An Algorithm for Convex Polytopes

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ABSTRACT. An algorithm, one that is economical and fast, for generating the convex polytope of a set S of points lying in an n-dimensional Euclidean space E^n is described. In the existing brute force method for determining the convex hull of a set of points lying in a two-dimensional space, one computes all possible straight lines joining each pair of points of S and tests whether the lines bound the given set S. This method can easily be generalized for computing the convex hull of a set $S \subset E^n$, $n > 2$. However, it turns out that this approach is not feasible due to excessive computer run time for a set of points lying in E^n when $n > 3$. The algorithm described in this paper avoids all the unnecessary calculations, and the convex polytope of a set $S \subset E^n$ is generated by systematically computing the faces from the edges of the desired convex polytope. A numerical comparison indicates that this new approach is far superior to the existing brute force technique.

KEY WORDS AND PHRASES: convex, polytope, algorithm, convex hull, faces, edges, hyperplanes, flats

CR CATEGORIES: 3.21, 5.19

Introduction

The study of convex polytopes has received a considerable impetus from its application to engineering problems. However, a search through the available published literature seems to indicate that practically no effort has been made to develop numerically efficient procedures to determine the convex hull of a finite set S of points lying in E^n.

The existing brute force technique for computing the faces of the convex polytope of a given set of points in E^n, $n > 2$, is not feasible due to excessive computer run time even when the size of the set is reasonably small. The approach presented here is essentially a systematic way for computing the faces sequentially from the edges of the desired convex polytope.

The algorithm is based on the observation that exactly two faces of the convex polytope $\mathbf{C}(S)$ of a set $S \subset E^n$ intersect along each edge of $\mathbf{C}(S)$. If one edge and one of the two faces containing this edge are known, then the second face can be computed by a process which is practically equivalent to a rotation of the known face about the known edge through an appropriate angle. The determination of each new face gives rise to at least $(n - 1)$ edges of $\mathbf{C}(S)$ that are different from the known edge. This process is continued until each edge is the intersection of two adjacent faces of the convex polytope. A method is presented for generating the first face that is needed to initiate the algorithm.

A FORTRAN program was developed, which is being used at Lockheed-Georgia Company. The computer run time is reasonably low to treat practical problems consisting of one thousand six-dimensional points.

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Mathematical Analysis

Let S be a set of $(m + 1)$ distinct points $P^i = (p_1^i, p_2^i, \cdots, p_n^i)$, $i = 0, 1, \cdots, m$ lying in E^n.

Definition 1. An r-flat is a region determined by $(r + 1)$ points and having dimension r. An r-flat $(r > n)$ is herein called a hyperplane of r dimensions and is denoted by $H^{T r}$.

Definition 2. A hyperplane H^{n-1} is the set of points $X = (x_1, x_2, \cdots, x_n)$ which satisfy an equation of the form $\sum_{i=1}^n \alpha_i x_i - \beta = 0$, where not all α_i are zero. A hyperplane H^{n-1} separates the space E^n into two half-spaces.

Definition 3. A normal to the hyperplane H^{n-1} is a vector parallel to $\mathbf{n}$, where $\mathbf{n} = (\alpha_1, \alpha_2, \cdots, \alpha_n)$. The unit normal to H^{n-1} is denoted by $\hat{\mathbf{n}}$ and is given by

$$\hat{\mathbf{n}} = \frac{1}{\left(\sum_{i=1}^{n} \alpha_i^2\right)^{\frac{1}{2}}} (\alpha_1, \alpha_2, \cdots, \alpha_n).$$

Definition 4. A hyperplane H^{n-1} bounds the set $S \subset E^n$ if and only if all points of S lie either on H^{n-1} or in one half-space. If $\hat{\mathbf{v}}_i$ denotes a unit vector along QP^i, $Q \in H^{n-1}$, and $P^i \in S$, then we say that H^{n-1} bounds the set S iff either the inner product $(\hat{\mathbf{n}} \cdot \hat{\mathbf{v}}_i) \geq 0$ for $i = 0, 1, \cdots, n$ or the inner product $(\hat{\mathbf{n}} \cdot \hat{\mathbf{v}}_i) \leq 0$ for $i = 0, 1, \cdots, m$; $\hat{\mathbf{n}}$ being the unit normal to H^{n-1}.

Definition 5. A hyperplane H^{n-1} is called a support plane of S if H^{n-1} bounds S and at least one point of S lies on H^{n-1}.

Definition 6. A system of $(l + 1)$ points is said to be linearly independent if no set of $(r + 1)$ points lies in the same $(r - 1)$-flat $(r \leq l)$.

Definition 7. The convex polytope $\mathbf{C}(S)$, to be called the n-polytope, of the set $S \subset E^n$ is the set of points which is the intersection of all the convex sets in E^n that contain S.

Definition 8. A support plane H^{n-1} of S is said to be an n-face of $\mathbf{C}(S)$ if n independent points of S lie on H^{n-1}.

Definition 9. An n-edge of $\mathbf{C}(S)$ is an $(n - 2)$-flat contained in a support plane of $\mathbf{C}(S)$ which is not an n-face of $\mathbf{C}(S)$.

THEOREM 1. *Each n-edge of the n-polytope $\mathbf{C}(S)$ lies in two and exactly two n-faces of $\mathbf{C}(S)$.*

PROOF. Let $\mathbf{E}$ be an n-edge of the n-polytope $\mathbf{C}(S)$. The proof of this theorem is constructive. That is, the two adjacent n-faces of $\mathbf{C}(S)$ intersecting along the edge $\mathbf{E}$ are constructed.

Let $P^0, P^1, \cdots, P^{n-2}$ be $(n - 1)$ linearly independent points of S that define the n-edge $\mathbf{E}$. The definition of $\mathbf{E}$ asserts the existence of a support plane H of S, containing $\mathbf{E}$, which is not an n-face of $\mathbf{C}(S)$. Let $\hat{\mathbf{n}}$ be the unit normal to the support plane H that is directed toward the points in S. That is, if $\hat{\mathbf{v}}_i$ denotes the unit vector along $p_0 p_i$ then

$$(\hat{\mathbf{n}} \cdot \hat{\mathbf{v}}_i) \geq 0, \qquad i = 1, 2, \cdots, m. \tag{1.1}$$

As pointed out in the Introduction, we need to compute the two appropriate angles through which the support plane H is to be rotated about $\mathbf{E}$ so as to generate the desired two adjacent n-faces of $\mathbf{C}(S)$. A three-dimensional case shown in Figure 1 suggests that we compute the angles between the support plane H and each of the hyperplane H_i's obtained by adjoining a point $P^i \in S$ to the points of $\mathbf{E}$ such

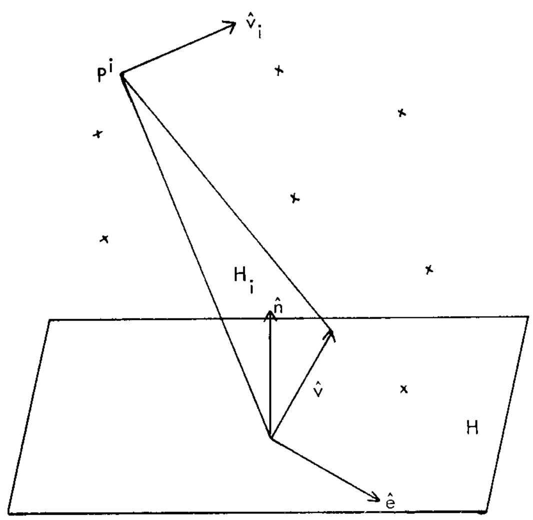

FIG. 1

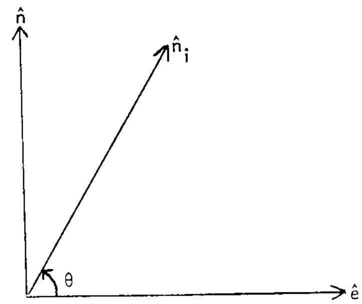

FIG. 2

If $\hat{\mathbf{n}}_i = \lambda\hat{\mathbf{n}} + \mu\hat{\mathbf{e}}$ then $\tan\theta = \lambda/\mu$.

that the new system has n linearly independent points. Clearly the hyperplanes corresponding to the maximum angle and the minimum angle are the desired n-faces of $\mathbf{C}(S)$.

To determine the angle between the support plane H and the hyperplane H_i one needs to know the normals $\hat{\mathbf{n}}$ and $\hat{\mathbf{n}}_i$ to H and H_i respectively, and a normal $\hat{\mathbf{n}}_i$ to H_i is determined by solving an $(n - 1) \times n$ linear homogeneous system. Thus to compute the desired maximum and minimum angles, one needs to generally solve the $(n - 1)*n$ linear system $(m - n - 1)$ times. But in the approach described here the desired two adjacent n-faces are computed by solving an $(n - 1)*n$ linear system exactly once.

To minimize numerical computation we extremize the tangent of the angle between the vector $\hat{\mathbf{e}}$, to be constructed, and the normals to hyperplanes H_i as follows (see Figure 2): Construct a unit vector $\hat{\mathbf{e}} = (e_1, e_2, \cdots, e_n)$ such that

$$\begin{aligned} &\text{(i)} \quad (\hat{\mathbf{e}} \cdot \hat{\mathbf{n}}) = 0, \\ &\text{(ii)} \quad (\hat{\mathbf{e}} \cdot \hat{\mathbf{v}}_i) = 0, \qquad i = 1, 2, \cdots, (n - 2). \end{aligned} \tag{1.2}$$

The components e_i of $\hat{\mathbf{e}}$ are calculated by solving the linear homogeneous system (1.2) of $(n - 1)$ equations in n variables.

Since the n vectors $\hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2, \cdots, \hat{\mathbf{v}}_{n-2}, \hat{\mathbf{n}}, \hat{\mathbf{e}}$ form a basis, any vector in E^n can be expressed linearly in terms of these. For each k such that $P^k \in S$ and $P^k \notin H$, let $\hat{\mathbf{n}}_k$ denote the unit normal to the hyperplane H_k spanned by n independent points $P^0, P^1, \cdots, P^{(n-2)}, P^k$. The direction of $\hat{\mathbf{n}}_k$ is specified by the relation

$$(\hat{\mathbf{n}}_k \cdot \hat{\mathbf{e}}) > 0. \tag{1.3}$$

Since $\boldsymbol{n}_k$ is normal to the n-edge $\mathbf{E}$ it must lie in the plane spanned by $\hat{\mathbf{n}}$ and $\hat{\mathbf{e}}$. Therefore, $\hat{\mathbf{n}}_k = \lambda_k\hat{\mathbf{n}} + \mu_k\hat{\mathbf{e}}$, where λ_k, μ_k are constants satisfying the relations:

$$\begin{aligned}
&\text{(i)} \quad \lambda_k^2 + \mu_k^2 = 1, \\
&\text{(ii)} \quad \lambda_k = (\hat{\mathbf{n}}_k \cdot \hat{\mathbf{n}}), \qquad\qquad (1.4)\\
&\text{(iii)} \quad \mu_k = (\hat{\mathbf{n}}_k \cdot \hat{\mathbf{e}}) > 0, \qquad \text{using (1.3).}
\end{aligned}$$

Further, $\hat{\mathbf{n}}_k$ being orthogonal to $\hat{\mathbf{v}}_k$ implies

$$([\lambda_k \hat{\mathbf{n}} + \mu_k \hat{\mathbf{e}}] \cdot \hat{\mathbf{v}}_k) = 0$$

or

$$\lambda_k (\hat{\mathbf{n}} \cdot \hat{\mathbf{v}}_k) = -\mu_k (\hat{\mathbf{e}} \cdot \hat{\mathbf{v}}_k).$$

Using (1.1) and (1.4) we obtain

$$\frac{\lambda_k}{\mu_k} = -\frac{(\hat{\mathbf{e}} \cdot \hat{\mathbf{v}}_k)}{(\hat{\mathbf{n}} \cdot \hat{\mathbf{v}}_k)}. \qquad (1.5)$$

We know a hyperplane H_m will be an n-face of $\mathbf{C}(S)$ if either $(\hat{\mathbf{n}}_m \cdot \hat{\mathbf{v}}_k) \geq 0$ or $(\hat{\mathbf{n}}_m \cdot \hat{\mathbf{v}}_k) \leq 0$ holds for each $P^k \in S$. But

$$\begin{aligned}
(\hat{\mathbf{n}}_m \cdot \hat{\mathbf{v}}_k) &= ([\lambda_m \hat{\mathbf{n}} + \mu_m \hat{\mathbf{e}}] \cdot \hat{\mathbf{v}}_k) \\
&= \mu_m (\hat{\mathbf{n}} \cdot \hat{\mathbf{v}}_k) \left[\frac{\lambda_m}{\mu_m} - \frac{\lambda_k}{\mu_k} \right], \qquad \text{using (1.5).}
\end{aligned}$$

Since $\mu_m > 0$ and $(\hat{\mathbf{n}} \cdot \hat{\mathbf{v}}_k) \geq 0$, for each $P^k \in S$, it follows that H_m will be an n-face of $\mathbf{C}(S)$ if and only if either $\lambda_m/\mu_m \geq \lambda_k/\mu_k$ or $\lambda_m/\mu_m \leq \lambda_k/\mu_k$ holds for each $P^k \in S$ and $P^k \notin H$. Thus we have shown the existence of exactly two n-faces, say F_1 and F_2, each containing the edge $\mathbf{E}$, whose normals $\hat{\mathbf{n}}_{\mathbf{E}1}$ and $\hat{\mathbf{n}}_{\mathbf{E}2}$ are given by the relations

$$\begin{aligned}
\hat{\mathbf{n}}_{\mathbf{E}1} &= \lambda_{\mathbf{E}1} \hat{\mathbf{n}} + \mu_{\mathbf{E}1} \hat{\mathbf{e}}, \\
\hat{\mathbf{n}}_{\mathbf{E}2} &= \lambda_{\mathbf{E}2} \hat{\mathbf{n}} + \mu_{\mathbf{E}2} \hat{\mathbf{e}},
\end{aligned}$$

where

$$\begin{aligned}
\frac{\lambda_{\mathbf{E}1}}{\mu_{\mathbf{E}1}} &= \min \left\{ \frac{\lambda_k}{\mu_k} \right\}, \\
\frac{\lambda_{\mathbf{E}2}}{\mu_{\mathbf{E}2}} &= \max \left\{ \frac{\lambda_k}{\mu_k} \right\};
\end{aligned}$$

the minimum and the maximum are taken over all k such that $P^k \in S$ and $P^k \notin H$;

$$\frac{\lambda_k}{\mu_k} = -\frac{(\hat{\mathbf{e}} \cdot \hat{\mathbf{v}}_k)}{(\hat{\mathbf{n}} \cdot \hat{\mathbf{v}}_k)};$$

and $\lambda_k^2 + \mu_k^2 = 1$.

THEOREM 2. *Let S_F be a subset of S consisting of points lying on an n-face F of $\mathbf{C}(S)$. Then an $(n-1)$-face of $\mathbf{C}(S_F)$ is an n-edge of $\mathbf{C}(S)$.*

PROOF. Let $\hat{\mathbf{n}}$ be the normal to the n-face F such that

$$(\hat{\mathbf{n}} \cdot \hat{\mathbf{v}}_k) \geq 0, \qquad k = 1, 2, \cdots, n, \qquad (2.1)$$

$\hat{\mathbf{v}}_k$ being a unit vector along $\overrightarrow{P^0 P^k}$.

Suppose $P^0, P^1, \cdots, P^r$ are $(r+1)$ points $(r \geq n-1)$ of S that form the subset S_F. Determine the convex hull $\mathbf{C}(S_F)$ of the set S_F in the $(n-1)$-dimensional subspace F. Suppose $P^0, P^1, \cdots, P^{n-2}$ are $(n-1)$ independent points that define an $(n-1)$-face f of $\mathbf{C}(S_F)$. Let $\hat{\mathbf{e}}$ be a unit vector normal to f such that $(\hat{\mathbf{e}} \cdot \hat{\mathbf{v}}_k) \geq 0$, $k = 1, 2, \cdots, r$. Since $\hat{\mathbf{e}}$ lies in the subspace F, it follows that $(\hat{\mathbf{e}} \cdot \hat{\mathbf{n}}) = 0$ and thus this vector $\hat{\mathbf{e}}$ is identical to the vector $\hat{\mathbf{e}}$ constructed in Theorem 1 and satisfies the extra condition

$$(\hat{\mathbf{e}} \cdot \hat{\mathbf{v}}_k) \geq 0, \qquad k = 1, 2, \cdots, m. \tag{2.2}$$

Proceeding as in the proof of Theorem 1, the unit normal $\hat{\mathbf{n}}_k$ to the hyperplane, generated by adjoining $P^k \in S$, $P^k \notin F$, to the points defining f, can be expressed linearly in terms of $\hat{\mathbf{n}}$ and $\hat{\mathbf{e}}$. That is,

$$\hat{\mathbf{n}}_k = \lambda_k \hat{\mathbf{n}} + \mu_k \hat{\mathbf{e}}, \tag{2.3}$$

where $1 = \lambda_k^2 + \mu_k^2$.

The direction of $\hat{\mathbf{n}}_k$ is specified by the relation

$$\mu_k = (\hat{\mathbf{n}}_k \cdot \hat{\mathbf{e}}) > 0. \tag{2.4}$$

The condition that $\hat{\mathbf{n}}_k$ is orthogonal to $\hat{\mathbf{v}}_k$ can be expressed in the form

$$\frac{\lambda_k}{\mu_k} = -\frac{(\hat{\mathbf{e}} \cdot \hat{\mathbf{v}}_k)}{(\hat{\mathbf{n}} \cdot \hat{\mathbf{v}}_k)}. \tag{2.5}$$

Let $P^m \in S$ be a point such that

$$\frac{\lambda_m}{\mu_m} = \max \left\{ \frac{\lambda_k}{\mu_k} \right\}, \tag{2.6}$$

where the maximum is taken over each k such that $P^k \in S$. Then, using (2.5),

$$\begin{aligned} (\hat{\mathbf{n}}_m \cdot \hat{\mathbf{v}}_k) &= ([\lambda_m \hat{\mathbf{n}} + \mu_m \hat{\mathbf{e}}] \cdot \hat{\mathbf{v}}_k) \\ &= \mu_m (\hat{\mathbf{n}} \cdot \hat{\mathbf{v}}_k) \left[\frac{\lambda_m}{\mu_m} - \frac{\lambda_k}{\mu_k} \right]. \end{aligned}$$

Using relations (2.1), (2.4), and (2.6) it follows that $(\hat{\mathbf{n}}_m \cdot \hat{\mathbf{v}}_k) \geq 0$ for each $P^k \in S$. This shows that the hyperplane obtained by adjoining the point $P^m \in S$ to the points defining f is an n-face F^* of $\mathbf{C}(S)$.

Consider the hyperplane H passing through the $(n-1)$ points $P^0, P^1, \cdots, P^{n-2}$ whose normal $\mathbf{n}^*$ is given by the relation $\mathbf{n}^* = a\hat{\mathbf{n}} + (1-a)\hat{\mathbf{n}}_m$, with $0 \leq a \leq 1$. Clearly the hyperplane H bounds the set S and contains the $(n-1)$-face f of $\mathbf{C}(S_F)$. Since H is not an n-face of $\mathbf{C}(S)$ it follows that the $(n-1)$-face f is an n-edge of $\mathbf{C}(S)$.

COROLLARY 1. *Suppose an n-edge* $\mathbf{E}$ *and an n-face* F *of* $\mathbf{C}(S)$ *containing* $\mathbf{E}$ *are given. Then an n-face* F^* *of* $\mathbf{C}(S)$, *distinct from* F *and adjacent to* F *along the n-edge* $\mathbf{E}$, *may be generated as follows. Let* $\hat{\mathbf{n}}$ *be the unit normal to* F *and* $\hat{\mathbf{e}}$ *be a unit vector orthogonal to* $\hat{\mathbf{n}}$ *and normal to* $\mathbf{E}$ *such that* $(\hat{\mathbf{n}} \cdot \hat{\mathbf{v}}_k) \geq 0$ *and* $(\hat{\mathbf{e}} \cdot \hat{\mathbf{v}}_k) \geq 0$, $k = 1, 2, \cdots, m$. *Define* $\lambda_k/\mu_k = -(\hat{\mathbf{e}} \cdot \hat{\mathbf{v}}_k)/(\hat{\mathbf{n}} \cdot \hat{\mathbf{v}}_k)$, $k = 1, 2, \cdots, m$. *Let* $P^m \in S$ *be a point such that* $\lambda_m/\mu_m = \max\{\lambda_k/\mu_k\}$. *Then the hyperplane obtained by adjoining the point* P^m *to the points defining* $\mathbf{E}$ *is an n-face of* $\mathbf{C}(S)$ *whose normal* $\hat{\mathbf{n}}^*$ *is given by the relation* $\hat{\mathbf{n}}^* = \lambda_m \hat{\mathbf{n}} + \mu_m \hat{\mathbf{e}}$, *with* $1 = \lambda_m^2 + \mu_m^2$.

Determination of Edges

If an n-edge of the desired convex polytope $\mathbf{C}(S)$ is given, and if a support plane H of S containing the given edge, which may or may not be an n-face of $\mathbf{C}(S)$, is known, then the proof of Theorem 1 or Corollary 1 determines a new n-face F of $\mathbf{C}(S)$ which intersects the support plane H along the known edge. The n-edges of $\mathbf{C}(S)$ lying on F can be found, according to Theorem 2, by computing the $(n - 1)$-faces of the convex polytope of points of S lying on F in the $(n - 1)$-dimensional subspace. If exactly n points of S lie on an n-face of $\mathbf{C}(S)$ then there exist exactly n n-edges of $\mathbf{C}(S)$ that lie on that n-face and these n n-edges are determined by taking all combinations of $(n - 1)$ points from these n points. Thus an n-edge and a support plane of S containing this n-edge of $\mathbf{C}(S)$ produces an n-face of $\mathbf{C}(S)$ which in turn gives rise to new n-edges of $\mathbf{C}(S)$. The process may be repeated by choosing one of the new n-edges and the n-face containing this n-edge to produce a new n-face of $\mathbf{C}(S)$ and thereby new n-edges. Theorem 1 asserts that each n-edge of $\mathbf{C}(S)$ lies in exactly two n-faces of $\mathbf{C}(S)$; therefore, once the two adjacent n-faces of $\mathbf{C}(S)$ containing an n-edge of $\mathbf{C}(S)$ are found, then that n-edge should be omitted from further consideration. Hence, a repeated application of Theorems 1 and 2 and Corollary 1, as suggested above, should generate all the n-faces of the desired convex polytope $\mathbf{C}(S)$, provided a starting n-edge and a support plane of S containing this n-edge are known.

THEOREM 3. *Let H be a support plane of S, containing r $(r < n)$ linearly independent points of S, whose normal $\hat{\mathbf{n}}$ is of the form $\hat{\mathbf{n}} = (\alpha_1, \alpha_2, \cdots, \alpha_r, 0, \cdots, 0)$. Then there exists at least one point of S which when adjoined to the r points of S on H form a linearly independent subset S^* of S which lies in a support plane H^* of S whose normal is of the form $\hat{\mathbf{n}}^* = (\beta_1, \beta_2, \cdots, \beta_r, \beta_{r+1}, 0, \cdots, 0)$.*

PROOF. Suppose $P^0, P^1, \cdots, P^{r-1}$ are r linearly independent points of S lying on the support plane H whose normal is the unit vector $\hat{\mathbf{n}}$ such that $(\hat{\mathbf{n}}\cdot\hat{\mathbf{v}}_k) \geq 0$. As in the proof of Theorem 1, construct a unit vector $\hat{\mathbf{e}}$ of the form $\hat{\mathbf{e}} = (e_1, e_2, \cdots, e_r, e_{r+1}, 0, \cdots, 0)$ such that

$$(\hat{\mathbf{e}}\cdot\hat{\mathbf{n}}) = 0 \tag{3.1}$$

and $(\hat{\mathbf{e}}\cdot\hat{\mathbf{v}}_i) = 0$, $i = 1, 2, \cdots, (r - 1)$, $\hat{\mathbf{v}}_i$ as usual being the unit vector along $\overrightarrow{p^0p^i}$. Observe (3.1) is a linear system of r equations in $(r + 1)$ variables.

For each point $P^k \in S$, $k > (r - 1)$, compute the ratio

$$\frac{\lambda_k}{\mu_k} = -\frac{(\hat{\mathbf{e}}\cdot\hat{\mathbf{v}}_k)}{(\hat{\mathbf{n}}\cdot\hat{\mathbf{v}}_k)} \tag{3.2}$$

provided $(\hat{\mathbf{n}}\cdot\hat{\mathbf{v}}_k) > 0$.

Let $P^r \in S$ be a point such that

$$\frac{\lambda_r}{\mu_r} = \max_{k>r}\left\{\frac{\lambda_k}{\mu_k}\right\}. \tag{3.3}$$

Consider the hyperplane H^* whose normal $\hat{\mathbf{n}}^*$ is given by

$$\hat{\mathbf{n}}^* = \lambda_r\hat{\mathbf{n}} + \mu_r\hat{\mathbf{e}},$$

with

$$1 = \lambda_r^2 + \mu_r^2 \tag{3.4}$$

and

$$\mu_r = (\hat{\mathbf{n}}^* \cdot \hat{\mathbf{e}}) > 0.$$

We show that H^* is the desired support hyperplane of S containing $r + 1$ linearly independent points of S and its normal is of the form $(\beta_1, \beta_2, \cdots, \beta_{r+1}, 0, \cdots, 0)$.

Clearly, for $k = 0, 1, \cdots, r - 1$

$$(\hat{\mathbf{n}}^* \cdot \hat{\mathbf{v}}_k) = \lambda_r(\hat{\mathbf{n}} \cdot \hat{\mathbf{v}}_k) + \mu_r(\hat{\mathbf{e}} \cdot \hat{\mathbf{v}}_k) = 0, \quad \text{using (3.1)},$$

and

$$(\hat{\mathbf{n}}^* \cdot \hat{\mathbf{v}}_r) = \lambda_r(\hat{\mathbf{n}} \cdot \hat{\mathbf{v}}_r) + \mu_r(\hat{\mathbf{e}} \cdot \hat{\mathbf{v}}_r) = 0, \quad \text{using (3.2)}.$$

Therefore, H passes through $(r + 1)$ points $P^0, P^1, \cdots, P^r$.

For any $P^k \in S$, $k > r - 1$,

$$\begin{aligned}(\hat{\mathbf{n}}^* \cdot \hat{\mathbf{v}}_k) &= ([\lambda_r \hat{\mathbf{n}} + \mu_r \hat{\mathbf{e}}] \cdot \hat{\mathbf{v}}_k) \\ &= \mu_r(\hat{\mathbf{n}} \cdot \hat{\mathbf{v}}_k)[\lambda_r/\mu_r - (\hat{\mathbf{e}} \cdot \hat{\mathbf{v}}_k)/(\hat{\mathbf{n}} \cdot \hat{\mathbf{v}}_k)].\end{aligned}$$

Using (3.2), (3.3), and (3.4), it follows that $(\hat{\mathbf{n}}^* \cdot \hat{\mathbf{v}}_k) > 0$. Thus $(\hat{\mathbf{n}}^* \cdot \hat{\mathbf{v}}_k) \geq 0$, $k = 0, 1, 2, \cdots, m$. Hence H is a support plane of S. Finally, from (3.4) it is evident that $\hat{\mathbf{n}}^*$ has the desired form $(\beta_1, \beta_2, \cdots, \beta_{r+1}, 0, \cdots, 0)$.

Starting Edge

The following procedure may be used for computing a starting n-edge of $\mathbf{C}(S)$.

Determine point(s) of the set S with least first component. Then the hyperplane H_1, defined by $x_1 = \min\{p_1^i\}$, passes through all points of S with least first component and its normal is of the form $\hat{\mathbf{n}}_1 = (1, 0, 0, \cdots, 0)$. Now Theorem 3 can be used repeatedly to generate support planes H_r of S whose normals are of the form $\hat{\mathbf{n}}_r = (\alpha_1, \alpha_2, \cdots, \alpha_r, 0, \cdots, 0)$ until $r = n$. Clearly H_n is an n-face of $\mathbf{C}(S)$ and the n-edges of $\mathbf{C}(S)$ lying on H_n can be found by using Theorem 2.

ALGORITHM

The convex polytope $\mathbf{C}(S)$ of the set S is generated by repeating a cycle of steps; each cycle computes a new face of the desired polytope until all the faces are determined.

Let S_c denote the subset of S whose convex polytope is being generated and let n_c denote the current dimension of the space. Let m_c denote the number of points of S_c.

Step 1. Set $S_c = S$, $n_c = n$, $m_c = m + 1$.

Step 2. Determine point(s) of S_c with least first component. Let S_b be the set consisting of points $P^I \in S_c$ such that $p_1^I = \min_{1 \leq i \leq m_c} \{p_1^i\}$. The hyperplane H, $x_1 = p^I$, is a support hyperplane of S and its normal is parallel to the vector $\hat{\mathbf{n}} = (1, 0, 0, \cdots, 0)$.

Step 3. Construct a unit vector $\hat{\mathbf{e}}$ such that $(\hat{\mathbf{e}} \cdot \hat{\mathbf{n}}) = 0$, $(\hat{\mathbf{e}} \cdot \hat{\mathbf{v}}_i) = 0$, $P_i \in S_b$, $\hat{\mathbf{v}}_i$ being a unit vector along $\overrightarrow{p^I p^i}$ and $(\hat{\mathbf{e}} \cdot \hat{\mathbf{v}}_k) \geq 0$, $P_k \in S_c$.

Step 4. For each point $P^k \in S_c$, compute the ratio

$$\frac{\lambda_k}{\mu_k} = -\frac{(\hat{\mathbf{e}} \cdot \hat{\mathbf{v}}_k)}{(\hat{\mathbf{n}} \cdot \hat{\mathbf{v}}_k)}$$

and determine point(s) $P^J \in S_c$ such that

$$\frac{\lambda_J}{\mu_J} = \max \left\{\frac{\lambda_k}{\mu_k}\right\}$$

where the maximum is taken over all k such that $P^k \in S_c$.

The normal to the d-flat defined by adjoining to S_b the points for which the ratio of λ and μ is maximum, is given by $\hat{\mathbf{n}}^* = \lambda_J\hat{\mathbf{n}} + \mu_J\hat{\mathbf{e}}$, where $1 = \lambda_J^2 + \mu_J^2$.

Step 5. If $d < n_c$ the starting face of $\mathbf{C}(S_c)$ has not been computed yet; therefore, replace S_c by the points on d-flat and return to step 3 with $\hat{\mathbf{n}} = \hat{\mathbf{n}}^*$. If $d \geq n_c$ an n_c-face of $\mathbf{C}(S_c)$ has been computed. In the case $d > n_c$ let S_c denote the points on the d-flat and return to step 2 with $n_c = n_c - 1$. When $d = n_c$ proceed to step 6.

Step 6. Check whether the n_c edges of the computed face are in storage. An edge in store implies that one face containing this edge was found before and now that the second face has been computed this edge will be omitted from further consideration. If some edges are not already in storage, store all except one. Return to step 3 with S_c consisting of points defining this edge and with $\hat{\mathbf{n}} = \hat{\mathbf{n}}^*$. However, if all edges are in storage, proceed to step 7.

Step 7. Search the storage for an edge. Pick an edge and compute the normal $\hat{\mathbf{n}}$ to the face containing this edge. Return to step 3 with S_c consisting of points on this edge. If no edge exists in the storage then the n_c-polytope has been computed; proceed to step 8.

Step 8. Check whether $n_c = n$. If yes, the desired convex polytope $\mathbf{C}(S)$ has been generated. If $n_c < n$ return to step 6 with the faces of the n_c-polytope being the edges of the $n_c = (n_c + 1)$-polytope.

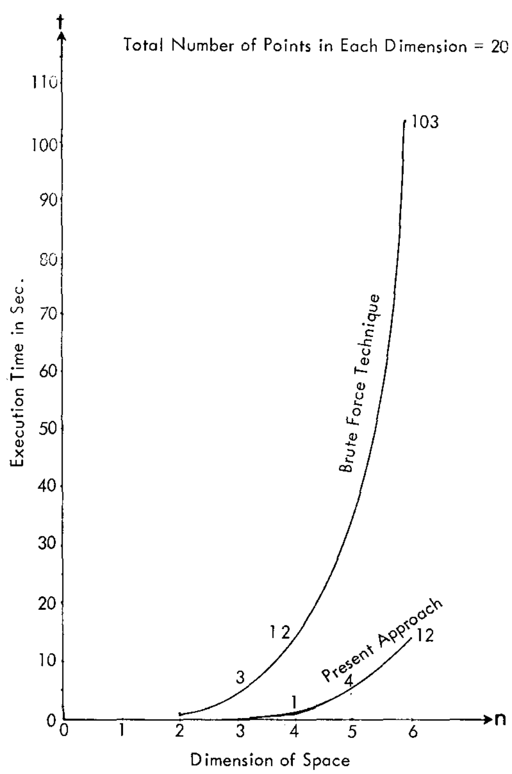

FIG. 3

Numerical Results

A FORTRAN program was developed and is being used as a production program at Lockheed-Georgia Company. Figure 3 shows that the results obtained are significantly superior to the brute force technique mentioned in the Introduction.

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