Translation:The Theory of the Rigid Electron in the Kinematics of the Principle of Relativity

The Theory of the Rigid Electron in the Kinematics of the Principle of Relativity.

by MAX BORN.

Dedicated to the memory of HERMANN MINKOWSKI.

Introduction.

The great importance of the concepts of rigid body and rigid connection in NEWTONian mechanics, is to the closest related with the fundamental views concerning space and time. Because the requirement that lengths shall be mutually comparable at different times, directly leads to the formation of the concept of measuring rods whose length is independent of time and motion, *i.e.*, which are rigid. Later, this concept of the rigid body proves to be fruitful for the development of dynamics itself; because the rigid body as a continuous mass system of only six degrees of freedom, is not only kinematically of the highest simplicity, but also dynamically by allowing the composition of the forces – which are acting at its points – to "resulting" forces and moments of the same magnitude, whose knowledge is sufficient for the description of motion. All these possibilities are principally based on the GALILEI-NEWTONian connection of space and time into a four-dimensional manifold[WS 1] (which I will call "world" following MINKOWSKI[1]); a connection essentially contained in the theorem, that the natural laws not only shall be independent from the choice of the origin and the unit of time, as well as from the location of the spatial reference system and the unit of length, but also from a uniform translation given to the reference system under maintenance of the measure of time.

Exactly these foundations of kinematics are the ones to be abandoned, when the electrodynamic relativity principle – as stated by LORENTZ, EINSTEIN, MINKOWSKI and others – comes into play. Because here, the connection of space and time into the "world" is different: the independence of the natural laws from the uniform translation of the spatial reference system only then takes place, when also the time parameter experiences a change, which not only tantamounts to a displacement of the origin and the choice of another unit. It is most closely connected to this, that measuring rods that maintain their length at uniform translation in the co-moving coordinate system, suffer a contraction in the direction of their velocity when viewed from a stationary system. By that, the concept of the rigid body fails, at least in its form adapted to NEWTONian kinematics.

However, a corresponding concept is by no means to be dispensed with in the new kinematics as well, since otherwise the comparison of lengths of moving bodies at different times becomes illusory. No difficulty arises at the formation of this concept for systems moving relative to each other, and the authors (mentioned above) of the foundational works on this theory, are using this circumstance without giving a particular definition of rigidity.

The difficulty only then arises, when accelerations are present. Only one attempt exists – made by Einstein[2] –, without completely clarifying the subject. Therefore I have undertaken the elaboration of the *kinematics of the rigid body on the basis of the relativity postulate*. Its possibility is probable from the outset, because the Newtonian kinematics represents in every relation a limiting case of the new kinematics, namely that one in which the speed of light c is seen as infinitely great. The method used by me, consists in defining rigidity by a differential law instead of an integral law.

Indeed, one arrives in this way at the general rigidity conditions in differential form, which are very analogous to the corresponding conditions of the old kinematics and also pass into them for $c = \infty$. The integration of these conditions, which is very easily executable in the old kinematics in general, and which leads to the constancy of the distance of rigidly connected points, was only executed by me for the case of uniformly accelerated translation; the result is hardly inferior to the old kinematics in terms of simplicity and illustratability, and makes the assumption near at hand, as to what may be the result at arbitrary curvilinear and rotatory motions; though I'm not discussing this. The main result is (at uniform motion), that the motion of a single point of a rigid body co-determines the motion of all other ones by a very simple law, *i.e.*, that the body thus only has *one degree of freedom*.

Now the question arises, whether (as in the old mechanics) the rigid body has simple properties in its dynamic behavior also in the new kinematics, and of course it will be about electromagnetic forces.

The practical value of the new definition of rigidity must therefore prove itself in the dynamics of the electron; the greater or lesser clarity of the results achieved there, is to be used to a certain degree also in favor or against the acceptance of the relativity principle *per se*, since experiments have probably given no definite decision and maybe won't give one.

The theory of Abraham, which studies the motion of an electron (being rigid in the ordinary sense) in the force field produced by itself, has not only led to a qualitatively satisfying explanation of the phenomena of inertia of free electrons on a pure electric basis, but has also led to a quantitative law for the dependency of the electrodynamic mass from velocity at very small accelerations, which is probably not to be seen as disproved by the experiments. Though this theory which superimposes the rigid body (which is suited to the old mechanics) upon electrodynamics, doesn't satisfy the relativity principle, and this is the reason why its further development – at which Sommerfeld,[3] P. Hertz,[4] Herglotz,[5] Schwarzschild[6] and others are participating – leads to extraordinary mathematical complications. Now, already Lorentz tried to adapt Abraham's theory to the relativity principle, and for that purpose he constructed his "deformable" electron. Exactly this electron is to be denoted as rigid according to the definition given by me. That despite of this agreement, Lorentz's theory gives rise to contradictions to which Abraham[7] has alluded, is due to the fact that the laws of the composition of forces at the rigid body into resulting forces, were taken over without criticism from the old mechanics; as to how these laws are to be modified, will be given by itself in the representation chosen here. Lorentz's formula for the dependency of mass from velocity, which represent the experiments as good as Abraham's formula, proves to be correct also in the more strict theory. Because this law, as it was already noticed by Einstein, and which was discussed by me in the paper[8] concerning "the inertial mass and the relativity principle" for arbitrary currents, is a direct consequence of kinematics and is not at all essentially connected with the actual electrodynamic mass, the "rest mass".

Yet, my theory strictly provides the dependency of the rest mass on acceleration for a class of motions, which corresponds – being the principally simplest accelerated motions – to the uniformly accelerated ones of the old mechanics, and which I call "hyperbolic motions", namely the rest mass proves to be constant up to enormous accelerations. Equations of motion in the form of the mechanical fundamental equations[9] which are adapted to the relativity principle, apply to this motion. Yet, since every accelerated motion can be approximated by such hyperbolic motions as long as their acceleration doesn't vary too suddenly, one achieves in this way an electrodynamic foundation of the fundamental equations of mechanics. This theory fails only for very rapidly changing accelerations; then also radiation resistances arise besides the inertial resistances. *It is remarkable, that an electron causes no actual radiation, as great as its acceleration may be, but it drags its field along with it*, which was up to now only known for uniformly moving electrons. *The radiation and the resistance of the radiation only arise at deviations from hyperbolic motion.*

My rigidity definition proves to be appropriate for the system of Maxwell's electrodynamics quite in the same way, as the old definition of rigidity for the system of Galilei-Newtonian mechanics. The rigid electron in this sense, represents the dynamically most simple motion of electricity. One can even go so far to assert, that the theory provides clear hints to an atomistic structure of electricity, which is not at all the case in Abraham's theory. Thus my theory is in agreement with the atomistic instinct of so many experimentalists, for which the interesting attempt of Levi-Civita[10] – to describe the motion of electricity as a freely moving fluid being bound by no kinematic conditions – will hardly find applause.

Since the simplicity of the dynamics is therefore not inferior to the simplicity of kinematics of the new rigid body, then one will ascribe to this concept of rigidity the same fundamental importance in the system of the electromagnetic world-picture, as the ordinary rigid body in the system of the mechanical world-picture.

First chapter. The kinematics of the rigid body.

§ 1. The rigid body of old mechanics.

With respect to the electrodynamic applications of the second and the third chapter, we won't concern ourselves with rigid systems of discrete points, but with continuous rigid bodies. A continuous current of matter can be represented in the way named after Lagrange, by giving the space coordinates x, y, z as functions of time t and of three parameters ξ, η, ζ – for example the values of x, y, z at time $t = 0$:

$$\left\{\begin{array}{l} x = x(\xi, \eta, \zeta, t), \\ y = y(\xi, \eta, \zeta, t), \\ z = z(\xi, \eta, \zeta, t). \end{array}\right. \tag{1}$$

The mass system is rigid, when the distance of any two of its points

$$r = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2} \tag{2}$$

is independent of time, thus equal to

$$\sqrt{(\xi_1 - \xi_2)^2 + (\eta_1 - \eta_2)^2 + (\zeta_1 - \zeta_2)^2}.$$

Then it follows from that, that equations (1) have the form

(3)
$$\begin{cases} x = a_1 + a_{11}\xi + a_{12}\eta + a_{13}\zeta, \\ y = a_2 + a_{21}\xi + a_{22}\eta + a_{23}\zeta, \\ z = a_3 + a_{31}\xi + a_{32}\eta + a_{33}\zeta, \end{cases}$$

where the quantities $a_\alpha,\ a_{\alpha\beta}$ are functions of time t, and the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = (a_{\alpha\beta})$$

is orthogonal[11]; *i.e.*, when $\overline{A}$ denotes the transposed matrix of A, and 1 is the unit matrix, then it is

(4)
$$\overline{A}A = 1$$

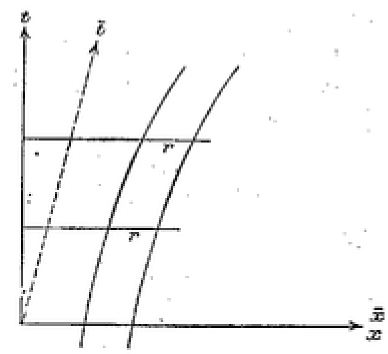

Fig. 1.

To oversee the generalization capability of this condition of the kinematics of the relativity principle, it is advantageous to use the interpretation (used by Minkowski in the work just cited) of the variables x, y, z, t as parallel coordinates in a space of four dimensions called "world". In the following, the figures shall always mean the plane cut $y = 0,\ z = 0$ through a four-dimensional space; within them, we draw the x-axis horizontally, and the t-axis upwards. The path of a point is represented in the $xyzt$-manifold (world) as a curve, the "world line", and the motion of a body is represented by a bundle of world lines. The previous condition $dr/dt = 0$ now means, that the connecting line of the passage points (of two world-lines each) through a three-dimensional structure t = const., has the same length for all those structures. Thus it is related to the three-dimensional points t = const. "parallel" to space $t = 0$.

The importance of this rigidity condition for Newtonian mechanics, lies in the fact that it is invariant with respect to transformations, which transfer the Newtonian equations of motion into themselves. These transformations have the form, when the origin is maintained:

(5)
$$\begin{cases} x = k_{11}\overline{x} + k_{12}\overline{y} + k_{13}\overline{z} + k_1 t, \\ y = k_{21}\overline{x} + k_{22}\overline{y} + k_{23}\overline{z} + k_2 t, \\ z = k_{31}\overline{x} + k_{32}\overline{y} + k_{33}\overline{z} + k_3 t, \end{cases}$$

where $k_{\alpha\beta},\ k_\alpha$ are constants, and the matrix

$$K = (k_{\alpha\beta})$$

is orthogonal:

(6)
$$\overline{K}K = 1$$

The orthogonal constituent only denotes the passage from the initial coordinate system to a system rotated around the origin; yet the second part denotes a uniform translation in time. This is represented in our four-dimensional world as passage from the initial t-axis to an inclined $\overline{t}$-axis. One immediately sees (Fig. 1), that the quantity r indeed remains unchanged at this occasion.

The relativity principle of electrodynamics states an invariance of natural laws with respect to *other* linear substitutions, and by that the meaning of quantity r becomes irrelevant. These "Lorentz transformations" connect the four magnitudes x, y, z, t with four new ones $\overline{x}, \overline{y}, \overline{z}, \overline{t}$ by such linear equations

$$\left\{\begin{aligned} x &= k_{11}\overline{x} + k_{12}\overline{y} + k_{13}\overline{z} + k_{14}\overline{t},\\ y &= k_{21}\overline{x} + k_{22}\overline{y} + k_{23}\overline{z} + k_{24}\overline{t},\\ z &= k_{31}\overline{x} + k_{32}\overline{y} + k_{33}\overline{z} + k_{34}\overline{t},\\ t &= k_{41}\overline{x} + k_{42}\overline{y} + k_{43}\overline{z} + k_{44}\overline{t}, \end{aligned}\right. \tag{7}$$

which transform the expression

$$x^2 + y^2 + z^2 - c^2t^2 \tag{8}$$

into itself, where c means the speed of light.

Here, the time (or rather the quantity $ct\sqrt{-1}$) is thus transformed with the coordinates in a symmetric way, and not only the t-axis

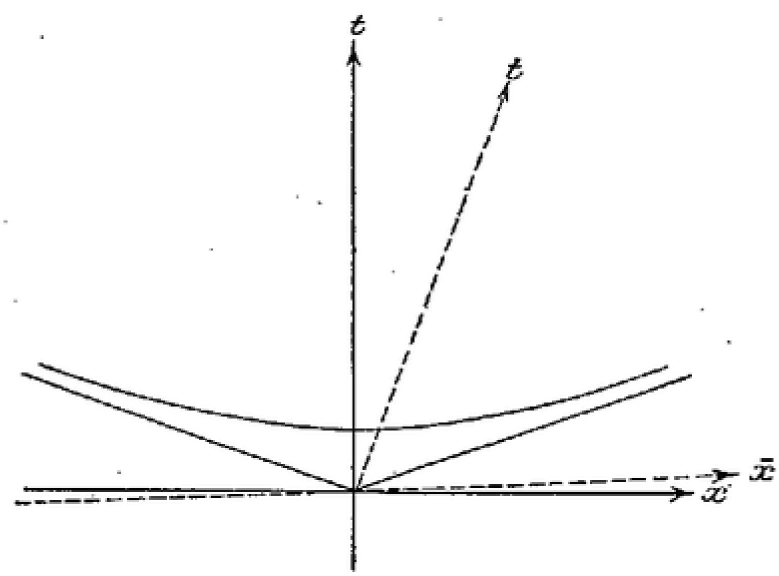

Fig. 2.

becomes inclined at this transformation, but also space $t = 0$ obtains another location in the four-dimensional world.[12] Since spaces t = const. don't go over to spaces $\overline{t}$ = const, then neither quantity r nor condition $dr/dt = 0$ is invariant.

At first it seems impossible as well, to provide an analogous condition between two world-lines, since there are no three-dimensional spaces with respect to transformations (7), (8), which are so preferred as previously the spaces t = const. with respect to (5).

Therefore for the sake of generalization, one has to look after another definition of rigidity in the old kinematics. For that, one can use the circumstance that one can replace condition r = const. (taking place between two finitely distant world lines) by a differential condition between infinitely adjacent world lines, so that, when the differential condition is satisfied in the whole space, it gives rise to equation r = const.

For that purpose, we consider at time t the distance of two infinitely adjacent world lines, *i.e.*, the line element

$$ds = \sqrt{dx^2 + dy^2 + dz^2}$$

If one sets this equal to a constant ϵ, then equation

$$ds^2 = \epsilon^2$$

represents an infinitely small sphere. This emerges from an infinitely small ellipsoid during the motion represented by (1), which one obtains when one represents the quantity ds^2 by means of the equations

(9)
$$\begin{cases} dx = \frac{\partial x}{\partial \xi} d\xi + \frac{\partial x}{\partial \eta} d\eta + \frac{\partial x}{\partial \zeta} d\zeta, \\ dy = \frac{\partial y}{\partial \xi} d\xi + \frac{\partial y}{\partial \eta} d\eta + \frac{\partial y}{\partial \zeta} d\zeta, \\ dz = \frac{\partial z}{\partial \xi} d\xi + \frac{\partial z}{\partial \eta} d\eta + \frac{\partial z}{\partial \zeta} d\zeta, \end{cases}$$

as quadratic form of $d\xi, d\eta, d\zeta$; let this form be:

(10)
$$\begin{cases} ds^2 = p_{11} d\xi^2 + p_{22} d\eta^2 + p_{33} d\zeta^2 \\ \qquad +2p_{12} d\xi\, d\eta + 2p_{13} d\xi\, d\zeta + 2p_{23} d\eta\, d\zeta \end{cases}$$

There, the matrix of the "deformation quantities" $p_{\alpha\beta}$

$$P = (p_{\alpha\beta})$$

from the matrix

(11)
$$A = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \zeta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} & \frac{\partial y}{\partial \zeta} \\ \frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \eta} & \frac{\partial z}{\partial \zeta} \end{pmatrix}$$

is composed in this way:

(12)
$$P = \overline{A} A$$

Now, we will call this motion *only in the smallest parts as rigid*, when an infinitely small structure is not changed during motion, thus when all $p_{\alpha\beta}$ are independent of time. Thus we have the *infinitesimal rigidity conditions*:

(13)
$$\frac{dp_{\alpha\beta}}{dt} = 0$$

When ξ, η, ζ are the initial values of x, y, z, then matrix A is equal to unit matrix 1 for $t = 0$, thus (12) reads:

$$P = \overline{A}A = 1$$

It is now an elementary theorem of infinitesimal geometry[13], that when these conditions are satisfied everywhere, it is about the motion of a rigid body.

This infinitesimal rigidity condition (13) can now easily be transfered to the kinematics of the relativity principle.

§ 2. The differential conditions of rigidity.

In the following, only such quantities shall have a meaning, which are invariant with respect to the Lorentz transformations (7), (8).

Now we consider a current, which we represent instead of equations of form (1), by the following equations which better correspond to the symmetry of quantities x, y, z, t required by the relativity principle:

(14)
$$\begin{cases} x = x(\xi, \eta, \zeta, \tau), \\ y = y(\xi, \eta, \zeta, \tau), \\ z = z(\xi, \eta, \zeta, \tau), \\ t = t(\xi, \eta, \zeta, \tau). \end{cases}$$

There, let τ be the *proper time*, *i.e.*, the identity exists:

(15)
$$\left(\frac{\partial x}{\partial \tau}\right)^2 + \left(\frac{\partial y}{\partial \tau}\right)^2 + \left(\frac{\partial z}{\partial \tau}\right)^2 - c^2\left(\frac{\partial t}{\partial \tau}\right)^2 = -c^2;$$

τ is measured starting at any "cross section" of the currents.

ξ, η, ζ shall characterize individual current-filaments, though we leave open their meaning. Now we set for the time being:

(16)
$$\begin{cases} x(0,0,0,\tau) = \mathfrak{x}(\tau), \\ y(0,0,0,\tau) = \mathfrak{y}(\tau), \\ z(0,0,0,\tau) = \mathfrak{z}(\tau), \\ t(0,0,0,\tau) = \mathfrak{t}(\tau), \end{cases}$$

and consider the filament of world lines, surrounding world line (16) $\xi = \eta = \zeta = 0$.

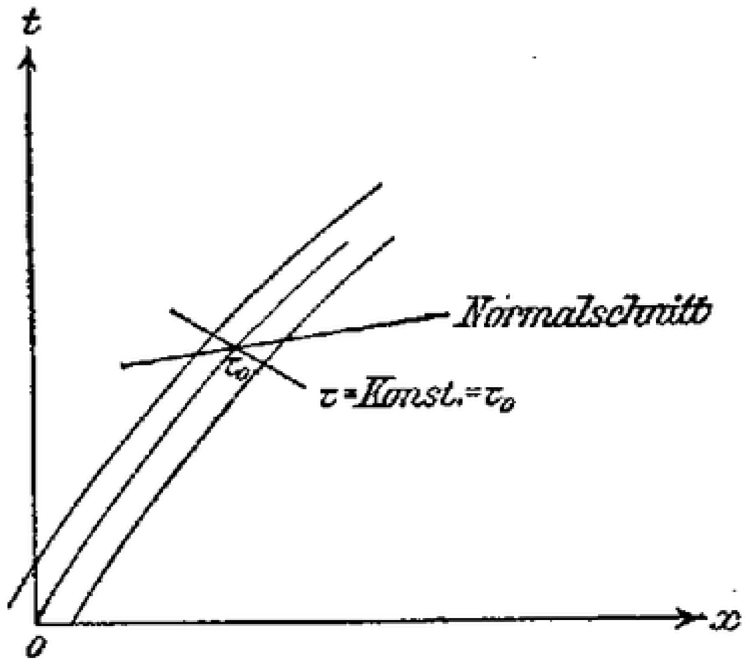

Fig. 3.

This can be represented as follows:

$$\text{(17)} \qquad \begin{cases} x = \mathfrak{x} + x_\xi d\xi + x_\eta d\eta + x_\zeta d\zeta + \dots, \\ y = \mathfrak{y} + y_\xi d\xi + y_\eta d\eta + y_\zeta d\zeta + \dots, \\ z = \mathfrak{z} + z_\xi d\xi + z_\eta d\eta + z_\zeta d\zeta + \dots, \\ t = \mathfrak{t} + t_\xi d\xi + t_\eta d\eta + t_\zeta d\zeta + \dots, \end{cases}$$

where we confine ourselves to terms being linear in the increments $d\xi$, $d\eta$, $d\zeta$ (which are first to be imagined as small, though finite). Here, $\mathfrak{x}, \mathfrak{y}, \mathfrak{z}, \mathfrak{t}$ are the functions defined by (16), and it is set:

$$x_\xi = \frac{\partial x}{\partial \xi}(0, 0, 0, \tau), \dots$$

Two spacetime vectors with components x_1, y_1, z_1, t_1 and x_2, y_2, z_2, t_2 are called *normal*, when their direction are conjugated with respect to the invariant hyperbolic structure

$$\text{(18)} \qquad x^2 + y^2 + z^2 - c^2 t^2 = -1,$$

thus when

$$\text{(19)} \qquad x_1 x_2 + y_1 y_2 + z_1 z_2 - c^2 t_1 t_2 = 0$$

All vectors being normal to a time-like vector[14] x_1, y_1, z_1, t_1 are satisfying a three-dimensional linear structure, which can be made to space $t = 0$ by a suitable Lorentz transformation; we call it the *normal cut* of the vector.

The concepts being so defined, are evidently invariant with respect to Lorentz transformations.

Now we consider a certain point P upon the world line $\xi=\eta=\zeta=0$, belonging to the value τ_0 of the proper time. Through this point P, we lay the normal cut to the velocity vector $\mathfrak{x}'_0, \mathfrak{y}'_0, \mathfrak{z}'_0, \mathfrak{t}'_0$ in P:

{{MathForm1|(20)|$\mathfrak{x}'_0\left(x-\mathfrak{x}_0\right)+\mathfrak{y}'_0\left(y-\mathfrak{y}_0\right)+\mathfrak{z}'_0\left(z-\mathfrak{z}_0\right)-c^2\mathfrak{t}'_0\left(t-\mathfrak{t}_0\right)=0$

where

$$\mathfrak{x}'=\frac{d\mathfrak{x}}{d\tau}=\left[\frac{\partial x}{\partial\tau}\right]_{\xi=\eta=\zeta=0},\dots$$

and index 0 means, that $\tau=\tau_0$ is to be inserted into the functions.

In (20), we replace x, y, z, t by their expressions (17) as functions of $d\xi, d\eta, d\zeta$ and τ:

$$\left\{\begin{array}{l}\mathfrak{x}'_0\left\{\mathfrak{x}-\mathfrak{x}_0+x_\xi d\xi+x_\eta d\eta+x_\zeta d\zeta+\dots\right\}+\dots\\ \qquad\dots-c^2\mathfrak{t}'_0\left\{\mathfrak{t}-\mathfrak{t}_0+t_\xi d\xi+t_\eta d\eta+t_\zeta d\zeta+\dots\right\}=0\end{array}\right.\tag{21}$$

We can see this as an equation for τ, from which one can calculate the values of proper time τ (belonging to normal cut τ_0) upon the neighboring line $d\xi, d\eta, d\zeta$. Since the difference $\tau-\tau_0=d\tau$ is small, then (21) will be a linear equation in $d\tau$. Namely, if one expands

$$\begin{cases}\mathfrak{x}=\mathfrak{x}_0+\mathfrak{x}'_0 d\tau+\dots,\\ \dots\dots\\ x_\xi=x^0_\xi+\left(x'_\xi\right)_0 d\tau+\dots,\\ \dots\dots\end{cases}\tag{22}$$

and if one considers, that according to (15) it is identical in τ

$$\mathfrak{x}'^2+\mathfrak{y}'^2+\mathfrak{z}'^2-c^2\mathfrak{t}'^2=-c^2,\tag{23}$$

then it follows from (21), when one neglects all quadratic terms in $d\xi, d\eta, d\zeta, d\tau$:

$$\left\{\begin{array}{l}c^2 d\tau=\mathfrak{x}'_0\left(x^0_\xi d\xi+x^0_\eta d\eta+x^0_\zeta d\zeta\right)+\dots\\ \qquad\dots-c^2\mathfrak{t}'_0\left(t^0_\xi d\xi+t^0_\eta d\eta+t^0_\zeta d\zeta\right),\end{array}\right.\tag{24}$$

or when we

$$\begin{cases}x^0_\xi d\xi+x^0_\eta d\eta+x^0_\zeta d\zeta=\Xi,\\ y^0_\xi d\xi+y^0_\eta d\eta+y^0_\zeta d\zeta=\mathrm{H},\\ x^0_\xi d\xi+z^0_\eta d\eta+z^0_\zeta d\zeta=\mathrm{Z},\\ x^0_\xi d\xi+t^0_\eta d\eta+t^0_\zeta d\zeta=\mathrm{T},\end{cases}\tag{25}$$

set as:

(26)
$$c^2 d\tau = \mathfrak{x}_0'\Xi + \mathfrak{y}_0'\mathrm{H} + \mathfrak{z}_0'\mathrm{Z} - c^2\mathfrak{t}_0'\mathrm{T}$$

Now we consider the (one-shell) hyperbolic structure located around the point $\xi = 0,\ \eta = 0,\ \zeta = 0,\ \tau = \tau_0$ as center:

(27)
$$(x - \mathfrak{x}_0)^2 + (y - \mathfrak{y}_0)^2 + (z - \mathfrak{z}_0)^2 - c^2(t - \mathfrak{t}_0)^2 = \epsilon^2$$

This cuts the normal cut (20) into a figure, which is to be seen as the "rest shape" of the filament at this place.

If we accordingly replace in (27), x, y, z, t by expressions (17), and then the quantities $\mathfrak{x}, \mathfrak{y}, \mathfrak{z}, \mathfrak{t}, x_\xi, \ldots$ by the expansions (22), it follows

(28)
$$\begin{cases} \left(\mathfrak{x}_0' d\tau + x_\xi^0 d\xi + x_\eta^0 d\eta + x_\zeta^0 d\zeta\right)^2 + \ldots \\ \qquad \cdots - c^2\left(\mathfrak{t}_0' d\tau + t_\xi^0 d\xi + t_\eta^0 d\eta + t_\zeta^0 d\zeta\right)^2 = \epsilon^2, \end{cases}$$

and herein is (upon the normal cut $d\tau$) the function of $d\xi, d\eta, d\zeta$ defined by (26); thus (28) goes over into:

(29)
$$\begin{cases} \left\{\left(1 + \frac{\mathfrak{x}_0'^2}{c^2}\right)\Xi + \frac{\mathfrak{x}_0'\mathfrak{y}_0'}{c^2}\mathrm{H} + \frac{\mathfrak{x}_0'\mathfrak{z}_0'}{c^2}\mathrm{Z} - \mathfrak{x}_0'\mathfrak{t}_0'\mathrm{T}\right\}^2 + \ldots \\ \qquad \cdots - c^2\left\{\frac{\mathfrak{t}_0'\mathfrak{x}_0'}{c^2}\Xi + \frac{\mathfrak{t}_0'\mathfrak{y}_0'}{c^2}\mathrm{H} + \frac{\mathfrak{t}_0'\mathfrak{z}_0'}{c^2}\mathrm{Z} + \left(1 - \mathfrak{x}_0'^2\right)\mathrm{T}\right\}^2 = \epsilon^2, \end{cases}$$

By that, the rest shape is given as a quadratic form in $d\xi, d\eta, d\zeta$. Since point $\xi = \eta = \zeta = 0$, $\tau = \tau_0$ was an arbitrary point of the current, one can omit indices 0 and replace $\mathfrak{x}' \ldots$ by $x_\tau, \ldots$. If we then write (29) in the form

(30)
$$\begin{cases} (c_{11}d\xi + c_{12}d\eta + c_{13}d\zeta)^2 + (c_{21}d\xi + c_{22}d\eta + c_{23}d\zeta)^2 \\ \qquad +(c_{31}d\xi + c_{32}d\eta + c_{33}d\zeta)^2 + (c_{41}d\xi + c_{42}d\eta + c_{43}d\zeta)^2 = \epsilon^2, \end{cases}$$

then the rectangular matrix for 4 rows and 3 columns $C = (c_{\alpha\beta})$ is equal to the product of two matrices S and A, which are formed from the derivatives of functions (14):

(31)
$$C = SA$$

namely it is:

(32)

$$S = \begin{pmatrix} 1+\frac{x_\tau^2}{c^2} & \frac{x_\tau y_\tau}{c^2} & \frac{x_\tau z_\tau}{c^2} & -\frac{x_\tau t_\tau}{ic} \\ \frac{y_\tau x_\tau}{c^2} & 1+\frac{y_\tau^2}{c^2} & \frac{y_\tau z_\tau}{c^2} & -\frac{y_\tau t_\tau}{ic} \\ \frac{z_\tau x_\tau}{c^2} & \frac{z_\tau y_\tau}{c^2} & 1+\frac{z_\tau^2}{c^2} & -\frac{z_\tau t_\tau}{ic} \\ -\frac{t_\tau x_\tau}{ic} & -\frac{t_\tau y_\tau}{ic} & -\frac{t_\tau z_\tau}{ic} & 1-t_\tau^2 \end{pmatrix}$$

(33) $$A = \begin{pmatrix} x_\xi & x_\eta & x_\zeta \\ y_\xi & y_\eta & y_\zeta \\ z_\xi & z_\eta & z_\zeta \\ ict_\xi & ict_\eta & ict_\zeta \end{pmatrix}$$

If we now develop the quadratic from (30) to $d\xi, d\eta, d\zeta$, then one has:

(34) $$\begin{cases} p_{11}d\xi^2 + p_{22}d\eta^2 + p_{33}d\zeta^2 \\ \qquad +2p_{12}d\xi\, d\eta + 2p_{13}d\xi\, d\zeta + 2p_{23}d\eta\, d\zeta \end{cases}$$

where it becomes

(35) $$P = (p_{\alpha\beta}) = \overline{C}C = \overline{A}\overline{S}SA$$

With the aid of equation (15) causing the determinant of S to vanish, this relation can still further be simplified; namely it is easily given by computation:

(36) $$\overline{S}S = S$$

and (33) thus goes over into:

(37) $$P = \overline{A}SA$$

This is analogues to equation (12) derived in § 1. The six quantities $p_{\alpha\beta}$ are to be denoted as "deformation quantities", and would be of importance in a theory of elasticity adapted to the relativity principle.

We will call a filament as rigid in the smallest parts, whose rest shape is independent from proper time τ, *i.e.*, for which the six equations

(38) $$\frac{\partial p_{\alpha\beta}}{\partial\tau} = 0$$

hold.

When these equations are satisfied in the whole space, then we are dealing with the motion of a rigid body.

By that, we have gained the general differential conditions of rigidity. Since they are solely formed by the aid of such concepts being invariant with respect to Lorentz transformations, they thus have necessarily the same property.

§ 3. The continuity equation and the incompressible current.

If ϱ is the density belonging to the current (1), then we know that it is connected with the velocity components

(39) $$w_x = \frac{\partial x}{\partial t},\ w_y = \frac{\partial y}{\partial t},\ w_z = \frac{\partial x}{\partial t}$$

by the continuity equation.

This can be formulated in two ways. According to the one of Euler, one sees ϱ, w_x, w_y, w_z as functions of x, y, z, t; then the continuity equation reads:

(40) $$\frac{\partial \varrho}{\partial t} + \frac{\partial \varrho w_x}{\partial x} + \frac{\partial \varrho w_y}{\partial y} + \frac{\partial \varrho w_z}{\partial z} = 0$$

According to the one of Lagrange, x, y, z, ϱ are seen as functions of ξ, η, ζ, t; then the condition reads:

(41) $$\frac{\partial \varrho \Theta}{\partial t} = 0$$

where Θ is the functional determinant

(42) $$\Theta = \begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \zeta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} & \frac{\partial y}{\partial \zeta} \\ \frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \eta} & \frac{\partial z}{\partial \zeta} \end{vmatrix}$$

The connection between both formulas is caused by the identity:[15]

(43) $$\frac{\partial \varrho}{\partial t} + \frac{\partial \varrho w_x}{\partial x} + \frac{\partial \varrho w_y}{\partial y} + \frac{\partial \varrho w_z}{\partial z} = \frac{1}{\Theta} \frac{d \varrho \Theta}{\partial t}$$

Both forms of the continuity equation can be transfered to the representation of the current with the aid of proper time by equations (14). First, it is evidently given:

(44) $$w_x = \frac{x_\tau}{t_\tau},\ w_y = \frac{y_\tau}{t_\tau},\ w_z = \frac{z_\tau}{t_\tau}$$

If we furthermore replace ϱ by the "rest density"

(45) $$\varrho^* = \frac{\varrho}{t_\tau}$$

then (40) goes over into:

(46) $$\frac{\partial \varrho^* x_\tau}{\partial x} + \frac{\partial \varrho^* y_\tau}{\partial y} + \frac{\partial \varrho^* z_\tau}{\partial z} + \frac{\partial \varrho^* t_\tau}{\partial t} = 0$$

We get the analogue to formula (41) by showing the correctness of the identity corresponding to (43)

(47) $$\frac{\partial \varrho^* x_\tau}{\partial x} + \frac{\partial \varrho^* y_\tau}{\partial y} + \frac{\partial \varrho^* z_\tau}{\partial z} + \frac{\partial \varrho^* t_\tau}{\partial t} = \frac{1}{D}\frac{\partial \varrho^* D}{\partial \tau}$$

where D means the functional determinant

(48) $$D = \begin{vmatrix} x_\xi & x_\eta & x_\zeta & x_\tau \\ y_\xi & y_\eta & y_\zeta & y_\tau \\ z_\xi & z_\eta & z_\zeta & z_\tau \\ t_\xi & t_\eta & t_\zeta & t_\tau \end{vmatrix}$$

For this purpose, we momentarily replace for the sake of shortness:

x, y, z, t by x_1, x_2, x_3, x_4

ξ, η, ζ, τ by $\xi_1, \xi_2, \xi_3, \xi_4$

Then we have for the left-hand side of (47):

$$\sum_\alpha \frac{\partial\left(\varrho^* \frac{\partial x_\alpha}{\partial \xi_4}\right)}{\partial x_\alpha} = \sum_{\alpha,\beta} \frac{\partial\left(\varrho^* \frac{\partial x_\alpha}{\partial \xi_4}\right)}{\partial \xi_\beta}\frac{\partial \xi_\beta}{\partial x_\alpha} = \sum_{\alpha,\beta}\left(\varrho^* \frac{\partial^2 x_\alpha}{\partial \xi_\beta \partial \xi_4} + \frac{\partial \varrho^*}{\partial \xi_\beta}\frac{\partial x_\alpha}{\partial \xi_4}\right)\frac{\partial \xi_\beta}{\partial x_\alpha}$$

If we now denote (in the scheme of determinant D) by $S\left(\partial x_\alpha / \partial \xi_\beta\right)$ the sub-determinant belonging to $\partial x_\alpha / \partial \xi_\beta$, then it is given by successive differentiation of equations (14) with respect to x_α, and by solving of the linear equations emerging in this way:

(49) $$\frac{\partial \xi_\beta}{\partial x_\alpha} = \frac{S\left(\frac{\partial x_\alpha}{\partial \xi_\beta}\right)}{D}$$

If this is inserted above, it follows

$$\begin{aligned}\sum_\alpha \frac{\partial\left(\varrho^* \frac{\partial x_\alpha}{\partial \xi_4}\right)}{\partial x_\alpha} &= \frac{1}{D}\sum_{\alpha,\beta}\left\{\varrho^* \frac{\partial^2 x_\alpha}{\partial \xi_\beta \partial \xi_4} S\left(\frac{\partial x_\alpha}{\partial \xi_\beta}\right) + \frac{\partial \varrho^*}{\partial \xi_\beta}\frac{\partial x_\alpha}{\partial \xi_4} S\left(\frac{\partial x_\alpha}{\partial \xi_\beta}\right)\frac{\partial \xi_\beta}{\partial \xi_\alpha}\right\} \\ &= \frac{1}{D}\left\{\varrho^* \frac{\partial D}{\partial \xi_4} + \frac{\partial \varrho^*}{\partial \xi_4} D\right\}\end{aligned}$$

according to general determinant theorems. Thus it follows

$$\sum_{\alpha}\frac{\partial\left(\varrho^{*}\frac{\partial x_{\alpha}}{\partial\xi_{4}}\right)}{\partial x_{\alpha}}=\frac{1}{D}\frac{\partial\varrho^{*}D}{\partial\xi_{4}}$$

which is the identity (47) that had to be proven.

Consequently, one can write the continuity condition in the form

(50)
$$\frac{\partial\varrho^{*}D}{\partial\tau}=0$$

Formulas (46), (47), (50) have invariant character with respect to Lorentz transformations

The quantity

$$\varrho^{*}D=\varrho_{0}$$

is only dependent from ξ,η,ζ; when D is equal to 1 for $\tau=0$ (which one can always assume), then ϱ_{0} is the "initial value of the rest density".

A current is called incompressible in the old kinematics, when ϱ is constant and independent from time t. In the new kinematics we will define it as follows:

A current is incompressible, when the rest density ϱ^{*} *is constant,* i.e., *independent from proper time* τ.

Two forms of the incompressibility condition are given from (46) and (50).

At first one can namely (46) write:

$$\varrho^{*}\left(\frac{\partial x_{\tau}}{\partial x}+\frac{\partial y_{\tau}}{\partial y}+\frac{\partial z_{\tau}}{\partial z}+\frac{\partial t_{\tau}}{\partial t}\right)+\frac{\partial\varrho^{*}}{\partial x}x_{\tau}+\frac{\partial\varrho^{*}}{\partial y}y_{\tau}+\frac{\partial\varrho^{*}}{\partial z}z_{\tau}+\frac{\partial\varrho^{*}}{\partial t}t_{\tau}=0$$

or

$$\varrho^{*}\left(\frac{\partial x_{\tau}}{\partial x}+\frac{\partial y_{\tau}}{\partial y}+\frac{\partial z_{\tau}}{\partial z}+\frac{\partial t_{\tau}}{\partial t}\right)+\frac{\partial\varrho^{*}}{\partial\tau}=0;$$

now, if ϱ^{*} shall not depend on τ, then *the first form of the incompressibility condition* follows:

(53)
$$\frac{\partial x_{\tau}}{\partial x}+\frac{\partial y_{\tau}}{\partial y}+\frac{\partial z_{\tau}}{\partial z}+\frac{\partial t_{\tau}}{\partial t}=0$$

The second form is immediately given from (50):

(54)
$$\frac{\partial D}{\partial\tau}=0$$

By that, when D is equal to 1 for $\tau=0$, D is identically equal to 1, and by (51):

$\varrho^* = \varrho_0(\xi, \eta, \zeta)$

§ 4. The uniform translation of the rigid body.

We now want to integrate the differential conditions of rigidity (38) for the simplest case of uniform translation. When we consider, that rigidity must be identical with incompressibility in this case, then we not only achieve by that a criterion as to what our rigidity definition means *mutatis mutandis*, but simultaneously also a method for integration.

Thus we set

(55) $$y = \eta,\ z = \zeta$$

and assume that x and t only depend on ξ and τ. Then we obtain from (32) and (33):

$$S = \begin{pmatrix} 1 + \frac{x_\tau^2}{c^2} & 0 & 0 & -\frac{x_\tau t_\tau}{ic} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{t_\tau x_\tau}{ic} & & & 1 - t_\tau^2 \end{pmatrix}$$

$$A = \begin{pmatrix} x_\xi & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ ict_\xi & 0 & 0 \end{pmatrix}$$

If one forms from that the matrix:

$P = \overline{A}SA,$

then one easily finds

$$P = \begin{pmatrix} (x_\xi t_\tau - x_\tau t_\xi)^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The six rigidity conditions thus are reduced to one equation:

(56) $$\frac{d}{d\tau}\left(x_\xi t_\tau - x_\tau t_\xi\right) = 0$$

On the other hand, determinant (48) becomes:

(57)
$$D = \begin{vmatrix} x_\xi & 0 & 0 & x_\tau \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ t_\xi & 0 & 0 & t_\tau \end{vmatrix} = \begin{vmatrix} x_\xi & x_\tau \\ t_\xi & t_\tau \end{vmatrix}$$

Thus the incompressibility condition

$$\frac{dD}{d\tau} = 0$$

is identical with the rigidity condition (56).

Consequently we can also replace the latter by the other form (53) of the incompressibility condition, which assumes the form here:

(58)
$$\frac{\partial x_\tau}{\partial x} + \frac{\partial t_\tau}{\partial t} = 0$$

The integration is now easily to be executed in this form.

If one puts:

(59)
$$x_\tau = p,\ t_\tau = -q,$$

then one obtains the two equations for q, q:

(60)
$$\begin{cases} \frac{\partial p}{\partial x} - \frac{\partial q}{\partial t} = 0, \\ \\ p^2 - c^2 q^2 = -c^2 \end{cases}$$

These are equivalent to a partial differential equation for a function of two independent variables. Namely, if one puts

(61)
$$p = \frac{\partial \varphi}{\partial t},\ q = \frac{\partial \varphi}{\partial x},$$

then the first equation (60) is satisfied, and the second one goes over into

(62)
$$\varphi_t^2 - c^2 \varphi_x^2 = -c^2$$

The simplest solution of these equation is obtained, when one puts φ_t and φ_x equal to constants γ and $-\delta$, which must satisfy the condition

(63)
$$\gamma^2 - c^2 \delta^2 = -c^2$$

Then it becomes

$$p = x_\tau = \gamma,\ q = -t_\tau = -\delta$$

from which it follows:

(64) $$\begin{cases} x = W(\xi) + \gamma\tau, \\ t = V(\xi) + \delta\tau, \end{cases}$$

where W and V are two arbitrary functions of ξ. Due to equation (63), the form of equations (64) is indeed conserved, when x, t are subjected to a Lorentz transformation.

Equations (64) together with (55) represent a rectilinear uniform motion. Functions $W(\xi),\ V(\xi)$ are determined by the value, which x and t shall have for $\tau = 0$. It is not convenient here to assume $x = \xi$ for $\tau = 0$, but functions $W(\xi),\ V(\xi)$ are so to be determined, that formulas (64) represent that Lorentz transformation which transforms the body into rest, *i.e.*, it is to be set:

(65) $$\begin{cases} x = \alpha\xi + \gamma\tau, \\ t = \beta\xi + \delta\tau, \end{cases}$$

where the conditions

(66) $$\alpha^2 - c^2\beta^2 = 1,\ \alpha\gamma - c^2\beta\delta = 0,\ \gamma^2 - c^2\delta^2 = -1$$

are satisfied.

As soon as one of the two quantities $\varphi_t,\ \varphi_x$ depends on t in (62), then this must also be the case for the other one. In this case, the integration of (62) can be easily executed by the aid of a Legendre transformation. Then, one can namely introduce the quantity

(67) $$\varphi_t = p$$

as independent variable besides x, and imagine t [from (67)] to be calculated as a function of x and p. If one then introduces (instead of φ) the new unknown function

$$\psi(p, x) = \varphi - pt,$$

then it is

(68) $$\begin{cases} \psi_p = \varphi_t t_p - p t_p - t = -t, \\ \psi_x = \varphi_x + \varphi_t t_x - p t_x = \varphi_x. \end{cases}$$

Consequently, (62) goes over into the following equation for $\psi(p, x)$:

$$p^2 - c^2\psi_x^2 = -c^2$$

this can be immediately integrated. It is given:

(69) $$\begin{cases} \psi_x = \sqrt{1 + \frac{p^2}{c^2}} = q, \\ \psi = qx - w(p), \end{cases}$$

where w means an arbitrary function. From that it follows by differentiation with respect to q under consideration of (68):

(70) $$\frac{p}{c^2 q}x - w'(p) = -t$$

If one consequently imagines p as being calculated as a function of t and inserted into $\varphi = \psi + pt$, then one has the desired most general solution of (62):

(71) $$\varphi = qx - w(p) + pt$$

According to (59) and (61), it is evidently

$$\frac{x_\tau}{t_\tau} = \frac{dx}{dt} = -\frac{\varphi_t}{\varphi_x}$$

from which it follows that any equation $\varphi = \mathbf{const.} = -\xi$ represents the world line of a point of the rigid body. From (70) and (71) we consequently find the following representation of the world lines:

(72) $$\begin{cases} \frac{p}{c^2}x + qt = qw', \\ qx + pt = w - \xi, \end{cases}$$

or, when solved with respect to x and t:

(73) $$\begin{cases} x = q(w-\xi) - pqw', \\ t = -\frac{p}{c^2}(w-\xi) + q^2 w'. \end{cases}$$

Here, the world lines of the rigid body are so described, that x and t are given as functions of the independent variables $\xi,\ p$. We now want to discuss this representation.

First it is to be noticed, that the rectilinear translatory motion only depends on *one arbitrary function of an argument $w(p)$*. Thus one can say, that also here *only one degree of freedom* (as in the old kinematics) is present. There, the usage of the independent variables $p = x_\tau$ is essential, which still will be of great importance. Furthermore, equations (73) go over into the corresponding representation of the uniform translation of the old kinematics, when $c = \infty$. Because $q = \sqrt{1 + \left(p^2/c^2\right)}$ becomes equal to 1 in this case; from the second equation (73) it follows for $c = \infty$, that p only depends on t, so that the first one assumes the form

$$x = \xi + a(t)$$

Finally we direct our attention to the characterization of the world lines in the xt-plane. One recognizes, that (72) and (73) have the form of a Lorentz transformation and their inverse ones, which transform the variables x, y into the variables $\overline{x} = w - \xi$, $\overline{t} = qw'$, and they read:

(74) $$\begin{cases} \overline{x} = qx + \frac{p}{c}ct, \quad x = q\overline{x} - \frac{p}{c}c\overline{t}, \\ \\ c\overline{t} = \frac{p}{c}x + qct, \quad ct = -\frac{p}{c}\overline{x} + qc\overline{t} \end{cases}$$

because the equations (66) between the coefficient are evidently satisfied due to (60).

Thus we are faced with a bundle of Lorentz transformations depending on the parameter p. The motion, or rather the corresponding bundle of world lines, can now be described as follows.

If one gives a certain value ξ_1 to ξ, then x and t are given by equations (73) as specific functions of p, which represent the world line of point ξ_1. The components of the velocity world-vector with respect to axes x and t, are p and $-q$. By that single curve ξ_1, all curves of the bundle are co-determined. One has to construct it as follows: To the tangent in a point p of the curve, one lays the line being normal to it in the sense of § 2 (p. 12); this forms (together with the tangent) the x- and t-axis of a transformed coordinate system. Upon this x-axis, one draws the distance $\xi_1 - \xi$ in the unit of this coordinate system[16]. If one now moves this coordinate system along curve ξ_1, then point ξ follows the world line belonging to the parameter value ξ. All points of such a normal (x-axis) belong to the same value p, thus they have the same velocity.

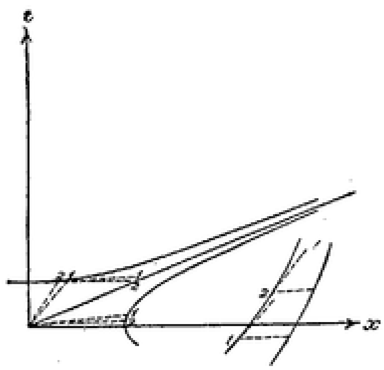

Fig. 4.

The uniform motion of a rigid body is so constituted, that – as soon as *one* point is transformed to rest – all of its points are transformed to rest by the same transformation. This rest transformation is exactly (74). The lines of same velocity p = const., except at uniform motion, always have an envelope; the regularity of motion stops at this one. At given dimensions of the body, the curvature of world lines thus cannot exceed a certain limit, and *vice versa. From that it follows, that a rigid body is necessarily extended into all directions, and has to be the smaller, the bigger the accelerations are that it should experience*. Here, we have the first hint at the fundamental importance of atomistics in the new dynamics. If the rigid body carries a substance of rest density ϱ^*, then it is independent of p, and it is a function of ξ, η, ζ, which we denote by

$$\varrho_0(\xi, \eta, \zeta)$$

§ 5. Hyperbolic motion.

The simplest motion different from uniform translation will be obtained by us, when we set the arbitrary function $w = 0$ in (72) and (73). Then it becomes

$$\left\{\begin{array}{l} x = -q\xi, \\ t = \frac{p}{c^2}\xi. \end{array}\right. \tag{75}$$

If one eliminates p therefrom, then it follows

$$x^2 - c^2t^2 = \xi^2 \tag{76}$$

From that one recognizes, that the corresponding world line in the xt-plane and the planes $y = \eta,\ z = \zeta$ being parallel to it, are hyperbolas having the lines corresponding to the speed of light as asymptotes, and which are cutting the x-axis in the distance ξ from the origin. A bundle of such hyperbolas represents a motion, at which the rigid body comes from infinity and is approaching the origin, then it is turning back and is moving away into infinity again, where its velocity decreases from the speed of light to zero first, and after the turning back increases again to c. This motion, to some extent analogous to the uniformly accelerated motion of old kinematics, we want to call *hyperbolic motion* shortly.

Since the origin is a quite arbitrary point, then the hyperbolas

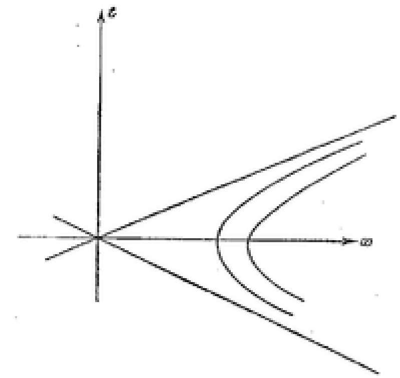

Fig. 5.

$$(x-\alpha)^2 - c^2(t-\beta)^2 = \xi^2$$

represent no essentially different motion; only the velocity is then different from zero for $t=0$. Thus we will be able to confine ourselves to formulas (75), (76).

This hyperbolic motion proves to be not only kinematically, but also dynamically as the simplest one. This is closely connected with the circumstance, that any world line is osculated by a hyperbola in any of its points P, the "curvature hyperbola", where the vector of magnitude $b = c^2/\xi$ directed from its center to point P, represents the acceleration vector of the world line in P.

Indeed, if we calculate the acceleration components of hyperbolic motion, then we find at first

(78) $$\frac{\partial^2 y}{\partial\tau^2} = 0,\ \frac{\partial^2 z}{\partial\tau^2} = 0$$

To calculate the x- and t-components, we consider the equations

(79) $$\begin{cases} \xi_t = -p, & p_t = \frac{c^2 q^2}{\xi} \\ \xi_x = -q, & p_x = \frac{pq}{\xi}. \end{cases}$$

Then it becomes

$$\frac{\partial^2 x}{\partial\tau^2} = p_\tau = p_x x_\tau + p_t t_\tau = \frac{pq}{\xi}p - \frac{c^2 q^2}{\xi}q$$

Thus we obtain

(80) $$\frac{\partial^2 x}{\partial\tau^2} = b_x = -qb$$

as well as

(81) $$\frac{\partial^2 t}{\partial\tau^2} = b_t = \frac{p}{c^2}b$$

where

(82) $$b=\sqrt{b_x^2-c^2b_t^2}=\frac{c^2}{\xi}$$

is the magnitude of acceleration. From (81), (81), (82), the previous assertion follows.[17]

The acceleration is thus constant for every world line of hyperbolic motion in terms of their magnitude; here lies the analogy with the uniformly accelerated motion of old mechanics represented by parabolic world lines. Thus it is the simplest accelerated motion, and every motion can approximated by hyperbolic motions. Based on that, we want to find out more precisely the dynamics of hyperbolic motions, above all we try to determine the force exerted by an electrically charged rigid body upon itself. The result (as approximation) will then also give information about all motions, in which the magnitude of the acceleration vector is only slightly changed.

Second chapter. The field of the rigid electron in hyperbolic motion.

§ 6. Retarded potentials and field strengths.

The forces exerted by moving electric charges, which enter into the equations of motion of these charges, are derived from certain auxiliary quantities, the retarded potentials and field strengths. We want to summarize the expression for these quantities, which are employed in the following.

Let an electric current be represented by equations of form (14); let the initial value of its rest density (see § 3, p. 18, (51)) be:

$$\varrho_0(\xi,\eta,\zeta)$$

Then the retarded potentials are given by the following expressions:

(83) $$\begin{cases} & 4\pi\Phi_x(x,y,z,t) \\ = & \iiint\left[\frac{\overline{\varrho}_0\overline{x}_\tau}{(x-\overline{x})\overline{x}_\tau+(y-\overline{y})\overline{y}_\tau+(z-\overline{z})\overline{z}_\tau-c^2(t-\overline{t})\overline{t}_\tau}\right]_{h=0} d\overline{\xi}\,d\overline{\eta}\,d\overline{\zeta} \\ & \cdots\cdots \\ & \cdots\cdots \\ & 4\pi\Phi(x,y,z,t) \\ = & \iiint\left[\frac{c\overline{\varrho}_0\overline{t}_\tau}{(x-\overline{x})\overline{x}_\tau+(y-\overline{y})\overline{y}_\tau+(z-\overline{z})\overline{z}_\tau-c^2(t-\overline{t})\overline{t}_\tau}\right]_{h=0} d\overline{\xi}\,d\overline{\eta}\,d\overline{\zeta} \end{cases}$$

where $\overline{x},\overline{y},\overline{z},\overline{t}$ and $\overline{x}_\tau,\overline{y}_\tau,\overline{z}_\tau,\overline{t}_\tau$ denote the functions (14) or their derivatives with respect to τ, taken from the arguments $\overline{\xi},\overline{\eta},\overline{\zeta}$, and the function of $x,y,z,t,\overline{\xi},\overline{\eta},\overline{\zeta}$ is to be inserted for τ into the brackets, which is given by solving the equation

(84) $$h=(x-\overline{x})^2+(y-\overline{y})^2+(z-\overline{z})^2-c^2(t-\overline{t})^2=0$$

with respect to τ; namely that solution of the equation which is definitely determined is to be taken[18], for which $t>\overline{t}$. As to how the expressions (83), which are surely not used yet for continuous currents in this form, are connected to the ordinary formulas for the retarded potentials, shall be shortly explained in the next paragraphs.

The electric field strength $\mathfrak{E}$ and the magnetic one $\mathfrak{M}$ are derived from the potentials according to the vector equations:

(85)
$$\begin{cases} \mathfrak{E} = -\frac{1}{c}\frac{\partial}{\partial t}\left(\Phi_x, \Phi_y, \Phi_z\right) - \mathbf{grad}\Phi, \\ \mathfrak{M} = \mathbf{curl}\left(\Phi_x, \Phi_y, \Phi_z\right). \end{cases}$$

The potentials (83) are solutions of equations[19]

(86)
$$\begin{cases} \frac{\partial}{\partial x}\mathrm{lor}\Phi + \frac{1}{c^2}\frac{\partial^2\Phi_x}{\partial t^2} - \Delta\Phi_x = \frac{\varrho^*}{c}x_\tau, \\ \dots\dots \\ \frac{\partial}{\partial t}\mathrm{lor}\Phi + \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} - \Delta\Phi = \varrho^* t_\tau, \end{cases}$$

namely especially such solutions, for which the quantity

(87)
$$\mathrm{lor}\Phi = \frac{\partial\Phi_x}{\partial x} + \frac{\partial\Phi_y}{\partial y} + \frac{\partial\Phi_z}{\partial z} + \frac{1}{c}\frac{\partial\Phi}{\partial t}$$

vanishes for itself.

Equations (86) are the Lagrangian equations regarding the variation problem[20], to find those functions $\Phi_x, \Phi_y, \Phi_z, \Phi$ for which the integral

(88)
$$W = \iiiint\left\{\frac{1}{2}\left(\mathfrak{M}^2 - \mathfrak{E}^2\right) - \frac{\varrho^*}{c}\left(\Phi_x x_\tau + \Phi_y y_\tau + \Phi_z z_\tau - \Phi t_\tau\right)\right\} dx\, dy\, dz\, dt$$

extended over an area G of the $xyzt$-manifold, becomes an extremum, where the current of electricity and the values of the potentials are given upon the boundary of G.

§ 7. Comparison of the expressions for retarded potentials.

The expressions (83) for the potentials can be seen as the superposition of the elementary potential, stemming from the individually moving points of the current. For the latter, one namely has the expressions according to Liénard and Wiechert[21]

(89)
$$\begin{cases} 4\pi\varphi_x = \left[\frac{ew_x}{cr\left(1-\frac{w_r}{c}\right)}\right]_{t-\bar{t}=\frac{r}{c}} \\ \dots\dots \\ \dots\dots \\ 4\pi\varphi = \left[\frac{e}{r\left(1-\frac{w_r}{c}\right)}\right]_{t-\bar{t}=\frac{r}{c}} \end{cases}$$

Where e denotes the charge of the acting point

(90)
$$x = \overline{x}(t),\ y = \overline{y}(t),\ z = \overline{z}(t);$$

furthermore

(91) $$r=\sqrt{(x-\overline{x})^2+(y-\overline{y})^2+(z-\overline{z})^2}$$

its distance from reference point x, y, z,

(92) $$w_r=\frac{1}{r}\left\{(x-\overline{x})\overline{w}_x+(y-\overline{y})\overline{w}_y+(z-\overline{z})\overline{w}_z\right\}$$

is the component of its velocity w_x, w_y, w_z in the direction of r, and $\overline{t}$ instead of t is to be set in the brackets, which is given from the equation

(93) $$t-\overline{t}=\frac{r}{c}$$

If one now has a continuous current, then world line (90) is to be replaced by a bundle of world lines, by bringing function (90) into the form (1) by inserting three parameters ξ, η, ζ, and by replacing e by density $\varrho(\xi, \eta, \zeta)$. Then, functions $\varphi_x, \varphi_y, \varphi_z, \varphi$ become also dependent on ξ, η, ζ, and one can integrate them over the entire space. There it is also to be noticed, that it is

$$dx\ dy\ dx=\Theta d\xi\ d\eta\ d\zeta$$

at the space integration, and that the functional determinant Θ connects itself with density ϱ to the initial density $\varrho_0=\varrho\Theta$ according to § 3, (41).

The emerged expressions can easily be brought into the form (83). For this, one only has to write the equations of motion of the acting point homogeneously in the form:

(94) $$x=\overline{x}(\tau),\ y=\overline{y}(\tau),\ z=\overline{z}(\tau),\ t=\overline{t}(\tau)$$

where τ denotes the proper time, and to replace ϱ by the rest density ϱ^*. Equation (93) then goes over into

(95) $$h=(x-\overline{x})^2+(y-\overline{y})^2+(z-\overline{z})^2-c^2(t-\overline{t})^2=0$$

from which τ is unequivocally given at the additional condition $t>\overline{t}$.[22]

The connection of expressions (83) with the otherwise ordinary expressions for the potentials is also easy to find out. The latter ones read:[23]

(96) $$\begin{cases} 4\pi\Phi_x=\iiint\frac{d\overline{x}\ d\overline{y}\ d\overline{z}}{r}\left[\frac{\varrho w_x}{c}\right]_{\overline{t}=t-\frac{r}{c}} \\ \cdots\cdots \\ \cdots\cdots \\ 4\pi\Phi=\iiint\frac{d\overline{x}\ d\overline{y}\ d\overline{z}}{r}\left[\varrho\right]_{\overline{t}=t-\frac{r}{c}} \end{cases}$$

There, the current is to be imagined as represented by equations of the form (1) (p. 6), furthermore it is

$r=\sqrt{(x-\overline{x})^2+(y-\overline{y})^2+(z-\overline{z})^2}$

and $\overline{x},\overline{y},\overline{z},\ \overline{t}=t-r/c$ are to be introduced as arguments in the brackets. The integrations in (95) are to be extended over all charges, *i.e.*, over temporally variable boundaries, since they are in motion. The passage from expressions (95) to expressions (83) now exactly consists in this, that one brings the integrals to invariable limits independent of time. This has to be happening in the following way.

In the equations of current (1), we replace t by $\overline{t}=t-r/c$, then we obtain equations of the form

(97)
$$\begin{cases}\overline{x}=\overline{x}\left\{\xi,\eta,\zeta,\bar{t}(\overline{x},\overline{y},\overline{z},\bar{t})\right\}\\ \overline{y}=\overline{y}\left\{\xi,\eta,\zeta,\bar{t}(\overline{x},\overline{y},\overline{z},\bar{t})\right\}\\ \bar{t}=\bar{t}\left\{\xi,\eta,\zeta,\bar{t}(\overline{x},\overline{y},\overline{z},\bar{t})\right\}\end{cases}$$

which connect $\overline{x},\overline{y},\overline{z}$ with ξ,η,ζ and evidently exactly represent the transformation, which brings the integrals to invariable limits when applied to (96); because this transformation (97) represents $\overline{x},\overline{y},\overline{z}$ as function of their initial values for the relevant instant in the brackets.

To calculate the functional determinant of transformation (97)

(98)
$$\Delta=\left[\frac{\partial(\overline{x},\overline{y},\overline{z})}{\partial(\xi,\eta,\zeta)}\right]_{t=\text{konst.}}$$

we want to denote the derivative from $\overline{x}$ to ξ at maintained $\bar{t}$ with $(\partial\overline{x}/\partial\xi)_{\bar{t}}$, and at maintained t with $(\partial\overline{x}/\partial\xi)_t$. If we differentiate then equations (97) (one after the other) with respect to ξ,η,ζ, then we obtain three equation systems with identical coefficients; for example at the differentiation with respect to ξ:

$$\left(\frac{\partial\overline{x}}{\partial\xi}\right)_t=\left(\frac{\partial\overline{x}}{\partial\xi}\right)_{\bar{t}}+\frac{\partial\overline{x}}{\partial\bar{t}}\left\{\frac{\partial\bar{t}}{\partial\overline{x}}\left(\frac{\partial\overline{x}}{\partial\xi}\right)_t+\frac{\partial\bar{t}}{\partial\overline{y}}\left(\frac{\partial\overline{y}}{\partial\xi}\right)_t+\frac{\partial\bar{t}}{\partial\overline{z}}\left(\frac{\partial\overline{z}}{\partial\xi}\right)_t\right\},$$

$$\left(\frac{\partial\overline{y}}{\partial\xi}\right)_t=\left(\frac{\partial\overline{y}}{\partial\xi}\right)_{\bar{t}}+\frac{\partial\overline{y}}{\partial\bar{t}}\left\{\frac{\partial\bar{t}}{\partial\overline{x}}\left(\frac{\partial\overline{x}}{\partial\xi}\right)_t+\frac{\partial\bar{t}}{\partial\overline{y}}\left(\frac{\partial\overline{y}}{\partial\xi}\right)_t+\frac{\partial\bar{t}}{\partial\overline{z}}\left(\frac{\partial\overline{z}}{\partial\xi}\right)_t\right\},$$

$$\left(\frac{\partial\overline{z}}{\partial\xi}\right)_t=\left(\frac{\partial\overline{z}}{\partial\xi}\right)_{\bar{t}}+\frac{\partial\overline{z}}{\partial\bar{t}}\left\{\frac{\partial\bar{t}}{\partial\overline{x}}\left(\frac{\partial\overline{x}}{\partial\xi}\right)_t+\frac{\partial\bar{t}}{\partial\overline{y}}\left(\frac{\partial\overline{y}}{\partial\xi}\right)_t+\frac{\partial\bar{t}}{\partial\overline{z}}\left(\frac{\partial\overline{z}}{\partial\xi}\right)_t\right\},$$

or

$$\begin{aligned}
\left(\frac{\partial\bar{x}}{\partial\xi}\right)_{t}\left(1+\frac{1}{c}\frac{\partial\bar{x}}{\partial\bar{t}}\frac{\partial r}{\partial\bar{x}}\right) & +\left(\frac{\partial\bar{y}}{\partial\xi}\right)_{t}\frac{1}{c}\frac{\partial\bar{x}}{\partial\bar{t}}\frac{\partial r}{\partial\bar{y}}\\
 & +\left(\frac{\partial\bar{z}}{\partial\xi}\right)_{t}\frac{1}{c}\frac{\partial\bar{x}}{\partial\bar{t}}\frac{\partial r}{\partial\bar{z}} & =\left(\frac{\partial\bar{x}}{\partial\xi}\right)_{\bar{t}},\\
\left(\frac{\partial\bar{x}}{\partial\xi}\right)_{t}\frac{1}{c}\frac{\partial\bar{y}}{\partial\bar{t}}\frac{\partial r}{\partial\bar{x}} & +\left(\frac{\partial\bar{y}}{\partial\xi}\right)_{t}\left(1+\frac{1}{c}\frac{\partial\bar{y}}{\partial\bar{t}}\frac{\partial r}{\partial\bar{y}}\right)\\
 & +\left(\frac{\partial\bar{z}}{\partial\xi}\right)_{t}\frac{1}{c}\frac{\partial\bar{y}}{\partial\bar{t}}\frac{\partial r}{\partial\bar{z}} & =\left(\frac{\partial\bar{y}}{\partial\xi}\right)_{\bar{t}},\\
\left(\frac{\partial\bar{x}}{\partial\xi}\right)_{t}\frac{1}{c}\frac{\partial\bar{z}}{\partial\bar{t}}\frac{\partial r}{\partial\bar{x}} & +\left(\frac{\partial\bar{y}}{\partial\xi}\right)_{t}\frac{1}{c}\frac{\partial\bar{z}}{\partial\bar{t}}\frac{\partial r}{\partial\bar{y}}\\
 & +\left(\frac{\partial\bar{z}}{\partial\xi}\right)_{t}\left(1+\frac{1}{c}\frac{\partial\bar{z}}{\partial\bar{t}}\frac{\partial r}{\partial\bar{z}}\right) & =\left(\frac{\partial\bar{z}}{\partial\xi}\right)_{\bar{t}},
\end{aligned}$$

Three equation are added twice, at which ξ is replaced by η or ζ. If we denote the matrices occurring here as follows:

(99) $$P=\begin{vmatrix}\left(\frac{\partial\bar{x}}{\partial\xi}\right)_{t} & \left(\frac{\partial\bar{y}}{\partial\xi}\right)_{t} & \left(\frac{\partial\bar{z}}{\partial\xi}\right)_{t}\\
\left(\frac{\partial\bar{x}}{\partial\eta}\right)_{t} & \left(\frac{\partial\bar{y}}{\partial\eta}\right)_{t} & \left(\frac{\partial\bar{z}}{\partial\eta}\right)_{t}\\
\left(\frac{\partial\bar{x}}{\partial\zeta}\right)_{t} & \left(\frac{\partial\bar{y}}{\partial\zeta}\right)_{t} & \left(\frac{\partial\bar{z}}{\partial\zeta}\right)_{t}
\end{vmatrix},$$

(100) $$Q=\begin{pmatrix}1+\frac{1}{c}\frac{\partial\bar{x}}{\partial\bar{t}}\frac{\partial r}{\partial\bar{x}} & \frac{1}{c}\frac{\partial\bar{x}}{\partial\bar{t}}\frac{\partial r}{\partial\bar{y}} & \frac{1}{c}\frac{\partial\bar{x}}{\partial\bar{t}}\frac{\partial r}{\partial\bar{z}}\\
\frac{1}{c}\frac{\partial\bar{y}}{\partial\bar{t}}\frac{\partial r}{\partial\bar{x}} & 1+\frac{1}{c}\frac{\partial\bar{y}}{\partial\bar{t}}\frac{\partial r}{\partial\bar{y}} & \frac{1}{c}\frac{\partial\bar{y}}{\partial\bar{t}}\frac{\partial r}{\partial\bar{z}}\\
\frac{1}{c}\frac{\partial\bar{z}}{\partial\bar{t}}\frac{\partial r}{\partial\bar{x}} & \frac{1}{c}\frac{\partial\bar{z}}{\partial\bar{t}}\frac{\partial r}{\partial\bar{y}} & 1+\frac{1}{c}\frac{\partial\bar{z}}{\partial\bar{t}}\frac{\partial r}{\partial\bar{z}}
\end{pmatrix},$$

(101) $$R=\begin{vmatrix}\left(\frac{\partial\bar{x}}{\partial\xi}\right)_{\bar{t}} & \left(\frac{\partial\bar{y}}{\partial\xi}\right)_{\bar{t}} & \left(\frac{\partial\bar{z}}{\partial\xi}\right)_{\bar{t}}\\
\left(\frac{\partial\bar{x}}{\partial\eta}\right)_{\bar{t}} & \left(\frac{\partial\bar{y}}{\partial\eta}\right)_{\bar{t}} & \left(\frac{\partial\bar{z}}{\partial\eta}\right)_{\bar{t}}\\
\left(\frac{\partial\bar{x}}{\partial\zeta}\right)_{\bar{t}} & \left(\frac{\partial\bar{y}}{\partial\zeta}\right)_{\bar{t}} & \left(\frac{\partial\bar{z}}{\partial\zeta}\right)_{\bar{t}}
\end{vmatrix},$$

then our nine equations can be summarized in the matrix equation:

$$P\overline{Q}=R$$

From that the relation of determinants follows:

$$|P| \cdot |Q| = |R| \tag{102}$$

Now it is evidently according to (98)

$$|P| = \Delta \tag{103}$$

furthermore it is according to § 3, (42), p. 16:

$$|R| = |\Theta|_{\bar{t}=t-\frac{r}{c}} \tag{104}$$

Finally one finds easily

$$\begin{aligned} |Q| &= 1 + \tfrac{1}{c}\left(\frac{\partial \bar{x}}{\partial \bar{t}}\frac{\partial r}{\partial \bar{x}} + \frac{\partial \bar{y}}{\partial \bar{t}}\frac{\partial r}{\partial \bar{y}} + \frac{\partial \bar{z}}{\partial \bar{t}}\frac{\partial r}{\partial \bar{z}}\right), \\ &= 1 + \tfrac{1}{c}\left(\overline{w}_x \frac{\partial r}{\partial \bar{x}} + \overline{w}_y \frac{\partial r}{\partial \bar{y}} + \overline{w}_z \frac{\partial r}{\partial \bar{z}}\right) \end{aligned}$$

which is to be written according to (92):

$$Q = \left[1 - \frac{w_r}{c}\right]_{\bar{t}=t-\frac{r}{c}} \tag{105}$$

Consequently it becomes:

$$\Delta = \left[\frac{\Theta}{1 - \frac{w_r}{c}}\right]_{\bar{t}=t-\frac{r}{c}} \tag{106}$$

If we insert this into (96) and consider that according to § 3, (41), p. 16, it is to be set:

$\varrho\Theta = \varrho(\xi, \eta, \zeta)$,

then it follows:

$$\left\{ \begin{array}{l} 4\pi\Phi_x = \iiint \varrho_0(\xi, \eta, \zeta) \left[\frac{w_x}{cr\left(1 - \frac{w_r}{c}\right)}\right]_{t-\bar{t}=\frac{r}{c}} d\xi\, d\eta\, d\zeta, \\ \ldots\ldots \\ 4\pi\Phi = \iiint \varrho_0(\xi, \eta, \zeta) \left[\frac{1}{r\left(1 - \frac{w_r}{c}\right)}\right]_{t-\bar{t}=\frac{r}{c}} d\xi\, d\eta\, d\zeta, \end{array} \right. \tag{107}$$

If we also replace w_x by x_τ / t_τ etc. as on p. 30, then formulas (107) go directly over into expressions (83). Indeed, *only the initial density* ϱ_0 occurs in them.

§ 8. Calculation of the potentials at hyperbolic motion.

We now want to evaluate the potentials (83) for hyperbolic motion

(108) $$x = -q\xi,\ y = \eta,\ z = \zeta,\ t = \frac{p}{c^2}\xi$$

Since $y_\tau = z_\tau = 0$, then also $\Phi_y = \Phi_z = 0$. Since we have in (108) the quantity p instead of τ as independent variable besides ξ, η, ζ, we will see equation (84) $h = 0$ as equation for p as well. Then it reads:

(109) $$\left(x + \overline{q}\overline{\xi}\right)^2 + (y - \overline{\eta})^2 + \left(z - \overline{\zeta}\right)^2 - c^2\left(t - \frac{\overline{q}\overline{\xi}}{c^2}\right)^2 = 0;$$

when one introduces the abbreviations

(110) $$\begin{cases} s = x^2 - c^2t^2 = \xi^2, \\ k = -\frac{1}{2\xi}\left(s + \overline{\xi}^2 + (y - \overline{\eta})^2 + \left(z - \overline{\zeta}\right)^2\right) \end{cases}$$

one can write (109) as follows:

$\overline{p}t + \overline{q}x = k$;

where it is added:

$\overline{p}^2 - c^2\overline{q}^2 = -c^2$

$\overline{p}$ is to be calculated from these equations; namely that value of $\overline{p}$ is to be chosen, for which $t > \overline{t}$. If one inserts the value following from the first equation

$$\overline{p} = \frac{k - \overline{q}x}{t}$$

into the second one, then the quadratic equation emerges for $\overline{q}$:

$$\overline{q}^2 - \overline{q}\frac{2kx}{s} = -\frac{k^2 + c^2t^2}{s}$$

From that if follows:

$$\overline{q} = \frac{1}{s}\left(kx + ct\sqrt{k^2 - s}\right)$$

If we also set (for abbreviation) for the positive square root

(111) $$B = \sqrt{k^2 - s}$$

and calculate $\overline{p}$, then we find

$$\overline{p} = \quad -\tfrac{c}{s}(kct \pm Bx)$$

$$\overline{q} = \quad \tfrac{1}{s}(kx \pm Bct)$$

Here, that sign is to be chosen, which corresponds to the smaller value of $\overline{t}$. Now, since $\overline{t} = \overline{q}\overline{\xi}/c^2$, then the following is given (presupposed, that the electron is moving on the right side of the origin $x = 0$, *i.e.*, $\overline{\xi} > 0$):

for all reference points, at which $x/s > 0$, the positive sign is to be taken,

for all reference points, at which $x/s < 0$, the negative sign is to be taken.

The distribution of these reference points can be derived from the figure.

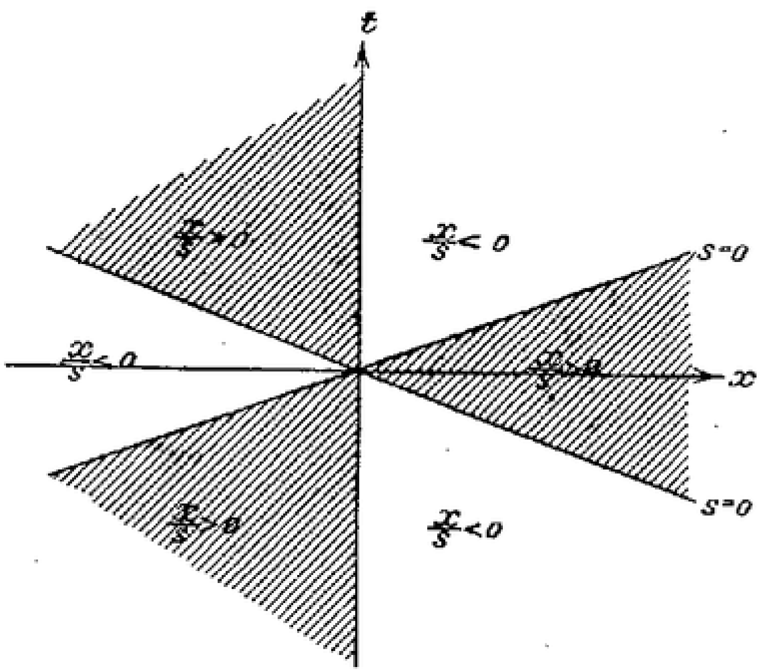

Fig. 6.

We want to presuppose mostly in the following, that $x/s > 0$; only such points can be interior points of the electron at hyperbolic motion. Thus the positive square root B is to be taken for them. When we sometimes also consider points for which $x/s < 0$, then we have to replace $+B$ by $-B$ everywhere.

Thus we have:

(112)
$$\begin{cases} \overline{p} = & -\tfrac{c}{s}(kct + Bx) \\ \overline{q} = & \tfrac{1}{s}(kx + Bct) \end{cases}$$

Now we calculate the denominator of integral (83) for these values of $\overline{p},\ \overline{q}$.

Due to $y_\tau = z_\tau = 0,\ x_\tau = p,\ t_\tau = -q$, it becomes:

$$(x+\overline{q}\overline{\xi})\overline{p}+c^2\left(t-\frac{\overline{p}}{c^2}\overline{\xi}\right)\overline{q}=x\overline{p}+c^2t\overline{q}=-cB$$

This is inserted into the integrals; it follows:

(113) $$\begin{cases} 4\pi\Phi_x(x,y,z,t)=\iiint\frac{\overline{\varrho}_0}{sB}(kct+Bx)d\overline{\xi}\,d\overline{\eta}\,d\overline{\zeta}, \\ 4\pi\Phi(x,y,z,t)=\iiint\frac{\overline{\varrho}_0}{sB}(kx+Bct)d\overline{\xi}\,d\overline{\eta}\,d\overline{\zeta}, \end{cases}$$

If we set for abbreviation:

(114) $$\begin{cases} \psi_1(s)=\frac{1}{s}\iiint\overline{\varrho}_0 d\overline{\xi}\,d\overline{\eta}\,d\overline{\zeta}=\frac{e}{s}, \\ \psi_2(s)=\frac{1}{s}\iiint\overline{\varrho}_0\frac{k}{B}d\overline{\xi}\,d\overline{\eta}\,d\overline{\zeta}, \end{cases}$$

where e denotes the total charge of the electron, then it simply becomes:

(115) $$\begin{cases} 4\pi\Phi_x=\psi_1(s)\cdot x+\psi_2(s)\cdot ct, \\ 4\pi\Phi=\psi_2(s)\cdot x+\psi_1(s)\cdot ct. \end{cases}$$

Here, ψ_1 and ψ_2 are functions of the connection s from x to t alone.

These potentials particularly satisfy equation (87) $\mathrm{lor}\Phi=0$; because one has, since $\partial s/\partial x=2x$, $\partial s/\partial t=-2c^2t$:

$$4\pi\frac{\partial\Phi_x}{\partial x}=\psi_1+2\psi_1'x^2+2\psi_2'ctx,$$

$$4\pi\frac{\partial\Phi}{\partial t}=\psi_1 c-2\psi_2'c^2tx-2\psi_1'c^3t^2$$

thus

$$\mathrm{lor}\Phi=\frac{\partial\Phi_x}{\partial x}+\frac{1}{c}\frac{\partial\Phi}{\partial t}=\frac{1}{2\pi}\left(\psi_1+s\psi_1'\right)$$

Now it is

$$\psi_1'=-\frac{e}{s^2}$$

thus it becomes

(116) $$\mathrm{lor}\Phi=0$$

For later purposes, we want to write potentials (115) in a still different way soon. For that, we consider that according to (108) it is to be set:

$$x=-q\xi,\ t=\frac{p}{c^2}\xi,\text{ thus }s=\xi^2$$

If we then introduce the following ones instead of the abbreviations $\psi_1,\ \psi_2$:

(117)
$$\begin{cases} 4\pi\overline{\Phi}_x = -\xi\psi_1 = -\frac{e}{\xi}, \\ 4\pi\overline{\Phi} = -\xi\psi_2 = -\frac{1}{\xi}\iiint \overline{\varrho}_0 \frac{k}{B} d\overline{\xi}\, d\overline{\eta}\, d\overline{\zeta} \end{cases}$$

then one can write instead of (115):

(118)
$$\begin{cases} \Phi_x = q\overline{\Phi}_x - \frac{p}{c}\overline{\Phi}, \\ \Phi = -\frac{p}{c}\overline{\Phi}_x + q\overline{\Phi} \end{cases}$$

Functions Φ_x, Φ thus are connected with the auxiliary functions $\overline{\Phi}_x, \overline{\Phi}$ by the same Lorentz transformation, which (9. 23, (74)) transforms the rigid body to rest. Therefore we will call $\overline{\Phi}_x, \overline{\Phi}$ the "rest potentials"; functions of ξ, η, ζ alone don't depend any more from p.

From relations (118) we see, that the electron is carrying its field; the rest potentials observed by an observer co-moving with the electron, only depend on the rest coordinates ξ, η, ζ.

We still want to provide the explicit expression for the scalar rest potential Φ:

(119)
$$4\pi\overline{\Phi}(\xi,\eta,\zeta) = -\frac{1}{\xi}\iiint \overline{\varrho}_0 \frac{1}{r}\frac{r^2 + 2\xi\overline{\xi}}{\sqrt{r^2 + 4\xi\overline{\xi}}} d\overline{\xi}\, d\overline{\eta}\, d\overline{\zeta}$$

where it is set

(120)
$$r^2 = (\xi - \overline{\xi})^2 + (\eta - \overline{\eta})^2 + (\zeta - \overline{\zeta})^2$$

If the reference point is located in the area $x/s < 0$, then the negative sign is to be chosen instead of the positive one. Both $\overline{\Phi}_x$ and $\overline{\Phi}$ become infinite at $\xi = 0$; the whole hyperbolic motion is singular for this value; yet the field strengths remain, as we will see, finite everywhere and are defined in the whole $xyzt$-manifold.

§ 9. The field strengths at hyperbolic motion.

From expressions (115), now we want to calculate the field strengths according to formulas (85). Here, they read:

(121)
$$\begin{cases} \mathfrak{E}_x = -\frac{1}{c}\frac{\partial\Phi_x}{\partial t} - \frac{\partial\Phi}{\partial x}, & \mathfrak{E}_y = -\frac{\partial\Phi}{\partial y}, & \mathfrak{E}_z = -\frac{\partial\Phi}{\partial z}, \\ \\ \mathfrak{M}_x = 0, & \mathfrak{M}_y = \frac{\partial\Phi_x}{\partial z}, & \mathfrak{M}_z = -\frac{\partial\Phi_x}{\partial y} \end{cases}$$

Now we find from (115) under consideration of (110):

$$4\pi\frac{\partial\Phi_x}{\partial t} = c\psi_2 - 2\psi_1' c^2 tx - 2\psi_2' c^3 t^2$$

$$4\pi\frac{\partial\Phi}{\partial x} = \psi_2 + 2\psi_2' x^2 + 2\psi_1' ctx.$$

Thus it becomes:

(122)
$$\mathfrak{E}_x = -\frac{1}{2\pi}\left(\psi_2 + s\psi_2'\right)$$

From that it follows, that $\mathfrak{E}_x$ only depends (besides η, ζ) on s, *i.e.*, on ξ, yet not on p. *The z-component of the electric field strength is thus constant along every world line of the electron.*

If one computes $\mathfrak{E}_x$, then it is given:

(123)
$$\mathfrak{E}_x = -\frac{1}{\xi}\iiint \overline{\varrho}_0 \frac{\overline{\xi}^2\left(r^2 - 2\xi(\xi-\overline{\xi})\right)}{r^3\left(r^2 + 4\xi\overline{\xi}\right)^{3/2}} d\overline{\xi}\, d\overline{\eta}\, d\overline{\zeta}$$

where r is defined by (120).

$\mathfrak{E}_x$ doesn't become infinite at $\xi = 0$. Yet one can continue $\mathfrak{E}_x$ also over the line $\xi = 0$, *i.e.*, $x + ct = 0$ and $x - ct = 0$. In the areas where $x/s < 0$, one has to give the opposite sign to the denominator in the same way as B^3. Furthermore it is to be considered that by equations (108), which represent the hyperbolas of the xt-plane normal to the x-axis for real values of ξ, those hyperbolas are represented which are normal to the t-axis for pure imaginary ξ. The parenthetical expressions in numerator and denominator of $\mathfrak{E}_x$, can be written in the form

$$\overline{\xi}^2 - \xi^2 + (\eta - \overline{\eta})^2 + (\zeta - \overline{\zeta})^2,$$
$$\left[\overline{\xi}^2 - \xi^2 + (\eta - \overline{\eta})^2 + (\zeta - \overline{\zeta})^2\right]^2 - 4\xi^2\overline{\xi}^2$$

only the square of ξ occur within them, so that they are real for purely imaginary values of ξ as well. Furthermore, the denominator expression in the integration area, *i.e.*, for $\overline{\xi}^2 > 0$, can never become zero; if one sets $\xi = i\alpha$, then it becomes

$$\left[-\alpha^2 + \overline{\xi}^2 + (\eta - \overline{\eta})^2 + (\zeta - \overline{\zeta})^2\right]^2 + 4\alpha^2\overline{\xi}^2 > 0$$

Thus $\mathfrak{E}_x$ is defined in the whole xt-plane.

Now we want to calculate the other field components. It becomes:

(124)
$$\begin{cases} \mathfrak{E}_y = -\frac{\partial\Phi}{\partial y} = -\frac{x}{4\pi}\frac{\partial\psi_2}{\partial\eta} = q\frac{\xi}{4\pi}\frac{\partial\psi_2}{\partial\eta}, \\ \mathfrak{E}_z = -\frac{\partial\Phi}{\partial z} = -\frac{x}{4\pi}\frac{\partial\psi_2}{\partial\zeta} = q\frac{\xi}{4\pi}\frac{\partial\psi_2}{\partial\zeta} \end{cases}$$

(125)
$$\begin{cases} \mathfrak{M}_y = \frac{\partial\Phi_x}{\partial z} = \frac{ct}{4\pi}\frac{\partial\psi_2}{\partial\zeta} = \frac{p}{c}\frac{\xi}{4\pi}\frac{\partial\psi_2}{\partial\zeta}, \\ \mathfrak{M}_z = -\frac{\partial\Phi_x}{\partial y} = -\frac{ct}{4\pi}\frac{\partial\psi_2}{\partial\eta} = -\frac{p}{c}\frac{\xi}{4\pi}\frac{\partial\psi_2}{\partial\zeta}, \end{cases}$$

Now one easily finds:

$$(126)\qquad \begin{cases} \frac{\partial\psi_2}{\partial\eta} = -8\iiint\overline{\varrho}_0 \frac{\overline{\xi}^2(\eta-\overline{\eta})}{r^3\left(r^2+4\xi\overline{\xi}\right)^{3/2}} d\overline{\xi}\, d\overline{\eta}\, d\overline{\zeta}, \\ \\ \frac{\partial\psi_2}{\partial\zeta} = -8\iiint\overline{\varrho}_0 \frac{\overline{\xi}^2(\zeta-\overline{\zeta})}{r^3\left(r^2+4\xi\overline{\xi}\right)^{3/2}} d\overline{\xi}\, d\overline{\eta}\, d\overline{\zeta}, \end{cases}$$

Consequently, the y- and z- components of the field strengths are not only dependent on the connection $\xi^2 = x^2 - c^2t^2$ (besides on η, ζ), but explicitly on x and t as well; thus one can see them as functions of ξ, η, ζ, p.

In areas where $x/s < 0$, the sign of expressions (126) is to be reversed again.

One recognizes, that also the y- and z-components of the field strengths, do not become infinite at $\xi = 0$. We won't discuss the field-path in a fixed coordinate system in more detail. Only that much can immediately be seen, that $\mathfrak{E}_y$ and $\mathfrak{E}_z$ are vanishing at $x = 0$, thus that the force lines are parallel to the x-axis there, and that $\mathfrak{M}_y$ and $\mathfrak{M}_z$ are vanishing at $t = 0$, *i.e.*, in the moment where the electron turns back and therefore is momentarily at rest. From that we see again, that the field is indeed carried along by the electron, because the magnetic field momentarily vanishes everywhere, when the electron is at rest for a moment.

§ 10. Transformation of the wave equation, the potentials, and field strengths of a co-moving coordinate system.

The form (118) given by us to the retarded potentials, leads us to transform the wave equation (86) itself to a coordinate system co-moving with the electron, *i.e.*, into the independent variables ξ, η, ζ, p.

To simplify the transformation, we transform the variation problem (88) instead of the differential equations (86).

Thus we first have to transform the components of field strengths (121).

Due to (79), it is:

$$\frac{1}{c}\frac{\partial\Phi_x}{\partial t} = \frac{1}{c}\frac{\partial\Phi_x}{\partial\xi}\xi_t + \frac{1}{c}\frac{\partial\Phi_x}{\partial p}p_t = -\frac{\partial\Phi_x}{\partial\xi}\frac{p}{c} + \frac{\partial\Phi_x}{\partial p}\frac{cq^2}{\xi},$$

$$\frac{\partial\Phi}{\partial x} = \frac{\partial\Phi}{\partial\xi}\xi_x + \frac{\partial\Phi}{\partial p}p_x = -\frac{\partial\Phi}{\partial\xi}q + \frac{\partial\Phi}{\partial p}\frac{pq}{\xi}.$$

We now introduce the rest potentials $\overline{\Phi}_x,\ \overline{\Phi}$ by the same relations as earlier:

$$(118)\qquad \begin{cases} \Phi_x = q\overline{\Phi}_x - \frac{p}{c}\overline{\Phi}, \\ \Phi = -\frac{p}{c}\overline{\Phi}_x - q\overline{\Phi} \end{cases}$$

Then it evidently is:

$$\frac{\partial\Phi_x}{\partial\xi} = q\frac{\partial\overline{\Phi}_x}{\partial\xi} - \frac{p}{c}\frac{\partial\overline{\Phi}}{\partial\xi},$$

$$\frac{\partial\Phi}{\partial\xi} = -\frac{p}{c}\frac{\partial\overline{\Phi}_x}{\partial\xi} + q\frac{\partial\overline{\Phi}}{\partial\xi},$$

and the corresponding holds for the derivatives with respect to η and ζ. On the other hand, it becomes

$$\frac{\partial\Phi_x}{\partial p} = \frac{p}{c^2 q}\overline{\Phi}_x - \frac{1}{c}\overline{\Phi} + q\frac{\partial\overline{\Phi}_x}{\partial p} - \frac{p}{c}\frac{\partial\overline{\Phi}}{\partial p}$$

$$\frac{\partial\Phi}{\partial p} = -\frac{1}{c}\overline{\Phi}_x + \frac{p}{c^2 q}\overline{\Phi} - \frac{p}{c}\frac{\partial\overline{\Phi}_x}{\partial p} + q\frac{\partial\overline{\Phi}}{\partial p}$$

Consequently it becomes:

(127)
$$\begin{cases} -\mathfrak{E}_x = \frac{1}{c}\frac{\partial\Phi_x}{\partial t} + \frac{\partial\Phi}{\partial x} = -\frac{\partial\overline{\Phi}}{\partial\xi} - \frac{1}{\xi}\left(\overline{\Phi} - cq\frac{\partial\overline{\Phi}_x}{\partial p}\right), \\ -\mathfrak{E}_y = \frac{\partial\Phi}{\partial y} = -\frac{p}{c}\frac{\partial\overline{\Phi}_x}{\partial\eta} + q\frac{\partial\overline{\Phi}}{\partial\eta}, \\ -\mathfrak{E}_z = \frac{\partial\Phi}{\partial z} = -\frac{p}{c}\frac{\partial\overline{\Phi}_x}{\partial\zeta} + q\frac{\partial\overline{\Phi}}{\partial\zeta}; \end{cases}$$

(128)
$$\begin{cases} \mathfrak{M}_x = 0 \\ \mathfrak{M}_y = \frac{\partial\Phi_x}{\partial z} = p\frac{\partial\overline{\Phi}_x}{\partial\zeta} - \frac{p}{c}\frac{\partial\overline{\Phi}}{\partial\zeta}, \\ -\mathfrak{M}_z = \frac{\partial\Phi_x}{\partial y} = q\frac{\partial\overline{\Phi}_x}{\partial\eta} - \frac{p}{c}\frac{\partial\overline{\Phi}}{\partial\eta}; \end{cases}$$

Now it is given:

(129)
$$\begin{aligned} \mathfrak{M}^2 - \mathfrak{E}^2 = & -\left[\frac{\partial\overline{\Phi}}{\partial\xi} + \frac{1}{\xi}\left(\overline{\Phi} - cq\frac{\partial\overline{\Phi}_x}{\partial p}\right)\right]^2 \\ & + \left(\frac{\partial\overline{\Phi}_x}{\partial\eta}\right)^2 - \left(\frac{\partial\overline{\Phi}}{\partial\eta}\right)^2 + \left(\frac{\partial\overline{\Phi}_x}{\partial\zeta}\right)^2 - \left(\frac{\partial\overline{\Phi}}{\partial\zeta}\right)^2 \end{aligned}$$

Furthermore it becomes:

(130)
$$\frac{\varrho^*}{c}\left(\Phi_x x_\tau + \Phi_y y_\tau + \Phi_z z_\tau - \Phi t_\tau\right) = -\varrho^*\left(\frac{p}{c}\Phi_x + q\Phi\right) = -\varrho^*\overline{\Phi}$$

Finally, the functional determinant becomes

(131)

$$\frac{\partial(x,y,z,t)}{\partial(\xi,\eta,\zeta,p)} = -\frac{\xi}{c^2 q}$$

Therefore the variation problem (88) goes over into the following one:

(132)
$$W = \iiiint \left\{ \frac{\xi}{2c^2 q} \left(\left[\frac{\partial\overline{\Phi}}{\partial\xi} + \frac{1}{\xi}\left(\overline{\Phi} - cq\frac{\partial\overline{\Phi}_x}{\partial p}\right)\right]^2 \right.\right.$$
$$\left. - \left[\left(\frac{\partial\overline{\Phi}_x}{\partial\eta}\right)^2 - \left(\frac{\partial\overline{\Phi}}{\partial\eta}\right)^2\right] - \left[\left(\frac{\partial\overline{\Phi}_x}{\partial\zeta}\right)^2 - \left(\frac{\partial\overline{\Phi}}{\partial\zeta}\right)^2\right]\right)$$
$$\left. + \frac{\xi}{c^2 q}\varrho^*\overline{\Phi}\right\} d\xi\, d\eta\, d\zeta\, dp = \mathrm{Min.}$$

From that, one takes the *differential equations of electrodynamics for a hyperbolically accelerated reference system*:

(133)
$$\begin{cases} \frac{1}{c}\frac{\partial}{\partial p}\left\{\frac{\partial\overline{\Phi}}{\partial\xi} + \frac{1}{\xi}\left(\overline{\Phi} - cq\frac{\partial\overline{\Phi}_x}{\partial p}\right)\right\} \\ \qquad -\frac{\xi}{q}\left(\frac{\partial^2\overline{\Phi}_x}{\partial\eta^2} + \frac{\partial^2\overline{\Phi}_x}{\partial\zeta^2}\right) = 0, \\ \frac{\partial}{\partial\xi}\left\{\xi\frac{\partial\overline{\Phi}}{\partial\xi} + \overline{\Phi} - cq\frac{\partial\overline{\Phi}_x}{\partial p}\right\} + \xi\left(\frac{\partial^2\overline{\Phi}}{\partial\eta^2} + \frac{\partial^2\overline{\Phi}}{\partial\zeta^2}\right) \\ \qquad -\left\{\frac{\partial\overline{\Phi}}{\partial\xi} + \frac{1}{\xi}\left(\overline{\Phi} - cq\frac{\partial\overline{\Phi}_x}{\partial p}\right)\right\} = \varrho^*\xi. \end{cases}$$

These equations necessarily must also satisfy the rest potentials (117) derived in § 8. Yet they do not depend on p, as well as the density $\varrho^* = \varrho_0(\xi,\eta,\zeta)$; thus they are "statical potentials" with respect to an accelerated coordinate system. By omission of the derivatives with respect to p, we obtain from (133) the *differential equations of electrostatics in a hyperbolically accelerated reference system*:

(134)
$$\begin{cases} \frac{\partial^2\overline{\Phi}_x}{\partial\eta^2} + \frac{\partial^2\overline{\Phi}_x}{\partial\zeta^2} = 0, \\ \frac{\partial}{\partial\xi}\left(\xi\frac{\partial\overline{\Phi}}{\partial\xi}\right) + \xi\left(\frac{\partial^2\overline{\Phi}}{\partial\eta^2} + \frac{\partial^2\overline{\Phi}}{\partial\zeta^2}\right) - \frac{\overline{\Phi}}{\xi} = \varrho_0\xi \end{cases}$$

Additionally, potentials $\Phi_x,\ \Phi$ satisfy equation $\mathrm{lor}\ \Phi = 0$; it goes over at the transformation into:

(135)
$$\frac{\partial\overline{\Phi}_x}{\partial\xi} + \frac{1}{\xi}\left(\overline{\Phi} - qc\frac{\partial\overline{\Phi}}{\partial p}\right) = 0$$

When $\overline{\Phi}_x,\ \overline{\Phi}$ are independent from p, this becomes:

(136) $$\frac{\partial\overline{\Phi}_x}{\partial\xi}+\frac{\overline{\Phi}_x}{\xi}=0$$

We now want to show directly, that expressions (117) or (119) indeed satisfy equations (134), (136).

This is clear from the outset for $\overline{\Phi}_x=-\frac{1}{4\pi}\frac{e}{\xi}$; both the first equation (134) as well as (136) are satisfied.

We use the explicit expression (119) for $\overline{\Phi}$. We will show, that it is the exact analogue to the electrostatic potential of given charges

$$4\pi u(\xi,\eta,\zeta)=\iiint\frac{\overline{\varrho}}{r}d\overline{\xi}\,d\overline{\eta}\,d\overline{\zeta}$$

and that it has exactly the same relation to the differential equation

(137) $$\frac{\partial}{\partial\xi}\left(\xi\frac{\partial\overline{\Phi}}{\partial\xi}\right)+\xi\left(\frac{\partial^2\overline{\Phi}}{\partial\eta^2}+\frac{\partial^2\overline{\Phi}}{\partial\zeta^2}\right)-\frac{\overline{\Phi}}{\xi}=f(\xi,\eta,\zeta)$$

as the ordinary potential u to the equation

$$\Delta u=f(\xi,\eta,\zeta)$$

First, the function being symmetric in the two series of variables $\xi,\eta,\zeta;\ \overline{\xi},\overline{\eta},\overline{\zeta}$

$$\frac{1}{r}\frac{1}{\xi\overline{\xi}}\frac{r^2+2\xi\overline{\xi}}{\sqrt{r^2+4\xi\overline{\xi}}}$$

a solution[24] of the homogeneous equation (137) $(f=0)$, which corresponds to the solution $1/r$ of $\Delta u=0$ That it indeed satisfies the equation, can directly be seen by a (though complicated) calculation. Furthermore, for $r=0$, *i.e.*, $\xi=\overline{\xi},\eta=\overline{\eta},\zeta=\overline{\zeta}$, it has a singularity of order $1/r$ and the factor of $1/r$ becomes equal to $1/\xi$ for $r=0$. However, it must excluded at that occasion, that ξ or $\overline{\xi}$ become zero; this value itself is naturally singular for differential equation (137). From this behavior of our fundamental solution, it follows exactly as in potential theory, that (when f is a function defined for $\xi\gtrless 0$) the expression

$$4\pi\overline{\Phi}(\xi,\eta,\zeta)=\iiint\frac{f(\overline{\xi},\overline{\eta},\overline{\zeta})}{r}\frac{1}{\xi\xi}\frac{r^2+2\xi\overline{\xi}}{\sqrt{r^2+4\xi\overline{\xi}}}d\overline{\xi}\,d\overline{\eta}\,d\overline{\zeta}$$

satisfies the inhomogeneous equation (137) (of course, only as long as ξ is different from zero). If one herein replaces f by its value $\varrho_0\xi$, which corresponded to the co-moving charges according to (134), then one is led back to (119). The different signs in different areas of the reference point are resulting from the considerations, that $\Phi_x,\ \Phi$ assembled by $\overline{\Phi}_x,\ \overline{\Phi}$ in the stationary coordinate system, are retarded potentials, not advanced ones. Consequently, one can unequivocally determine the rest potentials $\overline{\Phi}_x$, $\overline{\Phi}$ in an analogous way by differential equations (134), (136) and by their behavior in the infinite, as the ordinary static potential. Though I won't discuss this here more closely.

If one now considers, that $\overline{\Phi}_x$ doesn't depend on p, η, ζ, then one gains from (127), (128) *the following expressions for the field strengths by the rest potential* $\overline{\Phi}$ *alone*:

(138)
$$\begin{cases} \mathfrak{E}_x = \frac{\partial\overline{\Phi}}{\partial\xi} + \frac{\overline{\Phi}}{\xi}, & \mathfrak{M}_x = 0, \\ \mathfrak{E}_y = -q\frac{\partial\overline{\Phi}}{\partial\eta}, & \mathfrak{M}_y = -\frac{p}{c}\frac{\partial\overline{\Phi}}{\partial\zeta}, \\ \mathfrak{E}_z = -q\frac{\partial\overline{\Phi}}{\partial\zeta}, & \mathfrak{M}_z = \frac{p}{c}\frac{\partial\overline{\Phi}}{\partial\eta}. \end{cases}$$

One easily sees, that these expressions are identical with (122), (124), (125).

According to Minkowski,[25] we still introduce the *rest field strength* besides the rest potentials:

The *electrical rest force* is defined by

(139)
$$\begin{cases} \overline{\mathfrak{E}}_x = t_\tau \mathfrak{E}_x + \frac{1}{c}\left(y_\tau \mathfrak{M}_z - z_\tau \mathfrak{M}_y\right), \\ \overline{\mathfrak{E}}_y = t_\tau \mathfrak{E}_y + \frac{1}{c}\left(z_\tau \mathfrak{M}_x - x_\tau \mathfrak{M}_z\right), \\ \overline{\mathfrak{E}}_z = t_\tau \mathfrak{E}_z + \frac{1}{c}\left(x_\tau \mathfrak{M}_y - y_\tau \mathfrak{M}_x\right). \end{cases}$$

The expression for the *electrical rest work* is added:

(139*)
$$\overline{A} = x_\tau \mathfrak{E}_x + y_\tau \mathfrak{E}_y + z_\tau \mathfrak{E}_z$$

Furthermore, the *magnetic rest force* is defined by

(140)
$$\begin{cases} \overline{\mathfrak{M}}_x = t_\tau \mathfrak{M}_x - \frac{1}{c}\left(y_\tau \mathfrak{E}_z - z_\tau \mathfrak{E}_y\right), \\ \overline{\mathfrak{M}}_y = t_\tau \mathfrak{M}_y - \frac{1}{c}\left(z_\tau \mathfrak{E}_x - x_\tau \mathfrak{E}_z\right), \\ \overline{\mathfrak{M}}_z = t_\tau \mathfrak{M}_z - \frac{1}{c}\left(x_\tau \mathfrak{E}_y - y_\tau \mathfrak{E}_x\right). \end{cases}$$

and the *magnetic rest work*:

(140*)
$$\overline{B} = x_\tau \mathfrak{M}_x + y_\tau \mathfrak{M}_y + z_\tau \mathfrak{M}_z$$

If one inserts expressions (138) herein, and considers that it is to be set:

$x_\tau = p,\ t_\tau = -q,\ y_\tau = z_\tau = 0,$

then one obtains:

(141)
$$\begin{cases} \overline{\mathfrak{E}}_x = -q\left(\frac{\partial\overline{\Phi}}{\partial\xi} + \frac{\overline{\Phi}}{\xi}\right), & \overline{\mathfrak{M}}_x = 0, \\ \overline{\mathfrak{E}}_y = \frac{\partial\overline{\Phi}}{\partial\eta}, & \overline{\mathfrak{M}}_y = 0, \\ \overline{\mathfrak{E}}_z = \frac{\partial\overline{\Phi}}{\partial\zeta}, & \overline{\mathfrak{M}}_z = 0, \\ \overline{A} = p\left(\frac{\partial\overline{\Phi}}{\partial\xi} + \frac{\overline{\Phi}}{\xi}\right); & \overline{B} = 0. \end{cases}$$

The magnetic rest force and rest work are thus identically zero, as it was to be expected. The electric rest force and rest work, however, are solely derived from rest potential $\overline{\Phi}$. From the value of $\overline{\Phi}$ one finds the following explicit expressions for the electrical rest forces:

(142)
$$\begin{cases} \overline{\mathfrak{E}}_x = \frac{q}{n}\iiint\overline{\varrho}_0\frac{\overline{\xi}^2\left[r^2-2\xi(\xi-\overline{\xi})\right]}{r^3\left(r^2+4\xi\overline{\xi}\right)^{3/2}}d\overline{\xi}\,d\overline{\eta}\,d\overline{\zeta}, \\ \overline{\mathfrak{E}}_y = -\frac{2}{\pi}\xi\iiint\overline{\varrho}_0\frac{\overline{\xi}^2(\eta-\overline{\eta})}{r^3\left(r^2+4\xi\overline{\xi}\right)^{3/2}}d\overline{\xi}\,d\overline{\eta}\,d\overline{\zeta}, \\ \overline{\mathfrak{E}}_z = -\frac{2}{\pi}\xi\iiint\overline{\varrho}_0\frac{\overline{\xi}^2(\zeta-\overline{\zeta})}{r^3\left(r^2+4\xi\overline{\xi}\right)^{3/2}}d\overline{\xi}\,d\overline{\eta}\,d\overline{\zeta}, \end{cases}$$

and rest force $\overline{A}$ emerges from $\overline{\mathfrak{E}}_x$ by permutation of q with $-p$. These expressions in any case hold in the interior of the electron itself.

If one compares expressions (141) with equations (108), then one recognizes, that the quantities

$$\overline{\mathfrak{E}}_x,\ \overline{\mathfrak{E}}_y,\ \overline{\mathfrak{E}}_z,\ \frac{1}{c^2}\overline{A}$$

are in the same way composed of the quantities depending on ξ, η, ζ

$$\frac{\partial\overline{\Phi}}{\partial\xi} + \frac{\overline{\Phi}}{\xi},\ \frac{\partial\overline{\Phi}}{\partial\eta},\ \frac{\partial\overline{\Phi}}{\partial\zeta} \text{ and } p, q$$

as x, y, z, t are composed of ξ, η, ζ and p, q. From that it follows, that

$$\overline{\mathfrak{E}}_x,\ \overline{\mathfrak{E}}_y,\ \overline{\mathfrak{E}}_z,\ \frac{1}{c^2}\overline{A}$$

are exactly so transformed by a Lorentz transformation as x, y, z, t, that is, as a spacetime vector of first kind.

Third Chapter. The dynamics of the rigid electron at hyperbolic motion.

§ 11. The resulting forces and the equations of motion.

It's known that the product of rest density with the electric rest force defined in the previous paragraph, is denoted as ponderomotive force of the field and is seen as equivalent to the ordinary mechanical forces.

In ABRAHAM's theory of the rigid electron, as well as in LORENTZ's one of the "deformable" electron, the equations of motion of an electron free of ordinary mass, are formed in the following way: By integration over the space filled by an electron in a moment t, the resultants of those ponderomotive forces – both that of the external as well as the field generated by the electron itself (or the resulting moment when rotation is also considered) – are formed, and the sum of these resultants is set equal to zero.

This procedure of course is not in agreement with the way taken by us. Because the resultants so created, are evidently dependent on the reference system chosen. *We will try to state such equations of motion, which are invariant against Lorentz transformations.*

The form of the rest forces (141) and (142) produced by the electron itself, and the remark on p. 45 concerning the behavior under Lorentz transformations, however, lays it near at hand in which way one has to form the resultants, so that the equations of motion are invariant under Lorentz transformations, *i.e.*, so that the resultants themselves are transforming as spacetime vectors of first kind. *We will understand as the resulting force of a force field, the integral of the product of rest charge and rest force over the rest shape of the electron,* i.e.*, with respect to* ξ, η, ζ *at invariable* p.

As to how the resulting moments are to be defined in the case, where rotations are allowed, we won't discuss here.

Furthermore we want to state the *equations of motion of a rigid electron* as follows: *The rigid electron is so moving, that the resultants of its own field is oppositely equal to the resultant of the external field.*

Before we actually calculate the resultants of the interior field at hyperbolic motion, we want to make a remark concerning the momentum and energy theorems.

It's known that due to the electromagnetic fundamental equations, the following identities hold:

(143)
$$\begin{cases} \frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z} - \frac{1}{c^2}\frac{\partial \mathfrak{S}_x}{\partial t} = \varrho^* \overline{\mathfrak{E}}_x \\ \cdots\cdots \\ \cdots\cdots \\ \frac{\partial \mathfrak{S}_x}{\partial x} + \frac{\partial \mathfrak{S}_y}{\partial y} + \frac{\partial \mathfrak{S}_z}{\partial z} - \frac{1}{c^2}\frac{\partial W}{\partial t} = \varrho^* \overline{A} \end{cases}$$

Where

(144)

$$\begin{cases} X_x = \frac{1}{2}\left(\mathfrak{E}_x^2 - \mathfrak{E}_y^2 - \mathfrak{E}_z^2\right) + \frac{1}{2}\left(\mathfrak{M}_x^2 - \mathfrak{M}_y^2 - \mathfrak{M}_z^2\right), \\ X_y = \mathfrak{E}_x \mathfrak{E}_y + \mathfrak{M}_x \mathfrak{M}_y, \\ X_z = \mathfrak{E}_x \mathfrak{E}_z + \mathfrak{M}_x \mathfrak{M}_z \end{cases}$$

and (two times) three corresponding quantities of components of *Maxwell's stresses* , furthermore

(145) $$\mathfrak{S} = c[\mathfrak{E}\mathfrak{M}]$$

is the *ray vector*, and

(146) $$W = \frac{1}{2}\left(\mathfrak{E}^2 + \mathfrak{M}^2\right)$$

is the *energy density*. If one integrates these equations over a space limited by a closed surface, then one obtains the equations

(147) $$\begin{cases} \iiint \varrho^* \overline{\mathfrak{E}}_x dv = \iint T_x df - \frac{d}{dt} \iiint \frac{1}{c} \mathfrak{S}_x dv, \\ \ldots\ldots \\ \iiint \varrho^* \overline{A}\, dv = \iint \mathfrak{S}_n df - \frac{d}{dt} \iiint W\, dv, \end{cases}$$

where dv indicates the integration over space, df that over the boundary, and where $\mathfrak{S}_n$ is the normal component of $\mathfrak{S}$ and

$$T_x = X_x \cos(n, x) + X_y \cos(n, y) + X_z \cos(n, z)$$

is the normal component of the x-stresses at the boundary. These equations say, that the resulting force formed in the old sense, is equal to the decrease of the electromagnetic momentum $\frac{1}{c}\mathfrak{S}$ present in a volume, increased by the total stress upon the boundary of the volume; and that the work performed by the forces is equal to the decrease of the electromagnetic total energy, increased by the radiation passing through the boundary.

Let us now first consider Abraham's theory; in this one, the aether is assumed as a stationary absolute reference system, and the electron is rigid in the old sense. Here, the formation of the resultants by integration over the space at invariable t, is justified. Because, when we confine ourselves to the case of rectilinear translation, then all points of the electron have the same velocity w at a time t; the single points of the electron don't perform work relative to each other, the forces acting between them don't appear, and therefore one can see the integrals of the force components and the work over the volume of the electron, as resulting force components and total work. There, however, no relativity principle of any kind is satisfied.

In Lorentz's theory, the electron at quasi-stationary motion is seen as deformable, and namely by the same law according to which a rigid uniformly moving body (in the sense of the theory presented here) appears to be deformed by a stationary reference system. If one defines the resulting forces etc. as integrals at invariable t, then one will obtain quite different values, depending on the reference system employed; yet equations (143) are invariant, *i.e.*, they maintain their form when one subjects the coordinates to a Lorentz transformation, yet only then when the quantities $X_x, \ldots \mathfrak{S}_x, \ldots W$ are simultaneously transformed in a certain way. This circumstance causes the apparent occurrence of a deformation energy and -momentum, to which Planck and Abraham have alluded. Accordingly, the value for the resulting force, work, stresses, and of the ray, which are formed at invariable t, have no immediate meaning in the kinematics of the relativity principle. From the non-observance of this circumstance, the contradictions in LORENTZ's theory are immediately explained.

We can say, that the localization of energy and momentum in the aether in the old sense, in not in agreement with the relativity principle. However, we not at all need the energy and momentum theorems (in the considered form) for the formation of the equations of motion. Rather the definition given on p. 46 completely suffices. The energy theorem is then added to the three equations of motion as a statement depending on them, that the sum of the works of the resultants of the external and the interior field are always equal to zero.

§ 12. The resulting interior forces at hyperbolic motion.

From the expressions (141) or (142), we now form the resultants of the interior rest forces. If we denote the space element $d\xi\ d\eta,\ d\zeta$ by $d\omega$, then we obtain:

(148)
$$\begin{cases} K_x^{(i)} = \int \varrho_0 \overline{\mathfrak{E}}_x d\omega = & \frac{q}{\pi} \iint \varrho_0 \overline{\varrho}_0 \frac{\overline{\xi}^2 \left[r^2 - 2\xi(\xi - \overline{\xi})\right]}{r^3 \left(r^2 + 4\xi\overline{\xi}\right)^{3/2}} d\omega\ d\overline{\omega} \\ K_y^{(i)} = \int \varrho_0 \overline{\mathfrak{E}}_y d\omega = - & \frac{2}{\pi} \iint \varrho_0 \overline{\varrho}_0 \frac{\xi\overline{\xi}^2 (\eta - \overline{\eta})}{r^3 \left(r^2 + 4\xi\overline{\xi}\right)^{3/2}} d\omega\ d\overline{\omega}, \\ K_z^{(i)} = \int \varrho_0 \overline{\mathfrak{E}}_y d\omega = - & \frac{2}{\pi} \iint \varrho_0 \overline{\varrho}_0 \frac{\xi\overline{\xi}^2 (\zeta - \overline{\zeta})}{r^3 \left(r^2 + 4\xi\overline{\xi}\right)^{3/2}} d\omega\ d\overline{\omega} \end{cases}$$

The resulting force $K^{(i)}$ emerges from $K_x^{(i)}$ by permutation of q with $-p$.

First, we consider the integral $K_x^{(i)}$. Here, we will insert the coordinate of an arbitrary point a of the electron (instead of the coordinate ξ calculated from the asymptote of the hyperbolic motion), and therefore replace

ξ by $a + \xi$

$\overline{\xi}$ by $a + \overline{\xi}$.

We will immediately prove, that the electron must have a center. *Therefore we choose this center as reference point.*

Then we have to study (under the integral) the expression

$$\frac{(a + \overline{\xi})^2 \left[r^2 - 2(a + \xi)(\xi - \overline{\xi})\right]}{r^3 \left[r^2 + 4(a + \xi)(a + \overline{\xi})\right]^{3/2}}$$

Now, according to equation (82), the magnitude of acceleration of the center a is equal to

(149)

$$b = \frac{c^2}{a};$$

if we insert this into the expression above, then the following function of acceleration becomes:

(150) $$f = \frac{\left(c^2 + \bar{\xi}b\right)\left[br^2 - 2\left(c^2 + \xi b\right)\left(\xi - \bar{\xi}\right)\right]}{r^3\left[r^2 b^2 + 4(c^2 + \xi b)(c^2 + \bar{\xi}b)\right]^{3/2}}$$

If we denote this as a function of the two points $P(\xi, \eta, \zeta)$, $\overline{P}(\bar{\xi}, \bar{\eta}, \bar{\zeta})$ with $f(P, \overline{P})$, then f can be decomposed into a symmetric and antisymmetric part:

$$f(P, \overline{P}) = f_1(P, \overline{P}) + f_2(P, \overline{P})$$

where

$f_1 = \frac{1}{2}\left[f(P, \overline{P}) + f(\overline{P}, P)\right]$ symmetric,

$f_2 = \frac{1}{2}\left[f(P, \overline{P}) - f(\overline{P}, P)\right]$, antisymmetric

Now it is clear, that the integral

$$\iint \varrho\bar{\varrho}_0 f_2(P, \overline{P}) d\omega\, d\overline{\omega}$$

identically vanishes.

Consequently we can confine ourselves to the study of f_1. It is given:

(151) $$f_1 = \frac{b}{2}\frac{r^2\left[\left(c^2 + \xi b\right)^2 + \left(c^2 + \bar{\xi}b\right)^2\right] + 2(\xi - \bar{\xi})\left(c^2 + \xi b\right) + \left(c^2 + \bar{\xi}b\right)}{r^3\left[r^2 b^2 + 4(c^2 + \xi b)(c^2 + \bar{\xi}b)\right]^{3/2}}$$

Thus the six-fold integral in $K_x^{(i)}$ becomes proportional to b. If we combine b with the factor q [according to (80)] into $-b_x$, and combining b with p into $c^2 b_t$ by forming the work $K^{(i)}$, then we can write:

(152) $$\begin{cases} K_x^{(i)} &= -\mu b_x, \\ K^{(i)} &= -c^2 \mu b_t. \end{cases}$$

Where rest mass μ is the following quantity, which only depends on magnitude b of the acceleration of the center a:

(153)

$$\mu=\frac{1}{2\pi}\iint\frac{\varrho\bar{\varrho}}{r^3\left[r^2b^2+4(c^2+\xi b)(c^2+\bar{\xi}b)\right]^{3/2}}\left\{r^2\left[\left(c^2+\xi b\right)^2\right.\right.$$

$$\left.\left.+\left(c^2+\bar{\xi}b\right)^2\right]+2(\xi-\bar{\xi})^2\left(c^2+\xi b\right)\left(c^2+\bar{\xi}b\right)\right\}d\omega\,d\bar{\omega}$$

Since b depends on the initial coordinate a of the center only, it is constant at any hyperbolic motion. *Thus for any hyperbolic motion, μ is a constant which only depends on the shape and charge distribution of the electron.*

From (151), one obtains the equation of motion of the x-coordinate, by setting the sum of the interior force $K_x^{(i)}$ and the external force $K_x^{(a)}$ equal to zero; and $K^{(i)}+K^{(a)}=0$ is the expression for the energy equation as well. This gives:

(154) $$\begin{cases}\mu b_x = & K_x^{(a)},\\ \mu b_t = & \frac{1}{c^2}K^{(a)}.\end{cases}$$

It is to be shown now, that one can state an external force field, which is capable to sustain hyperbolic motion. This is performed by an electric force E_x acting in the x-direction, and which is independent from location and time. Then, the rest force according to (139) is namely:

$$\overline{E}_x=t_\tau E_x=-qE_x$$

as well as the rest work

$$\overline{A}=x_\tau E_x=qE_x$$

and when E_x is constant, then the integration of $\varrho_0\overline{E}_x$ and $\varrho_0\overline{A}$ with respect to ξ,η,ζ is simply

$$\begin{aligned}K_x^{(a)}&=-qeE_x,\\ K^{(a)}&=qeE_x.\end{aligned}$$

This force will be capable of sustaining the hyperbolic motion with acceleration b, when one chooses:

(155) $$E_x=\frac{\mu}{e}b$$

If the external force field is only slightly variable, so that one can see it as constant in the interior of the electron, then it will generate a motion which only slightly deviates from hyperbolic motion. When we see equations (154) as valid also in this case, we are neglecting radiation.

Thus we obtain for slightly variable, yet arbitrary great accelerations, the following equations of motion and energy:

(156) $$\begin{cases}\mu\frac{\partial^2x}{\partial\tau^2}=t_\tau eE_x,\\ \\ \mu\frac{\partial^2t}{\partial\tau^2}=\frac{1}{c^2}x_\tau eE_x,\end{cases}$$

where x and t are related to the center of the electron. *These equations are invariant under Lorentz transformations and have the form of the mechanical equations of motion of a mass point*.

If one sees μ as constant (which is justified by the next paragraph), and if one introduces the "ordinary" mass by the relation

(157) $$m = \mu t_\tau = \frac{\mu}{\sqrt{1 - \frac{w^2}{c^2}}}$$

and replaces the derivatives with respect to τ by derivatives with respect to t, then one obtains:

(158) $$\begin{cases} \frac{\partial m w_x}{\partial t} = eE_x, \\ \frac{\partial m}{\partial t} = \frac{1}{c^2} eE_x w_x. \end{cases}$$

The first of these equations, is the equation of motion in an analogous form of one of the NEWTONian equations of old mechanics, the second one is the energy equation. The quantity $c^2 m$ thus corresponds to the kinetic energy of old mechanics. *The dependency of mass m from velocity is given by LORENTZ's formula (157)*; more essential than this one (which is also valid for the ordinary (non-electromagnetic) mass in the new kinematics), is the dependency of rest mass μ from the magnitude b of acceleration according to formula (153). This dependency shall be studied more closely in the next paragraph.

Before that, we still have to consider the y- and z-component of the interior force.

If we apply to $K_y^{(i)}$ the same considerations as to $K_x^{(i)}$, by splitting the integration into a symmetric and an antisymmetric part, then we obtain:

(159) $$K_y^{(i)} = -\frac{2}{\pi} b \iint \varrho_0 \overline{\varrho}_0 \frac{(\eta - \overline{\eta})(\xi - \overline{\xi}) \left(c^2 + b\xi\right) \left(c^2 + b\overline{\xi}\right)}{r^3 \left\{r^2 b^2 + 4(c^2 + b\xi)(c^2 + b\overline{\xi})\right\}} d\omega \, d\overline{\omega}$$

and an analogous expression hold for $K_z^{(i)}$.

If one presupposes acceleration b as small, then it becomes:

(160) $$\left[K_y^{(i)}\right]_0 = -\frac{b}{4\pi c^2} \iint \varrho_0 \overline{\varrho}_0 \frac{(\eta - \overline{\eta})(\xi - \overline{\xi})}{r^3} d\omega \, d\overline{\omega}$$

We now will postulate, that at vanishingly small acceleration, the electron exerts no lateral forces upon itself. Namely, if this were the case, then external lateral forces would already be required at quasi-stationary translation, to sustain the motion. Yet this contradicts the observation at cathode- and Becquerel rays, which are moving rectilinear by themselves without external lateral influence.

Yet, the charge distribution must be symmetrical to one of the planes $\xi = 0$ or $\eta = 0$, so that $\left[K_y^{(i)}\right]_0 = 0$. One also sees, that it must also be symmetrical to one of the planes $\xi = 0$ or $\zeta = 0$, so that $\left[K_z^{(i)}\right]_0 = 0$.

Since furthermore the direction of motion is an arbitrary direction in the electron, then the charge must be symmetrical to every plane passing through the center. Thus it is located in concentric layers around the center.

From the observational fact, that no external lateral forces are necessary to sustain the quasi-stationary translation, it thus follows, that the electron has a center around which the charge is distributed in concentric layers.

Though if this is the case, then it is given from (159) without further ado, that $K_y^{(i)}$ then vanishes for arbitrary values of b; and the same applies to $K_z^{(i)}$. Consequently we have the result:

The electron exerts no lateral force upon itself at arbitrary accelerated hyperbolic motion.

By that, also the still missing part of the law of inertia is derived electrodynamically. With the same approximation, by which equations (156) and (158) hold for motions with slight changes of accelerations, we can also transfer this result upon such motions.

The insight, that one can conclude the existence of a center and of the charge distribution in concentric layers from the behavior of electrons at quasi-stationary translation, is an additional contribution to fortify the atomistic view of electricity. I don't believe, that any other theory can give such a close connection between atomistics and the principle of dynamics.

§ 13. The electrodynamic rest mass.

First we want to calculate the value of the rest mass for quasi-stationary motions, *i.e.*, for vanishingly small values of b. If we set $b = 0$ in expression (153), then it goes over into

$$\mu_0 = \frac{1}{8\pi c^2}\left\{\iint \frac{\varrho_0 \overline{\varrho}_0}{r} d\omega\, d\overline{\omega} + \iint \varrho_0 \overline{\varrho}_0 \frac{(\xi - \overline{\xi})^2}{r^3} d\omega\, d\overline{\omega}\right\}$$

The first of these two integrals is the electrostatic energy of the electron

$$4\pi U = \iint \frac{\varrho_0 \overline{\varrho}_0}{r} d\omega\, d\overline{\omega} \tag{161}$$

The second integral can also be represented (because of the centric symmetry of the electron) in the forms

$$\iint \varrho_0 \overline{\varrho}_0 \frac{(\eta - \overline{\eta})^2}{r^3} d\omega\, d\overline{\omega}$$

and

$$\iint \varrho_0 \overline{\varrho}_0 \frac{(\zeta - \overline{\zeta})^2}{r^3} d\omega\, d\overline{\omega}$$

If we add up these three expressions, then we obtain $4\pi U$ again. Thus the second integral is equal to $\frac{4\pi}{3} U$, and we obtain for the *rest mass at quasi-stationary motion*:

(162) $$\mu_0 = \frac{2}{3c^2} U$$

If the electron is particularly a homogeneously charged sphere of radius R, it is given:

(163) $$\mu_0 = \frac{1}{5\pi} \frac{e^2}{Rc^2}$$

where e is the total charge.

This expression agrees with the values given by all other theories.[26]

If the motion is not quasi-stationary any more, then one has to use the general expression (153) for μ . Then, one will develop μ with respect to powers of b:

(164) $$\mu = \mu_0 + b\mu_1 + b^2 \mu_2 + \dots$$

We prove now, that the coefficient μ_1 of b is equal to zero. Namely, one finds the following expression for it:

$$\mu_1 = -\frac{1}{16\pi c^2} \iiint \varrho_0 \overline{\varrho}_0 \frac{(\xi - \overline{\xi})\left(r^2 + (\xi - \overline{\xi})^2\right)}{r^3} d\omega\, d\overline{\omega}$$

Now, since the charge of the electron is distributed in concentric layers, then $\xi, \eta, \zeta;\ \overline{\xi}, \overline{\eta}, \overline{\zeta}$ corresponds to another $-\xi, \eta, \zeta;\ -\overline{\xi}, \overline{\eta}, \overline{\zeta}$ in any value system, for which the integrand assumes the opposite value. Consequently it is $\mu_1 = 0$.

Furthermore one finds:

(165) $$\begin{cases} \mu_2 = -\frac{1}{32\pi c^6} \iiint \varrho_0 \overline{\varrho}_0 \left\{ 3r + 2\frac{\xi^2 + \overline{\xi}^2 + 6\xi\overline{\xi}}{r} \right. \\ \left. \qquad + \frac{(\xi - \overline{\xi})^2 \left(3\xi^2 + 3\overline{\xi}^2 + 10\xi\overline{\xi}\right)}{r^3} \right\} d\omega\, d\overline{\omega} \end{cases}$$

This value is extraordinarily small compared with the value of μ_0; because, while the latter is converging into infinity at decreasing radius R of the electron, μ_2 is converging (like R) into zero. Furthermore, μ_2 has the sixth power of the speed of light in the denominator. Thus we can say:

In the series for the rest mass

(166) $$\mu = \mu_0 + b^2\mu_2 + \dots$$

the coefficient of μ_2 *is so extraordinarily small compared to* μ_0*, that already the quadratic term in the acceleration, can in no way become noticeable at any observation.*

Therefore, one can see the rest mass as constant in all practical cases. Its value is given by expression (162) for μ_0.

By that, the basic features of the dynamics of the rectilinear moving electron are given on an electromagnetic basis. Of course, the area of applicability of the theory immediately is extended by the consideration, that any uniform translation in any direction can be superimposed over rectilinear motions; because this is only about the passage from one coordinate system of another one by the aid of a Lorentz transformation, where our equations of motions are invariant. *Therefore, this theory encloses the deflection of the electrons by electric fields, which have an arbitrary direction with respect to their velocity and are not changing too rapidly in terms of location and time. On the other hand, it is not immediately valid for the magnetic deflection*; though it can be easily seen, that (for quasi-stationary motion) also the magnetic deflection is reproduced by the theory.

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Correction to the paper: The theory of the rigid electron in the kinematics of the principle of relativity.

by MAX BORN.

In § 13, p. 54, of the paper published in Ann. d. Phys. 30. p. 1. 1909, $4\pi U$ is to be replaced by $8\pi U$, in formula (161) as well as in some lines below. Consequently, formula (162) must read:

(162) $$\mu_0 = \frac{4}{3c^2}U$$

(Received October 30, 1909.)

Annotations by Wikisource

1. This means the four-dimensional representation of Newtonian mechanics in which $c = \infty$.

Footnotes by the author

1. H. MINKOWSKI, Raum und Zeit, Physk. Zeitschr. 10. p. 104, 1909, and Jahresber. d. deutsch. Mathematiker-Vereinigung, 18. (Also published separately in print.) The knowledge of this work is presupposed at some discussions.
2. A. EINSTEIN, Jahrb. der Radioakt. und Elektronik 4. Heft 4, § 18, 1907.
3. A. SOMMERFELD, Nachr. d. k. Ges. d. Wissensch. zu Göttingen, math.-physik. Kl. Heft 2 a. 5, 1904
4. P. HERTZ, Math. Ann. 68. p. 1. 1907.

5. G. Herglotz, Nachr. d. k. Ges. d. Wissensch. zu Göttingen, math.-physik. Kl. Heft 6, 1903.
6. K. Schwarzschild, Nachr. d. k. Ges. d. Wissensch. zu Göttingen, math.-physik. Kl. p. 125, 1903.
7. See M. Abraham, Theorie der Elektrizität 2. Aufl. Vol. 2. § 22.
8. M. Born, Ann. d. Phys. 28. p. 571. 1909.
9. See. A. Einstein, Ann. d. Phys. 17. p. 891. 1905 (http://www.fourmilab.ch/etexts/einstein/specrel/www/); M. Planck, Verh. d. Deutsch. Phys. Ges. 8. p. 136. 1906; H. Minkowski, Nachr. d. k. Ges. d. Wissensch. zu Göttingen, math.-physik. Kl. p. 54. 1908; see M. Born, l.c.
10. T. Levi-Civita, Sui campi elettromagnetici puri, at C. Ferrari, Venezia 1908; Sulle azione meccaniche etc., Rendiconti d. R. Acad. dei Lincei 18. 5a. This theory also appears to be leading to contradictions with experiments, when applied to cathode rays.
11. To avoid laborious expressions, I use the matrix calculus which is most suitable for these considerations; a very simple representation which is understandable without previous knowledge, can be found in § 11 of the work of Minkowski concerning "die Grundgleichungen für die elektromagnetischen Vorgänge in bewegten Körpern" (cited in note 2, p. 5).
12. For a closer geometric description of the Lorentz transformation, see H. Minkowski, Raum und Zeit, l.c. (Note 1, p. 2)
13. This theorem is seldom formulated explicitly, yet it is an immediate consequence of the simplest projection theorems.
14. *i.e.* a vector hitting the structure (18) in a real way.
15. See for example Weber-Riemann, Die partiellen Differentialgleichungen der mathematischen Physik 2. § 146. 1901.
16. See H. Minkowski, Raum u. Zeit, l.c. (Note p. 2)
17. H. Minkowski, Raum und Zeit, l.c. (Note p. 2.)
18. The equation has one and only one such solution, because the velocity of the current cannot reach the speed of light. See H. Minkowski, l. c. (Note p. 7).
19. In this representation, I follow the procedure of W. Ritz, by seeing the potentials as the effects of charge to charge as primary throughout, and by moving the partial differential equation into the second row at first. It is characteristic, that in my whole theory no use is at all made of the actual field equations of Maxwell for $\mathfrak{E}, \mathfrak{M}$.
20. See. K. Schwarzschild, l.c. (Note p. 4); M. Born, l.c. (Note p. 5).
21. E. Wiechert, Arch. néerl. (2) 5. p. 549. 1900. A. Liénard, L'éclairage électrique 16. p. 5, 53, 106. 1898
22. See l.c. (Note. 1 p. 28).
23. See for example M. Abraham, Theorie der Elektrizität 2, 2. Ed., Formulas (51b), (51c), p. 57.
24. It is (exactly like $1/r$ for $\Delta u = 0$) Green's function of differential equation (137) for the infinite space with the limiting condition, that the solution shall vanish at infinity, and that the derivative

multiplied by r^2 shall remain finite.

25. H. Minkowski, l.c. (Note p. 5., see p. 32ff.)
26. See for example M. ABRAHAM, Theorie d. Elektrizität 2. 2. ed. p. 179, formula (117c). There, another unit was used; our formula (163) goes over into that of ABRAHAM

$$\mu_0 = \frac{4}{5}\frac{e^2}{Rc^2}$$

when e is replaced by $\sqrt{4\pi}e$.
