the event  $k$ , while in the latter case several observers  $S_k$  measure that the events  $k$  and  $k + 1$  in their local surroundings happen at different points of time, which does not imply any problem.

By definition a *Born rigid motion* of a body leaves lengths unchanged, when measured in the body's proper frame. As made clear by Cavalleri and Spinelli<sup>20</sup> and by Newburgh,<sup>21</sup> a Born rigid motion is not a material property of the body, but the result of a specific program of forces designed to set the body in motion without introducing stresses. The result of the analysis given above shows that a transition of the disk from rest to rotational motion, while it satisfies Born's definition of rigidity, is a *kinematic impossibility*. This is the kinematic resolution of Ehrenfest's paradox.

#### V. DESCRIPTION BY THE ROTATING OBSERVER

##### A. On the transformation of time

Let  $S'$  be an observer on the periphery of the disk, rotating with it. He is not an inertial observer. Assume that the measurements of  $S$  and  $S'$  are connected with a transformation that has the Lorentz transformation between  $S$  and the inertial observer  $S_k$ , which is instantaneously at rest relative to  $S'$ , as limit for small  $\omega$  and large

$R$ , such that the uniform rotation tends to a uniform translation.<sup>22-24</sup> Then if one regards events at  $n$  points on the periphery of the disk, which happen simultaneously as measured by  $S'$ , they happen at different points of time as measured by  $S$ . This implies that an event  $k$  happens at two different points of time, measured by  $S$ , which is impossible. A corresponding difficulty arises for events that happen simultaneously as measured by  $S$ . Then one and the same event is measured by  $S'$  to happen at different points of time—again, an impossibility.

The conclusion is that events measured as simultaneous by  $S$  must also be measured as simultaneous by  $S'$ . This is attained by using the following transformation between the coordinates of the observations of  $S$  and those of  $S'$ :

$$r' = r, \quad \theta' = \theta - \omega t, \quad t' = t. \quad (24)$$

As mentioned by Tangherlini<sup>25</sup> and Atwater,<sup>26</sup> this transformation has a Galilean character. This is due to the fact that angular velocity, as opposed to translational velocity, is a quantity with an absolute value that can be locally measured, both mechanically (by use of Foucault's pendulum) and optically (Sagnac's experiment<sup>27-31</sup>). The transformation (24) is discussed by Berenda<sup>32</sup> and Møller.<sup>33</sup> It implies that Einstein's synchronization convention cannot be used by mutual synchronization of the clocks on the disk. The clocks on the disk indicate always the same point of time as the clocks in an inertial system where the center of the disk is at rest. This means that the clocks on the disk are synchronized by means of a time signal sent out from the center of the disk, with the convention that a clock at a distance  $r$  from the center of the disk, which receives a time signal, emitted at a point of time  $t$ , is adjusted to read  $t + \Delta t$ , where  $\Delta t$  is the time used by the signal from the center of the disk to the clock. The formula for the time interval  $\Delta t$  is deduced below [Eq. (56)] to make the convention explicit.<sup>34</sup>

The line element on the rotating disk based on the transformation (24) is<sup>35</sup>

$$ds^2 = dr^2 + r^2 d\theta'^2 + 2\omega r^2 d\theta' dt - (1 - \omega^2 r^2/c^2) c^2 dt^2. \quad (25)$$

Now consider a standard clock inserted at rest on the disk at a distance  $r$  from its center. The line element of the time track of this clock is then given by

$$ds^2 = - (1 - \omega^2 r^2/c^2) c^2 dt^2. \quad (26)$$

The increase  $d\tau_0$  in the time of the standard clock is given by

$$ds^2 = - c^2 d\tau_0^2. \quad (27)$$

From Eqs. (26) and (27) there follows

$$d\tau_0 = (1 - \omega^2 r^2/c^2)^{1/2} dt. \quad (28)$$

Thus a standard clock at a point on the disk goes slower

than the coordinate clock on it.

During an angular increase of  $2\pi$  for a radius on the disk, the coordinate clock ages  $2\pi/\omega$  like the laboratory clock, while the standard clock ages  $(2\pi/\omega) \times (1 - \omega^2 r^2/c^2)^{1/2}$ . This difference corresponds to the different aging of the resting and the traveling brother in the twin paradox; the coordinate clock measures the aging of the resting brother, and the standard clock measures the aging of the traveling brother.

The slowing down of the standard clocks relative to the coordinate clocks on the disk increases with  $r$ , according to Eq. (28). Thus, events at different distances from the center of the disk, measured as simultaneous on the coordinate clocks, are not measured as simultaneous on the standard clocks. Although the clocks in  $SS_k$  are momentarily at rest relative to those in  $SS'$ , they do not agree with any of the two sets of clocks in  $SS'$  as to the simultaneity of events. This is due to the different synchronization procedures for the clocks in  $SS'$  and in  $SS_k$ ; the latter are synchronized by the Einstein convention.

## B. Spatial geometry on the rotating disk

The spatial geometry on the rotating disk is characterized by the differential spatial line element on it. Several such line elements can be found, depending on the measuring rods and the clocks used. In this sense the spatial geometry on the disk has a conventional character. However, the *proper* spatial line element  $d\sigma$  is of special significance, since it defines a coordinate-independent spatial geometry on the rotating disk. As shown by Møller<sup>36</sup> the element  $d\sigma$  is invariant under a transformation connecting two different coordinate systems inside the same system of reference. The geometry characterized by this proper spatial element is then *the* spatial geometry on the rotating disk.

One can define the proper spatial line element by the following operations. Use the radar method and measure the time  $d\tau$  taken by a light signal between emission and absorption on a standard clock. The spatial line element in the direction  $l$  between the clock and the reflector is then given by

$$d\sigma_l = \frac{1}{2} c d\tau. \quad (29)$$

This equation, being a definition of  $d\sigma_l$ , does not tell anything about the velocity of light measured locally on the disk. This question is treated in Sec. V C.

Let the clock and the reflector be placed on the same radius on the disk with radial coordinates  $r$  and  $r + dr$ . The four-dimensional line element between two events connected with a light signal—that is, between the emission and the absorption—is zero. Using Eq. (25) for this case gives

$$dr^2 - (1 - \omega^2 r^2/c^2) c^2 dt^2 = 0. \quad (30)$$

The time interval between emission and absorption, measured on the coordinate clock, is then

$$dt_r = \frac{2 dr}{c(1 - \omega^2 r^2/c^2)^{1/2}}. \quad (31)$$

Using Eq. (28), one gets for the corresponding interval measured on a standard clock

$$d\tau_r = 2 dr/c. \quad (32)$$

Substitution into Eq. (29) gives the radial proper spatial line element

$$d\sigma_r = dr. \quad (33)$$

Now let the clock and the reflector be placed on the same circle around the center of the disk, with angular coordinates  $\theta'$  and  $\theta' + d\theta'$ . The time interval between emission and absorption measured on the coordinate clock is

$$dt_t = \frac{2r d\theta'}{c(1 - \omega^2 r^2/c^2)}. \quad (34)$$

The corresponding interval measured on the standard clock is

$$d\tau_t = \frac{2r d\theta'}{c(1 - \omega^2 r^2/c^2)^{1/2}}. \quad (35)$$

Thus the tangential proper spatial line element is

$$d\sigma_t = (1 - \omega^2 r^2/c^2)^{-1/2} r d\theta'. \quad (36)$$

From Eqs. (33) and (36) it follows that the spatial geometry on the disk is characterized by the proper spatial line element

$$d\sigma^2 = dr^2 + (1 - \omega^2 r^2/c^2)^{-1} r^2 d\theta'^2. \quad (37)$$

It can be shown by using the above method that the proper spatial line element is generally given by<sup>37</sup>

$$d\sigma^2 = (g_{1k} - g_{14}g_{k4}/g_{44}) dx^k dx^k, \quad l, k = 1, 2, 3 \quad (38)$$

where  $g_{\mu\nu}$  are the elements of the metric tensor. From Eq. (25) one gets

$$\begin{aligned} g_{11} &= g_{22} = 1, & g_{14} &= 0, & g_{24} &= \omega r/c, \\ g_{44} &= -(1 - \omega^2 r^2/c^2). \end{aligned} \quad (39)$$

Substitution into Eq. (38) gives Eq. (37).

The fact that the proper spatial line element between two points depends on the angular velocity  $\omega$  of the rotating disk illustrates the absolute or Galilean character of angular velocity. It also shows that the disk cannot pass from rest to rotation in such a way that both the radial and the tangential proper spatial line elements remain unchanged.<sup>38</sup>

We note that the spatial line element  $dl$  obtained by putting  $dt = 0$  in Eq. (25),

$$dl^2 = dr^2 + r^2 d\theta'^2, \quad (40)$$

is different from  $d\sigma$ . The line element (40) based on simultaneity on the coordinate clocks—that is, on simultaneity on the laboratory clocks—characterizes Euclidean of flat spatial geometry.

Equation (36) for the tangential spatial line element can be written

$$d\theta' = (1 - \omega^2 r^2/c^2)^{1/2} d\sigma_t r^{-1}. \quad (41)$$

Thus on the rotating disk the angle  $d\theta'$  is not measured as the arc length divided by the radius.

From the second of Eqs. (24), it follows that at a certain moment in time, measured on the coordinate clocks,  $d\theta' = d\theta$ . Accordingly, the angle around the periphery of the disk, measured on it, is equal to  $2\pi$ . From Eq. (36) it then follows that the circumference of the disk has the proper length

$$\sigma_t = 2\pi R(1 - \omega^2 R^2/c^2)^{-1/2}. \quad (42)$$

From the first of Eqs. (24) and Eq. (42), it follows that the ratio between the circumference of the disk and its radius, measured on the disk, is

$$f' = 2\pi(1 - \omega^2 R^2/c^2)^{-1/2}. \quad (43)$$

The observer  $S'$  on the disk describes this by saying that the geometry on the disk is non-Euclidian. As remarked by Arzelies,<sup>39</sup> when we transfer the various points of the disk onto an Euclidian plane, giving them the coordinates of the system  $SS'$ , the result is as if the measurements were being made with a meterstick, the length of which varied with its orientation.

### C. The velocity of light on the rotating disk

It is here shown that even locally the velocity of light on the rotating disk is different from  $c$ .

The velocity of a particle is given by

$$w = d\sigma/dt. \quad (44)$$

From Eq. (33) it follows that by radial movement the velocity of a particle, measured on the rotating disk, is given by

$$w_r = dr/dt. \quad (45)$$

The velocity of a photon is now found by putting  $ds = 0$  in Eq. (25). By radial movement this gives

$$dr/dt = (1 - \omega^2 r^2/c^2)^{1/2} c. \quad (46)$$

Thus the velocity of light in the radial direction is

$$c_r = (1 - \omega^2 r^2/c^2)^{1/2} c. \quad (47)$$

From Eq. (36) it follows that by tangential movement the velocity of a particle is given by

$$w_t = (1 - \omega^2 r^2 / c^2)^{-1/2} r d\theta' / dt. \quad (48)$$

Equation (25) gives, for the velocity of a photon moving in tangential direction,

$$r d\theta' / dt = c - \omega r. \quad (49)$$

The velocity of light in tangential direction is then

$$c_t = (1 - \omega^2 r^2 / c^2)^{-1/2} (c - \omega r). \quad (50)$$

Using the above method, one can easily deduce the general formula for the velocity of light in the direction  $l$ , expressed by the components  $g_{\mu\nu}$  of the metric tensor. From Eq. (38) one finds the invariant spatial line element in the direction  $l$ ,

$$d\sigma_l = (g_{11} - g_{14}^2 / g_{44})^{1/2} dx^1. \quad (51)$$

The velocity of a particle moving in the direction  $l$  in a stationary field is

$$w_l = \frac{d\sigma_l}{dt} = \left[ \left( g_{11} - \frac{g_{14}^2}{g_{44}} \right)^{1/2} \frac{dx^1}{dx^4} \right] c. \quad (52)$$

The line element between two points in space-time having coordinates  $(x^1, x^4)$ ,  $(x^1 + dx^1, x^4 + dx^4)$  can be written

$$ds^2 = g_{11}(dx^1)^2 + 2g_{14}dx^1 dx^4 + g_{44}(dx^4)^2. \quad (53)$$

The light velocity in the direction  $l$  is given by putting  $ds = 0$  in Eq. (53), which leads to

$$dx^1 / dx^4 = g_{44} / [(g_{14}^2 - g_{11}g_{44})^{1/2} + g_{14}]. \quad (54)$$

Substitution into Eq. (52) gives the light velocity in the direction  $l$ ,

$$c_l = \frac{(-g_{44})^{1/2}}{g_{14} / (g_{11}g_{44} - g_{14}^2)^{1/2} + 1} c. \quad (55)$$

It can be shown that this equation is in agreement with the corresponding equation given by Møller.<sup>35</sup> Inserting the values of  $g_{\mu\nu}$  from Eq. (39) gives the two equations (47) and (50).

The reason that the velocity of light is locally different from  $c$  is that the clocks on the disk are not synchronized by the Einstein convention, since this convention, if used on the disk, would lead to a kinematic paradox as shown in Sec. V A. Instead, they are synchronized by a time signal emitted from the center of the disk. We are now able to calculate the time taken for this to reach a circle with radius  $r$  about the center. Using Eq. (47), we find<sup>40</sup>

$$\begin{aligned} \Delta t &= [(1/c) \int_0^r dr] (1 - \omega^2 r^2 / c^2)^{-1/2} \\ &= (1/\omega) \arcsin(\omega r / c). \end{aligned} \quad (56)$$

## VI. CONCLUSION

The measurements of a rotating disk by an inertial observer  $S_k$  momentarily at rest relative to the disk and by an accelerated observer  $S'$  rotating with the disk are examined. It is found that, although  $S_k$  and  $S'$  are momentarily at rest relative to each other, their measurements of the disk differ radically.  $S_k$  observes a disk that rolls. He measures an elliptical shape for the circumference of the disk, and he finds that each point of it describes a cycloid-like path, while its center moves along a straight line with constant velocity.  $S'$  observes a disk at rest, while the surroundings are rotating. He measures a circular shape for the circumference of the disk.

It should be noted that the elliptical shape of the periphery as measured by  $S_k$  is not equal to its appearance as photographed by  $S_k$ . After McCrea,<sup>41</sup> Terrell,<sup>42</sup> Penrose,<sup>43</sup> and Weisskopf<sup>44</sup> drew attention to the difference between observation by measurement and visual appearance, much work was done on this topic. Recent investigations and further references are found in the articles by McGill<sup>45</sup> and by Matthews and Lakshmanan.<sup>46</sup> A photograph of the periphery taken by  $S_k$  at the moment he is at rest relative to the disk has a shape exhibited by Scott and van Driel,<sup>47</sup> while a photograph of the periphery taken by  $S'$  permanently at rest relative to the disk has circular shape.

Nor do the observers  $S_k$  and  $S'$  agree as to the simultaneity of events. We regard the situation illustrated in Fig. 4. Light signals 1 and 2 are emitted so that they simultaneously reach the point where  $S_1$  and  $S'$  stay. Both  $S_1$  and  $S'$  are regarding themselves as at rest and halfway between the emission points of 1 and 2. Since  $S_1$  measures the light velocity to be isotropic, he concludes that events 1 and 2 happen simultaneously. This is not the case for  $S'$ . He measures a greater light velocity in the direction in which he observes that the surroundings rotate than in the opposite direction [Eq. (50)]. Thus, light from 2 moves towards him with greater velocity than light from 1. Accordingly, light from 1 has taken a longer time to reach him than light from 2.  $S'$  therefore concludes that event 1 happens earlier than 2.

Assume that  $S$  and  $S'$  measure that the  $n$  points on the periphery of the rotating disk get simultaneous blows. Then the inertial observers  $S_k$ , using clocks synchronized by the Einstein convention, observe that the points  $k$  and  $k + 1$  in their local surroundings get blows at different points of time. Now, the proper length of an accelerated body is measured by inertial observers momentarily at rest relative to the object, measuring its coordinates by simultaneity on their clocks synchronized by the Einstein convention. Thus an observer permanently at rest relative to an accelerated body generally does not measure its proper length, since his clocks need not be synchronized in the above sense. With regard to the rotating disk, we may conclude that the proper lengths of the elements on the periphery change owing to accelerations which are measured as simultaneous by  $S'$ . Moreover, it is shown in Sec. IV that it is kinematically impossible to give the points accelerations that leave the proper lengths of all the elements of the periphery unchanged.

It is found that  $S'$  on the disk observes a non-Euclidian spatial geometry and a variable velocity of light. These conclusions are arrived at within the framework of the special theory of relativity by using a covariant formulation of the equations.

The dynamic aspects concerning the relativistic description of a rotating disk are not touched on here. They are treated by Clark,<sup>48</sup> Cavalleri,<sup>16</sup> Brotas,<sup>49</sup> and McCrea.<sup>50</sup>

The kinematic description of a rotating disk given above does not suggest any nonlinear dependence of the speed of a point on the disk, observed by  $S$ , upon the distance from its axis, as proposed by Hill,<sup>51</sup> Rosen,<sup>52</sup> and Weinstein.<sup>53</sup> This is in accordance with the results of a recent experiment performed by Phipps.<sup>54</sup>

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