domain (evidently bordering the region) appears to keep individual experimental observation unlikely.

The foregoing treatment remains independent of fundamental constants by replacing Hall's quantum mechanical parameter $\nu=(\hbar/2m)\ln(b/a)$, in terms of Planck's constant $\hbar$, the mass m of the helium atom, and the inner and outer radii of quantized vortices a and b, respectively. Implications are not intended, however, that rotating helium remains devoid of macroscopic quantum effects. In imagining such features, the domain structure indicating ability to preserve the irrotational state[10] (quantum index, $n=0$) over prescribed regions may in fact, constitute a form of macroscopic uncertainty principle. Such considerations will be discussed more fully in a later detailed treatment.

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[7]Such "magnus force" $2\rho_S(\vec{\Omega}\times\vec{v})$ exerted per unit volume against vortex lines constituted the product of total circulation $(2\vec{\Omega})$ residing per unit area in vortex lines times the relative momentum density $(\rho_S\vec{v})$.

[8]Procedures employed by Hall in reference 6 for evaluating curvature in terms of the circulation vector $\nabla\times\vec{v}_S$ retain validity here. Application of the $\vec{k}\cdot\nabla$ operation to $\nabla\times\vec{v}_S$ results in the same differential functional dependence of domain axis velocity $\vec{v}$ on system coordinate position as appear in Eqs. (5)-(7) of that reference for vortices.

[9]Space forbids equation-by-equation detail, and the reader is therefore referred to the following equations of the Hall 1958 treatise, reference (6): (a) Eq. (1), (b) Eqs. (5)-(8), (c) Eqs. (9)-(10), and (d) Eqs. (16)-(17).

[10]In this connection, note the interesting observations that magnitude of the angular momentum and rotational energy content for counter-rotating domains become, respectively,

$$\text{angular momentum}=\tfrac{1}{2}\pi\rho_s a^4\Omega/\rho_s\pi a^2 =3.4\times10^{-3}\ \text{cm}^2/\text{sec}$$

per gram of superfluid, and

$$\text{rotational energy}=\tfrac{1}{4}\pi\rho_s a^4\Omega =3.5\times10^{-5}\rho_s\,\text{g-cm/sec}^2$$

per unit length of domain, each evidently independent of angular rotation rate Ω.

PHOTON CORRELATIONS*

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In 1956 Hanbury Brown and Twiss[1] reported that the photons of a light beam of narrow spectral width have a tendency to arrive in correlated pairs. We have developed general quantum mechanical methods for the investigation of such correlation effects and shall present here results for the distribution of the number of photons counted in an incoherent beam. The fact that photon correlations are enhanced by narrowing the spectral bandwidth has led to a prediction[2] of large-scale correlations to be observed in the beam of an optical maser. We shall indicate that this prediction is misleading and follows from an inappropriate model of the maser beam. In considering these problems we shall outline a method of describing the photon field which appears particularly well suited to the discussion of experiments performed with light beams, whether coherent or incoherent.

The correlations observed in the photoionization processes induced by a light beam were given a simple semiclassical explanation by Purcell,[3] who made use of the methods of microwave noise theory. More recently, a number of papers have been written examining the correlations in considerably greater detail. These papers[2,4-6] retain the assumption that the electric field in a light beam can be described as a classical Gaussian stochastic process. In actuality, the behavior of the photon field is considerably more varied than such an assumption would indicate. Whereas a stationary Gaussian stochastic process is described completely by its frequency-dependent power spectrum, a great deal more information in the form of amplitude and phase relations between differing quantum states may be required to describe a steady light beam. Beams of identical spectral distributions may exhibit altogether different photon correlations or, alternatively, none at all. There is ultimately no substitute for the quantum theory in describing quanta.

We assume, for convenience, that the field has discrete propagation modes labeled by an index k (which in free space, for example, specifies propagation vector and polarization). To describe the quantum state of the kth mode we must specify an infinite set of complex amplitudes, one for each quantum occupation state $|n_k\rangle$, $n_k = 0, 1, 2, \cdots$. Since the states we wish to describe include ones in which the phase of the kth mode is fairly well defined, and a large number of states $|n_k\rangle$ must then be superposed, it is preferable to use an altogether different set of basis states. We take these to be of the form

$$|\alpha_k\rangle = \exp(-\tfrac{1}{2}|\alpha_k|^2)\sum_n[\alpha_k{}^n/(n!)^{1/2}]|n\rangle, \qquad (1)$$

where α_k is an arbitrary complex amplitude. We shall call the $|\alpha_k\rangle$ coherent states; their use is well known in discussions of the harmonic oscillator in the classical limit. The expectation value in the state $|\alpha_k\rangle$ of the contribution of the kth mode to the total field is a monochromatic wave with complex amplitude proportional to α_k. The coherent states $|\alpha_k\rangle$, for all complex α_k, form a complete set in a sense best expressed by the relation

$$(1/\pi)\int|\alpha_k\rangle\langle\alpha_k|d^{(2)}\alpha_k = 1, \qquad (2)$$

where $d^{(2)}\alpha_k$ is a real element of area of the complex α_k plane. It follows that any state may be expanded linearly in terms of coherent states. The most general light beam can thus be described by a density operator of the form

$$\rho = \int \mathcal{P}(\{\alpha_k, \alpha_k'\})\prod_k|\alpha_k\rangle\langle\alpha_k'|d^{(2)}\alpha_k d^{(2)}\alpha_k', \qquad (3)$$

which deals with all the modes of the field at once.

An incoherent light beam must be described as a statistical mixture of all the excitation states available for each mode excited. For the kth mode, the probability to be associated with the state $|n_k\rangle$ is proportional to $\{\langle N_k\rangle/(1+\langle N_k\rangle)\}^{n_k}$, where $\langle N_k\rangle$ is the mean number of photons occupying the mode. A simple theorem expresses this mixture in terms of the coherent states defined earlier: The density operator (3) reduces to a product of operators of the form

$$\rho_k = \int p(\alpha_k)|\alpha_k\rangle\langle\alpha_k|d^{(2)}\alpha_k, \qquad (4)$$

where the probability $p(\alpha_k)$ is a Gaussian function,

$$p(\alpha_k) = \{\pi\langle N_k\rangle\}^{-1}\exp(-|\alpha_k|^2/\langle N_k\rangle). \qquad (5)$$

In particular, blackbody radiation may be described as a mixture of coherent waves by substituting for $\langle N_k\rangle$ the familiar value for thermal excitation of a field oscillator.

To discuss photon correlations we examine the photoionization probability of a pair of atoms, labeled 1 and 2, which lie at $\vec{r}_1$ and $\vec{r}_2$ within the light beam. We assume that the incident beam is of narrow enough spectral bandwidth that any variation of frequency-dependent parameters entering the photoionization probabilities may be neglected. Then, if we sum the transition probabilities over final electron energies, there is no difficulty in defining a time at which each electron emission takes place. The probability density for ionization of atom 1 at time t_1 and for atom 2 at t_2 may be written as

$$w(t_1t_2) = w_1w_2C(\vec{r}_1t_1\vec{r}_2t_2), \qquad (6)$$

where w_1 and w_2 are the constant transition probabilities for each atom placed individually in the beam, and C is the function whose departure from unity expresses a tendency for the two events to be correlated.

We assume, to simplify notation, that photons of only one polarization are present. The appropriate vector component of the electric field operator has a positive-frequency part $\mathcal{E}^{(+)}(\vec{r}, t)$ and a negative-frequency part $\mathcal{E}^{(-)}(\vec{r}, t)$. The correlation function may be expressed in terms of these operators as

$$C(\vec{r}_1t_1\vec{r}_2t_2) = \frac{\mathrm{tr}\{\rho\, \mathcal{E}^{(-)}(\vec{r}_1t_1)\, \mathcal{E}^{(-)}(\vec{r}_2t_2)\, \mathcal{E}^{(+)}(\vec{r}_1t_1)\, \mathcal{E}^{(+)}(\vec{r}_2t_2)\}}{\mathrm{tr}\{\rho\, \mathcal{E}^{(-)}(\vec{r}_1t_1)\, \mathcal{E}^{(+)}(\vec{r}_1t_1)\}\mathrm{tr}\{\rho\, \mathcal{E}^{(-)}(\vec{r}_2t_2)\, \mathcal{E}^{(+)}(\vec{r}_2t_2)\}}, \qquad (7)$$

where tr stands for trace, and ρ is a density operator of the general form (3).

It is easily shown that coherent states of the field lead to no photoionization correlations at all. If the state of the field is specified by any density operator of the form $\prod_k |\alpha_k\rangle\langle\alpha_k|$, the correlation function C reduces to unity. A correlation between photons only appears when incoherent mixtures or superpositions of the coherent states are present. For collimated, completely incoherent beams of the type described earlier (e.g., filtered thermal radiation), we find

$$C = 1 + |R(t_1 - t_2 - c^{-1}(x_1 - x_2))|^2, \tag{8}$$

where the coordinates x_j are the components of $\vec{r}_j$ in the propagation direction, and the function R is given by

$$R(t) = \sum_k \langle N_k \rangle \exp(-i\omega_k t) / \sum_k \langle N_k \rangle. \tag{9}$$

This result, for incoherent beams, corresponds in the classical limit, when the modes are treated as forming a continuum, to that derived using stochastic models.[2-6] It may be associated, in this limit, with a tendency of the complex total field strength of the beam to fluctuate in modulus with time.

The density operator which represents an actual maser beam is not yet known. It is clear that such a beam cannot be represented by a product of individual coherent states, $\prod_k |\alpha_k\rangle\langle\alpha_k|$, unless the phase and amplitude stability of the device is perfect. On the other hand, a maser beam is not at all likely to be described by the ideally incoherent classical model which underlies the calculation of Mandel and Wolf,[2] and leads them to results corresponding to Eqs. (8) and (9). More plausible models for a steady maser beam are much closer in behavior to the ideal coherent states. They may be shown to lead to photon correlations only to the extent that random amplitude modulation is present in the statistically averaged beam.

If photoionization processes tend to be correlated in time, the distribution of the number of photons recorded by a counter in a fixed interval of time should differ from the Poisson distribution. We have developed a general technique for finding this distribution for incoherent light beams.[7] The result for the important case in which the spectral distribution has the Lorentz line shape, $\langle N_k\rangle \sim \{(\omega_k - \omega)^2 + \gamma^2\}^{-1}$, with central frequency ω, may be stated as follows: Let W be the average rate at which photons are recorded; then the probability that n photons are recorded in a time t, long compared to $1/\gamma$, is

$$P(n,t) = \frac{1}{n!} \frac{(\gamma W t)^n}{(\gamma^2 + 2\gamma W)^{\frac{1}{2}n}} S_n\left[t(\gamma^2 + 2\gamma W)^{1/2}\right] \times \exp\{-t[(\gamma^2 + 2\gamma W)^{1/2} - \gamma]\}. \tag{10}$$

The functions $S_n(x)$ are nth order polynomials in x^{-1}, familiar in the theory of modified Bessel functions of half-integral order. They may be found from the relations $S_0 = S_1 = 1$ and $S_{n+1} = -S_n' + (1 + nx^{-1})S_n$. The full set of moments of the distribution is given by the averages of products of the form $n!/(n-j)!$. These are

$$\langle n!/(n-j)!\rangle_{\text{av}} = (Wt)^j S_j(\gamma t). \tag{11}$$

When the number of photons recorded during a relaxation time is small, $W \ll \gamma$, e.g., when the linewidth becomes appreciable, the distribution (10) approaches the Poisson distribution as a limit.

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