

# Testing Quantum Mechanics

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This paper presents a general framework for introducing nonlinear corrections into ordinary quantum mechanics, that can serve as a guide to experiments that would be sensitive to such corrections. In the class of generalized theories described here, the equations that determine the time-dependence of the wave function are no longer linear, but are of Hamiltonian type. Also, wave functions that differ by a constant factor represent the same physical state and satisfy the same time-dependence equations. As a result, there is no difficulty in combining separated subsystems. Prescriptions are given for determining the states in which observables have definite values and for calculating the expectation values of observables for general states, but the calculation of probabilities requires detailed analysis of the method of measurement. A study is presented of various experimental possibilities, including the precession of spinning particles in external fields, experiments of Stern-Gerlach type, and the broadening and de-tuning of absorption lines. © 1989 Academic Press, Inc.

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## 1. INTRODUCTION AND SUMMARY

Considering the pervasive importance of quantum mechanics in modern physics, it is odd how rarely one hears of efforts to test quantum mechanics experimentally with high precision. It is true that over the last decade there have been a number of experimental tests [1] of predictions that distinguish quantum mechanics from

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local hidden variable theories, and these experiments have conclusively ruled out the hidden variable theories, but they do not provide tests of quantum mechanics itself to a precision better than about 1%.

In principle, the test of any specific quantum mechanical theory like quantum electrodynamics is also a test of quantum mechanics itself, but it ought to be possible to test quantum mechanics more stringently than any individual quantum theory. Or, to put this more open-mindedly, perhaps we can formulate experiments that would show up departures from quantum mechanics itself at a much tinier level than departures from any specific theory.

The problem is to know what to test. Usually we can get guidance on how to test a theory like general relativity or the standard model of elementary particle interactions, by first inventing some generalized theory as a foil, such as general relativity with extra massless scalars, or the standard model with extra gauge bosons. By a “generalized” theory, I mean one that differs from the theory we want to test, but reduces to it when some parameters become very small. (In this sense, the local hidden variable theories do not qualify as generalizations of quantum mechanics.) We can set upper bounds on these parameters by doing experiments to look for new effects that could arise in the generalized theory, and in this way we get a sense of how accurate is the theory we want to test.

The trouble is that it is very difficult to find any logically consistent generalization of quantum mechanics. One obvious target for generalization is the linearity of quantum mechanics, but if we arbitrarily add nonlinear terms to the Schrödinger equation, how do we know that the theory we obtain will have a sensible physical interpretation? At least in part, it is the dearth of generalized versions of quantum mechanics that has made it so hard to plan experimental tests of quantum mechanics.

This paper will outline a possible nonlinear generalization of quantum mechanics, that can be used as a guide in subjecting quantum mechanics to experimental tests. (A brief description has already been given in Ref. [2].) It is not our purpose here to find the most general such theory, but only to find *some* generalization of quantum mechanics, introducing only such new features as seem inevitable once we give up the principle of linear superposition.

Section 2 describes our general formalism, without yet specifying a physical interpretation. As in quantum mechanics, physical states are represented by rays in a complex vector space and observables by the generating functions for transformations of these rays, but these transformations are not in general linear. (This section also comments briefly on the differences between the work of this paper and other work on possible non-linear corrections to quantum mechanics.) The first step toward a physical interpretation is taken in Section 3, where we describe the “eigenvalue” condition satisfied when a state has a definite value for some observable. Eigenstates of the energy have a simple harmonic time dependence, as in ordinary quantum mechanics, but more general states have a richer of possible variation with time, including possible chaotic behavior. In Sections 2 and 3, the rules of the theory are illustrated by reference to a simple case, the general two-component

theory. In Section 4 we turn to a more realistic class of examples, in which a single particle with spin is subjected to various external fields. One of the striking consequences of the inclusion of non-linearities is that some of the theorems derived from symmetry principles in ordinary quantum mechanics no longer apply; for instance, the energy levels of a particle of spin  $j \geq 1$  in an arbitrarily weak magnetic field are no longer equally spaced, and a spin  $\frac{1}{2}$  particle can have a linear interaction with a weak quadrupole electric field. Also, spin precession frequencies can be very accurately measured, so it is noteworthy that non-linearities give these frequencies an anomalous dependence on the mixture of states in the initial wave function. We return in Section 5 to the problem of interpreting this formalism and make a general proposal regarding expectation values and some tentative remarks about probabilities. In Section 6 we take up the question of calculating the resonant frequencies and line shapes for the absorption of radiation. In the simplest case, a single photon striking an atom or nucleus in an energy eigenstate, the resonant absorption frequencies are equal to differences of energy eigenvalues, just as in ordinary quantum mechanics. However, this is not generally the case here. In a classical electromagnetic field, containing unlimited numbers of photons, resonant absorption frequencies turn out to be indirectly related to energy differences, and there is an anomalous line broadening, arising either from mixtures of states in the initial wave function, or from changes in the wave function as it evolves. The latter effect leads to a characteristic "de-tuning," which would prevent resonant transitions in sufficiently weak external fields. The observation of such resonant transitions between hyperfine levels of the  $\text{Be}^{9+}$  ion [3] has already been used [4] to set stringent upper bounds on the contribution of nonlinearities to the internal energy of the  $\text{Be}^9$  nucleus. Finally, an Appendix describes a useful transformation of the components of the wave function, which allows us to put the time-dependent Schrödinger equation in a convenient form when it is integrable, and allows us to apply a version of the KAM theorem to this equation when it is not integrable.

## 2. FORMALISM

### (a) *Wave Functions*

As in ordinary quantum mechanics, the state of a system is assumed to be represented by a complex-valued wave function  $\psi$ . For illustrative purposes we will mostly take  $\psi$  here to be a function  $\psi_k$  of a discrete variable  $k$  that takes values  $1, 2, \dots, N$ , but later we shall consider also cases where the wave function is a function  $\Psi_k(\mathbf{x})$  also of a continuous position variable  $\mathbf{x}$ . A special feature of quantum mechanics that will be required here as well is that for an arbitrary complex number  $Z$ , the wave functions  $\psi_k$  and  $Z\psi_k$  represent the same physical state.

### (b) *Observables*

In ordinary quantum mechanics, observables are represented by Hermitian matrices  $A_{kl}$ . Equivalently, one could describe observables in quantum mechanics

in terms of real bilinear functions  $\psi_k^* A_{kl} \psi_l$ . We generalize this here, representing observables by real *non-bilinear* functions  $a(\psi, \psi^*)$ . (These functions will be interpreted in Section 5 in terms of expectation values.) However, in order to retain the feature of quantum mechanics that for an arbitrary non-zero complex number  $Z$ , the wave functions  $\psi$  and  $Z\psi$  represent the same physical state, we require that the functions representing observables are (like  $\psi_k^* A_{kl} \psi_l$ ) homogeneous of degree one in both  $\psi^*$  and  $\psi$

$$\psi_k \frac{\partial a}{\partial \psi_k} = \psi_k^* \frac{\partial a}{\partial \psi_k^*} = a. \tag{2.1}$$

Such functions (without a reality condition) form an algebra. The sum is defined in the obvious way

$$(a + b)(\psi, \psi^*) \equiv a(\psi, \psi^*) + b(\psi, \psi^*), \tag{2.2}$$

as is multiplication with a complex scalar  $\xi$ ,

$$(\xi a)(\psi, \psi^*) \equiv \xi a(\psi, \psi^*). \tag{2.3}$$

Multiplication, on the other hand, is defined by

$$a * b \equiv \frac{\partial a}{\partial \psi_k} \frac{\partial b}{\partial \psi_k^*}. \tag{2.4}$$

(This is an obvious generalization of the matrix multiplication in ordinary quantum mechanics: if  $a = \psi_n^* A_{nm} \psi_m$  and  $b = \psi_n^* B_{nm} \psi_m$  then  $a * b = \psi_k^* A_{kl} B_{lm} \psi_m$ .) One bilinear function will be important here as in quantum mechanics; it is the norm

$$n = \psi_k^* \psi_k \tag{2.5}$$

which acts as a unit element for the product (2.4):

$$n * a = a * n = a. \tag{2.6}$$

The product (2.4) is distributive, but neither commutative nor associative. From the failure of associativity will follow most of the differences between this formalism and ordinary quantum mechanics.

(c) *Symmetries*

One important class of functions are those that generate symmetry transformations. To generalize the linear transformations  $\delta\psi_k = -i\varepsilon A_{kl} \psi_l$  of quantum mechanics, we suppose that the change in the wave function associated with an infinitesimal function  $\varepsilon a(\psi, \psi^*)$  is

$$\varepsilon \delta_a \psi_k \equiv -i\varepsilon \frac{\partial a}{\partial \psi_k^*}. \tag{2.7}$$

Then the change in some other function  $b$  induced by  $\varepsilon a$  is

$$\begin{aligned}\varepsilon\delta_a b &= \frac{\partial b}{\partial\psi_k} \varepsilon\delta_a\psi_k + \frac{\partial b}{\partial\psi_k^*} \varepsilon\delta_a\psi_k^* \\ &= -i\varepsilon \left[ \frac{\partial b}{\partial\psi_k} \frac{\partial a}{\partial\psi_k^*} - \frac{\partial b}{\partial\psi_k^*} \frac{\partial a}{\partial\psi_k} \right]\end{aligned}$$

or, compactly,

$$\delta_a b = i[a, b] \equiv i(a * b - b * a). \quad (2.8)$$

(Equations (2.6) and (2.8) imply that the norm  $n$  is invariant under all symmetries.) This “commutator” is obviously antisymmetric:

$$[a, b] = -[b, a]. \quad (2.9)$$

A little less obviously, even though the product (2.4) is non-associative, the commutator (2.8) satisfies the Jacobi identity

$$[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0. \quad (2.10)$$

Thus, as in quantum mechanics, we can introduce Lie algebras of symmetry transformations (2.7), in which we require that

$$[\lambda_i, \lambda_j] = iC_{ijh}\lambda_h \quad (2.11)$$

with  $C_{ijh}$  the structure constants of some ordinary Lie algebra.

#### (d) Time Dependence

One symmetry of special importance is that of time translation, generated by a real “Hamiltonian” function  $h(\psi, \psi^*)$ . We define the time dependence of the wave function so that

$$\psi_k(t + \varepsilon) = \psi_k(t) + \varepsilon\delta_h\psi_k(t)$$

or, in other words,

$$\frac{d\psi_k}{dt} = -i \frac{\partial h}{\partial\psi_k^*}. \quad (2.12)$$

This is our time-dependent nonlinear Schrödinger equation.

Equation (2.12) and its complex conjugate show that this is a classical Hamiltonian system, with real  $q$ 's and  $p$ 's given by

$$\psi_k = (q_k + ip_k)/\sqrt{2}. \quad (2.13)$$

Indeed, the “commutator” (2.8) is just proportional to the classical Poisson

bracket, so it is not surprising that it satisfies the Jacobi identity (2.10). All this, of course, trivially also true in ordinary quantum mechanics, where  $h$  is the bilinear  $\psi_k^* H_{kl} \psi_l$ . We shall call  $h$  the ‘‘Hamiltonian function,’’ to distinguish it from the quantum-mechanical Hamiltonian, which is  $H_{kl}$ , rather than  $\psi_k^* H_{kl} \psi_l$ .

Any function  $a$  of a wave function  $\psi(t)$  (and its complex conjugate) that satisfies (2.12) has the time dependence

$$\frac{da}{dt} = -i[a, h] \tag{2.14}$$

just as in quantum mechanics. We immediately note two conserved functions:  $h$  itself and the unit element  $n$ , which according to (2.6) commutes with everything. However, because of the failure of associativity, a product  $a * b$  of conserved functions is not necessarily itself conserved.

This leads to an important difference between ordinary quantum mechanics and the nonlinear formalism described here. In ordinary  $N$ -component quantum mechanics, the time-dependence of the wave function is at worst quasi-periodic; it is a sum of  $N$  simple harmonic complex terms, in general with incommensurate frequencies. A large class of non-linear Hamiltonian systems display quasi-periodicity of a more general sort; the  $2N$  quantities  $\text{Re } \psi_k$  and  $\text{Im } \psi_k$  move on an  $N$ -torus, with

$$\psi_k(t) = \sum_{n_1 \dots n_N} c_k(n_1 \dots n_N) \exp\left(-i \sum_v n_v \omega_v t\right), \tag{2.15}$$

the difference being that in the non-linear case the sum runs over all positive and negative integers and the frequencies  $\omega_v$ , as well as the constants  $c_k$  depend on the initial conditions. This behavior is guaranteed for Hamiltonian systems that are *integrable*; that is, for which there are  $N$  functionally independent quantities  $b_v(\psi, \psi^*)$  that commute (in the sense of Eq. (2.8)) with each other and with  $h(\psi, \psi^*)$ . In ordinary quantum mechanics, this is always the case; with  $h = \psi_k^* H_{kl} \psi_l$ , we can take the  $N$  conserved quantities as  $b_v = \psi_k^* (H^v)_{kl} \psi_l$ , with  $v = 0, 1, \dots, N - 1$ , or equivalently as the individual  $|\psi_k|^2$  in a basis in which  $H_{kl}$  is diagonal. (This is one way of understanding why the wave function always has quasi-periodic time-dependence in ordinary quantum mechanics.) As already mentioned, in the generalized theory described here there are always at least two conserved commuting quantities  $b_v$ , namely, the norm  $n$  and the Hamiltonian function itself, so such theories are integrable for  $N \leq 2$ . However, the ‘‘powers’’  $h * h, h * (h * h)$ , etc. do not necessarily commute with  $h$ , so these systems are not in general integrable with  $N > 2$  components.

This opens up the possibility that the time-dependent wave function may not move quasi-periodically over an  $N$ -torus, as in ordinary quantum mechanics, but may instead move chaotically through the whole  $2N$ -dimensional space of the variables  $\text{Re } \psi_k$  and  $\text{Im } \psi_k$ . Actually, such chaos is rare in Hamiltonian systems

that (like those considered here) are close to integrable systems. Suppose that the Hamiltonian takes the form

$$h = h_0 + h_1,$$

where  $h_0$  is integrable and  $h_1$  is in some sense small. Because  $h_0$  is integrable, we can define a set of canonical action and angle variables,  $J_k$  and  $\Theta_k$ , with rates of change,

$$\begin{aligned}\frac{dJ_k}{dt} &= -\frac{\partial h(J, \Theta)}{\partial \Theta_k}, \\ \frac{d\Theta_k}{dt} &= \omega_k(J) + \frac{\partial h_1(J, \Theta)}{\partial J_k},\end{aligned}$$

where

$$\omega_k(J) \equiv \frac{\partial h_0(J)}{\partial J_k}.$$

According to the “averaging principle” [5], since  $\Theta_k$  changes much more rapidly than  $J_k$ , for sufficiently incommensurate frequencies  $\omega_k$ , the right-hand sides may be replaced with their averages over  $\Theta$ :

$$\begin{aligned}\frac{dJ_k}{dt} &= -\left\langle \frac{\partial h_1}{\partial \Theta_k} \right\rangle = 0 \\ \frac{d\Theta_k}{dt} &= \omega_k(J) + \frac{\partial}{\partial J_k} \langle h_1(J) \rangle.\end{aligned}$$

These averaged equations are just the same as if the Hamiltonian function had the integrable form

$$h_{\text{EFF}} = h_0(J) + \langle h_1(J) \rangle.$$

Hence as long as the departures from quantum mechanics are small compared with the incommensurate frequency differences in the quantum limit, the time-dependent Schrödinger equation may be expected to behave as if it were integrable.

This argument is not rigorous, and not always reliable. However, there are rigorous theorems that in some cases lead to a similar conclusion. A profound result, the KAM theorem [6], tells us that the motion of a Hamiltonian system in phase space is quasi-periodic except for  $q$ 's and  $p$ 's close to certain resonant tori. For  $N > 1$ , the “Arnol'd diffusion” produced by small perturbations will eventually destroy quasi-periodicity. However, a theorem of Nekhoroshev [7] states that for all initial conditions, the action variables remain close to their initial values for an exponentially long time. Unfortunately, in proving both theorems it is assumed not only that the departures from integrability are small, but also that the frequencies

$\omega_k(J)$  are sufficiently “steep” functions of the  $J$ 's. This condition is *not* satisfied if we take  $h_0(J)$  to be the Hamiltonian function of ordinary quantum mechanics, for which the  $\omega_k(J)$  are just constants. In the Appendix we show that a version of the KAM theorem applies in the theories of this paper at least in a neighborhood of each of the solutions of (2.12) with simple harmonic time-dependence. The numerical calculations described in Section 4 suggest that quasi-periodic behavior tends to persist even in regions of phase space where no version of the KAM theorem applies.

(e) *Galilean Invariance: One Particle Realizations*

Time-translation invariance can be easily embedded in a larger group of symmetries, that of Galilean invariance. For this, we need functions  $h$ ,  $\mathbf{p}$ ,  $\mathbf{j}$ , and  $\mathbf{k}$  that respectively generate time translations, space translations, rotations, and boosts, and satisfy the commutation relations

$$\begin{aligned} [\mathbf{p}, h] &= [\mathbf{j}, h] = [k_i, k_j] = 0 \\ [\mathbf{k}, h] &= i\mathbf{p} \quad [k_i, p_j] = i\delta_{ij}\mathcal{M} \\ [j_i, p_j] &= i\epsilon_{ijk}p_k \quad [j_i, k_j] = i\epsilon_{ijk}k_k \\ [j_i, j_j] &= i\epsilon_{ijk}j_k, \end{aligned} \tag{2.16}$$

where  $\mathcal{M}$  commutes with everything. (All “commutators” are defined as in Eq. (2.8).) It is not easy (though perhaps not impossible) to find realizations of the subalgebra spanned by  $\mathbf{j}$ ,  $\mathbf{k}$ ,  $\mathbf{p}$ , and  $\mathcal{M}$  other than the conventional bilinear ones (or equivalent realizations obtained by nonlinear homogeneous canonical transformations) so it is natural to adopt these conventional realizations here. For a single free particle of mass  $m$ , described by wave function  $\Psi_k(\mathbf{x})$ , these are

$$\mathbf{p} = -i \int \Psi_k^*(\mathbf{x}) \nabla \Psi_k(\mathbf{x}) d^3x \tag{2.17}$$

$$\mathbf{k} = m \int \Psi_k^*(\mathbf{x}) \mathbf{x} \Psi_k(\mathbf{x}) d^3x \tag{2.18}$$

$$\mathbf{j} = \int \Psi_k^*(\mathbf{x}) [-i(\mathbf{x} \times \nabla) \delta_{kl} + \mathbf{S}_{kl}] \Psi_l(\mathbf{x}) \tag{2.19}$$

$$\mathcal{M} = m \int d^3x \Psi_k^*(\mathbf{x}) \Psi_k(\mathbf{x}) \equiv mn, \tag{2.20}$$

where  $\mathbf{S}$  are Hermitian spin matrices satisfying the algebra  $\mathbf{S} \times \mathbf{S} = i\mathbf{S}$ . But since  $h$  nowhere appears on the right-hand side of any commutation relation, there is no

obstacle to constructing a wide variety of non-bilinear Hamiltonian functions, by adding a suitable internal energy term  $h'$  to the conventional bilinear kinetic energy:

$$h = -\frac{1}{2m} \int \Psi_k^*(\mathbf{x}) \nabla^2 \Psi_k(\mathbf{x}) d^3x + h'. \quad (2.21)$$

Any  $h'$  that commutes with  $\mathbf{j}$ ,  $\mathbf{p}$ , and  $\mathbf{k}$  provides a realization of the Galilean algebra.

The simplest possibility is that the internal energy  $h'$  is the integral over  $\mathbf{x}$  of a homogeneous scalar function of  $\Psi_k(\mathbf{x})$  and  $\Psi_k^*(\mathbf{x})$

$$h' = \int d^3x \mathcal{H}(\Psi(\mathbf{x}), \Psi^*(\mathbf{x})). \quad (2.22)$$

This will yield a satisfactory realization of the Galilean algebra provided that the function  $\mathcal{H}$  satisfies the rotational invariance and homogeneity conditions

$$0 = \frac{\partial \mathcal{H}}{\partial \Psi_k} \mathbf{S}_{kl} \Psi_l - \frac{\partial \mathcal{H}}{\partial \Psi_k^*} \mathbf{S}_{kl}^* \Psi_l^* \quad (2.23)$$

$$\frac{\partial \mathcal{H}}{\partial \Psi_k} \Psi_k = \frac{\partial \mathcal{H}}{\partial \Psi_k^*} \Psi_k^* = \mathcal{H}. \quad (2.24)$$

With total Hamiltonian function given by (2.21) and (2.22), the time-dependent free-particle Schrödinger equation is

$$i \frac{\partial \Psi_k}{\partial t} = \frac{\delta h}{\delta \Psi_k^*} = -\frac{1}{2m} \nabla^2 \Psi_k + \frac{\partial \mathcal{H}(\Psi, \Psi^*)}{\partial \Psi_k^*}. \quad (2.25)$$

For instance,  $\mathcal{H}$  could be taken in the form

$$\mathcal{H} = \sum_s \rho^{-s} \mathcal{H}_s. \quad (2.26)$$

Here the sum runs over integer  $s = 0, 1, 2, \dots$ ;  $\rho$  is the density

$$\rho(\Psi, \Psi^*) = \Psi_k^* \Psi_k \quad (2.27)$$

and  $\mathcal{H}_s$  is a sum of terms proportional to  $1 + s$  factors of  $\Psi$ 's and  $\Psi^*$ 's, e.g.,

$$\mathcal{H}_0(\Psi, \Psi^*) = H_{kl} \Psi_k^* \Psi_l, \quad (2.28)$$

$$\mathcal{H}_1(\Psi, \Psi^*) = G_{klmn} \Psi_k^* \Psi_l^* \Psi_m \Psi_n, \quad \text{etc.} \quad (2.29)$$

with constant coefficients  $H_{kl}$ ,  $G_{klmn}$ , etc., chosen to make each  $\mathcal{H}_s$  a scalar in the sense of (2.23). If the spin matrices  $\mathbf{S}_{kl}$  furnish an irreducible representation of the rotation algebra, then by Schur's lemma, the  $H_{kl}$  must be proportional to  $\delta_{kl}$ , giving a contribution to  $h'$  simply proportional to the norm  $n$ . Thus the bilinear

( $s = 0$ ) terms in  $h'$  have no effect other than to set the zero of energy for each spin. The internal energy function  $h'$  acquires a dynamical significance only when we include non-bilinear terms, with  $s \geq 1$ . (This is why it plays no role in ordinary non-relativistic quantum mechanics.) The particular form (2.26) for these non-bilinear terms has the advantage, that although the density  $\rho$  in the denominators can vanish at particular spacetime points, it can do so only where all components  $\Psi_k$  vanish, and at such points the numerators in (2.26) vanish even more strongly than  $\rho^s$ . Even the variational derivative of  $h'$  with respect to  $\Psi_k$  or  $\Psi_k^*$  vanishes where  $\rho = 0$ , so there are no infinities in the Schrödinger equation (2.25) at such points. It is only the second and higher variational derivatives of  $h'$  that become ill-defined where  $\rho$  vanishes.

Because of the homogeneity (2.24) of the Hamiltonian density  $\mathcal{H}$ , the time-dependent Schrödinger equation (2.25) allows separable solutions

$$\Psi_k(\mathbf{x}, t) = \varphi(\mathbf{x}, t) \psi_k(t) \tag{2.30}$$

with

$$i \frac{d\varphi}{dt} = -\frac{1}{2m} \nabla^2 \varphi \tag{2.31}$$

$$i \frac{d\psi_k}{dt} = \frac{\partial \mathcal{H}(\psi, \psi^*)}{\partial \psi_k^*}. \tag{2.32}$$

For the most part in this paper we shall simply ignore the translational degree of freedom, and write  $h$  instead of  $\mathcal{H}$  in the time-dependent Schrödinger equation (2.32), as if  $\mathcal{H}$  were the total Hamiltonian function.

(f) *Another Option*

We might try to avoid the denominators  $\rho^{-s}$  in (2.26), which introduce singularities though not infinities in (2.25) at points where all components of the wave function vanish, by writing

$$h' = \sum_s n^{-s} \int d^2x \mathcal{H}_s(\Psi(\mathbf{x}), \Psi^*(\mathbf{x})) \tag{2.33}$$

instead of (2.22). (This is the case discussed in Ref. [2].) As before, the sum ranges over integers  $s = 0, 1, 2, \dots$ ; the functions  $\mathcal{H}_s$  are scalars in the sense of Eq. (2.23), formed as a sum of products of  $1 + s$  factors of  $\Psi$ 's and  $1 + s$  factors of  $\Psi^*$ 's; and  $n$  is the norm

$$n = \int d^3x \Psi_k^*(\mathbf{x}) \Psi_k(\mathbf{x}). \tag{2.34}$$

The norm  $n$  is time-independent, so here there is no possibility of a singularity. On the other hand, (2.33) is not really local, because  $n$  is itself an integral over  $\mathbf{x}$ . For instance, if the wave function differs from zero only in two widely separated regions

$\mathcal{A}$ ,  $\mathcal{B}$ , then the Schrödinger equation in region  $\mathcal{A}$  depends on the wave function in region  $\mathcal{B}$ , though only through the coefficients  $n^{-s} = (n_{\mathcal{A}} + n_{\mathcal{B}})^{-s}$ , where  $n_{\mathcal{A}}$  and  $n_{\mathcal{B}}$  are the integrals of  $\Psi_k^*(\mathbf{x}) \Psi_k(\mathbf{x})$  over the regions  $\mathcal{A}$  and  $\mathcal{B}$ . This non-locality is not in itself so disturbing; we are not attempting a relativistic formulation, and locality is a tricky matter even in ordinary quantum mechanics. However, a particle that has undergone many measurement episodes (like Stern–Gerlach experiments) will have its wave function spread out over so many different regions  $\mathcal{A}$ ,  $\mathcal{B}$ ,  $\mathcal{C}$ , ... that the total value of  $n$  will be enormously larger than the contribution, say  $n_{\mathcal{A}}$ , from any one region. This will correspondingly decrease the effect of any non-bilinear ( $s \geq 1$ ) terms in (2.33), perhaps to the point of non-observability, with the  $s = 1$  term the least suppressed. Indeed, according to the “many-worlds” interpretation [8] of quantum mechanics, the wave function of the universe has since the beginning been spread over vastly many configurations (in most of which we do not even exist), so that even in a thoroughly non-linear theory, the world would now be well described by linear quantum mechanics [9]. This suggests the disturbing reflection, that our linear quantum mechanical theories may give no insight whatever to the physics relevant to the origin of the universe. However, these considerations only arise in non-local theories such as those described by (2.33), and not in the local theories discussed in item (e) above. (Another important difference between the two classes of theory is that the time-dependent Schrödinger equation derived from (2.33) does not generally have separable solutions like (2.30).)

(g) *Separated Systems*

The above discussion of the Galilean group has been for a single particle. However, in constructing realistic physical theories, it is essential to be able to combine separate isolated systems, in a way that does not introduce unphysical correlations between them.

Consider two systems I and II, taken now for simplicity to have only a finite number of degrees of freedom, with states represented by wave functions  $\psi_k$  and  $\varphi_l$ , respectively, and with Hamiltonians  $h_I(\psi, \psi^*)$  and  $h_{II}(\varphi, \varphi^*)$ . We want to combine these systems in a compound system I + II, with wave function  $\Psi_{kl}$ , but for the present without introducing any interaction between the subsystems. In ordinary quantum mechanics  $h_I$  and  $h_{II}$  are bilinear, and the Hamiltonian  $h_{I+II}$  of the combined system is constructed as a bilinear with a matrix coefficient given by the direct sum of the matrices in  $h_I$  and  $h_{II}$ . As a generalization of this direct sum, we take the Hamiltonian here as

$$h_{I+II}(\Psi, \Psi^*) = \sum_l h_I(\psi^{(l)}, \psi^{(l)*}) + \sum_k h_{II}(\varphi^{(k)}, \varphi^{(k)*}), \tag{2.35}$$

where

$$\psi^{(l)}_k \equiv \varphi^{(k)}_l \equiv \Psi_{kl}. \tag{2.36}$$

(It is easy to see that in the special case where  $h_I$  and  $h_{II}$  are bilinear, so is  $h_I + h_{II}$ , with a matrix kernel equal to the direct sum of those in  $h_I$  and  $h_{II}$ .) More generally,

any observable that is additive for separated subsystems will be represented by a function like (2.35).

In general, (2.35) gives the wave function  $\Psi_{kl}$  a time-dependence governed by

$$i \frac{d}{dt} \Psi_{kl} = \frac{\partial h_{I+II}}{\partial \Psi_{kl}^*} = \frac{\partial h_I(\psi^{(l)}, \psi^{(l)*})}{\partial \psi_k^{(l)*}} + \frac{\partial h_{II}(\varphi^{(k)}, \varphi^{(k)*})}{\partial \varphi_l^{(k)*}}. \quad (2.37)$$

Suppose we try a separable solution

$$\Psi_{kl} = \psi_k \varphi_l. \quad (2.38)$$

The homogeneity of the Hamiltonian functions tells us that then

$$\begin{aligned} \frac{\partial h_I(\psi^{(l)}, \psi^{(l)*})}{\partial \psi_k^{(l)*}} &= \varphi_l \frac{\partial h_I(\psi, \psi^*)}{\partial \psi_k^*} \\ \frac{\partial h_{II}(\varphi^{(k)}, \varphi^{(k)*})}{\partial \varphi_l^{(k)*}} &= \psi_k \frac{\partial h_{II}(\varphi, \varphi^*)}{\partial \varphi_l^*}. \end{aligned}$$

Hence the Schrödinger equation (2.37) has a solution of the form (2.38), provided that the factors  $\psi_k$  and  $\varphi_l$  satisfy the separate Schrödinger equations for systems I and II

$$i \frac{d}{dt} \psi_k = \frac{\partial h_I(\psi, \psi^*)}{\partial \psi_k^*} \quad (2.39)$$

$$i \frac{d}{dt} \varphi_l = \frac{\partial h_{II}(\varphi, \varphi^*)}{\partial \varphi_l^*}. \quad (2.40)$$

The result, that the Schrödinger equation has a solution of the separable form (2.38), shows that (2.35) represents the correct way to combine noninteracting systems.

For example, the Hamiltonian function given by (2.21) and (2.22) is of the form (2.35) (with indices  $l, k$  replaced with  $k, \mathbf{x}$ ) which explains why the time-dependent Schrödinger equation here has separable solutions of the form (2.30). This is not the case for a Hamiltonian function with  $h'$  given by (2.33), and so in this case we do not find separable solutions.

(h) *Changes of Basis*

Commutation relations and the time-dependent Schrödinger equation are left unchanged if we subject the wave function to a transformation

$$\psi_k \rightarrow \psi'_k(\psi, \psi^*) \quad (2.41)$$

which is canonical in the sense that

$$[\psi'_k, \psi'_i] = \delta_{ki} \quad (2.42)$$

$$[\psi'_k, \psi'_i] = [\psi'^*_k, \psi'^*_i] = 0 \quad (2.43)$$

with “commutators” defined by Eq. (2.8). Such canonical transformations can be generated by real functions  $f(\psi, \psi^*)$ ; we take the transformed wave functions as

$$\psi'_k = \tilde{\psi}_k(1), \quad (2.44)$$

where

$$i \frac{d}{ds} \tilde{\psi}_k(s) = \frac{\partial f(\tilde{\psi}(s), \tilde{\psi}^*(s))}{\partial \tilde{\psi}^*_k(s)} \quad (2.45)$$

$$\tilde{\psi}_k(0) = \psi_k. \quad (2.46)$$

In ordinary quantum mechanics we would require these transformations to be linear, in which case  $\psi_k$  would be transformed by an element of  $\text{Sp}(2N)$ .

Here as in quantum mechanics, we must require that functions like  $h$  would have the same homogeneity properties when expressed in terms of  $\psi'_k$  and  $\psi'^*_k$  as when expressed in terms of  $\psi_k$  and  $\psi^*_k$ . Then  $\psi'_k$  must be homogeneous of degree  $(1, 0)$  in  $\psi$  and  $\psi^*$ :

$$\frac{\partial \psi'_k}{\partial \psi_l} \psi_l = \psi'_k, \quad \frac{\partial \psi'_k}{\partial \psi'^*_l} \psi'^*_l = 0. \quad (2.47)$$

Equivalently, the generating function  $f(\psi, \psi^*)$  must be homogeneous of degree one in both  $\psi$  and  $\psi^*$ . This incidentally also shows that the norm  $n$  is form-invariant,

$$n \equiv \psi^*_k \psi_k = \psi'^*_k \psi'_k. \quad (2.48)$$

In ordinary quantum mechanics this requirement restricts these transformations to the subgroup  $U(N)$  of  $\text{Sp}(2N)$ , and all interpretative postulates are required to be invariant under such unitary transformations of the wave functions. The reader will be able to check that the interpretative postulates to be proposed further on in this paper are invariant under the larger set of canonical transformations (2.44) with  $f$  real and homogeneous of degree one in the  $\psi$  and  $\psi^*$ , but not necessarily bilinear. Thus we can freely use such transformations to change the choice of our wave functions.

Before continuing with these generalities, it may be helpful to look briefly at one very simple system that can exhibit departures from ordinary quantum mechanics. The wave function has just two components,  $\psi_1$  and  $\psi_2$ . For a 2-component system it is always possible to carry out a canonical homogeneous transformation (as described in Appendix A) so that the Hamiltonian function is of the form

$$h = n\bar{h}(a), \quad (2.49)$$

where

$$n \equiv |\psi_1|^2 + |\psi_2|^2 \tag{2.50}$$

$$a \equiv |\psi_2|^2/n. \tag{2.51}$$

In ordinary quantum mechanics  $h$  is bilinear in  $\psi_k$  and  $\psi_k^*$ , so  $\bar{h}(a)$  is linear in  $a$ . Departures from ordinary quantum mechanics are represented by non-linear terms in  $\bar{h}(a)$ .

With Hamiltonian function (2.49), the non-linear time-dependent Schrödinger equation (2.12) takes the form

$$i \frac{d\psi}{dt} = \psi_1 [\bar{h}(a) - a\bar{h}'(a)]$$

$$i \frac{d\psi_2}{dt} = \psi_2 [\bar{h}(a) + (1 - a)\bar{h}'(a)].$$

The solution is immediate,

$$\psi_k(t) = c_k \exp(-i\omega_k(a)t) \tag{2.52}$$

with

$$\omega_1(a) = \bar{h}(a) - a\bar{h}'(a), \tag{2.53}$$

$$\omega_2(a) = \bar{h}(a) + (1 - a)\bar{h}'(a), \tag{2.54}$$

and  $a$  is now a constant of motion. If  $\bar{h}(a)$  is linear, then  $\omega_1$  and  $\omega_2$  are independent of  $a$ . Otherwise, the frequencies depend on the initial conditions, as expected for non-linear oscillators. This is a particularly simple example of quasi-periodic behavior, which is expected here because these two-component systems are automatically integrable.

\* \* \*

Now that our formalism has been outlined, it is time to say a few words comparing the work described in this paper with other work on possible non-linear corrections to quantum mechanics. There is a large literature on what is called the “non-linear Schrödinger equation,” typically taken in the form

$$i \frac{\partial \psi}{\partial t} = -\frac{1}{2m} \nabla^2 \psi + \varepsilon |\psi|^2 \psi.$$

In most of this work, this equation is studied as a test case in non-linear dynamics and not as a serious proposal for modifications to quantum mechanics. The results obtained by the mathematical studies of this equation are unfortunately of no use to us here, because this equation lacks the crucial property of homogeneity, according to which if  $\psi$  is a solution then so is  $Z\psi$  for arbitrary complex constant  $Z$ . Some authors [10-12] have considered non-linear corrections to quantum

mechanics as possible means of resolving the difficulties of quantum mechanical measurement theory, but without imposing the homogeneity conditions and without considering specific experimental tests. The problem of combining separate systems in non-linear theories was taken up by Bialynicki-Birula and Mycielski [13], with the conclusion that the non-linear terms in the one-particle Schrödinger equation should take a logarithmic form, as in the free-particle equation

$$i \frac{\partial \psi}{\partial t} = -\frac{1}{2m} \nabla^2 \psi - b \psi \ln |\psi|^2.$$

Shimony [14] suggested that this equation could be tested by neutron interferometry, exploiting the feature that a partial absorption of one of the neutron beams would reduce the value of  $|\psi|^2$  and hence affect the subsequent change in phase of the neutron wave function. An experiment of Shull *et al.* [15] subsequently put a bound of  $3.4 \times 10^{-13}$  eV on  $b$ . This was subsequently improved to  $3 \times 10^{-15}$  eV by measurements of neutron Fresnel diffraction [16]. It should be emphasized that this sort of experiment cannot detect non-linearities in local homogeneous equations, like Eq. (2.25) of this paper, where the overall scale of the wave function has no effect on the evolution of the phase. (Also, these experiments are more limited in accuracy than those discussed here in Sections 4 and 6, such as that carried out in Ref. [3], because it is possible to observe atomic transitions or precession over times much longer than even a very cold neutron.) It has been pointed out to me that homogeneous non-linear Schrödinger equations have been considered by Haag and Banner [17] and Kibble [18]. In order that  $\psi$  and  $Z\psi$  should represent the same state, Haag and Banner considered a modification of the non-linear Schrödinger equation of Ref. [12], to read (in our notation)

$$i \frac{\partial \psi}{\partial t} = -\frac{1}{2m} \nabla^2 \psi + \varepsilon \psi \mathbf{A} \cdot \nabla \text{Arg } \psi.$$

Homogeneity here is achieved at the cost of introducing a mysterious extraneous vector potential  $\mathbf{A}$ . Kibble briefly considered a homogeneous non-linear Schrödinger equation of the form

$$i \frac{\partial \psi}{\partial t} = -\frac{1}{2m} \nabla^2 \psi + \lambda |\nabla \psi|^2 \psi / |\psi|^2$$

but he rejected this possibility in favor of that of Ref. [13], because Kibble's equation violates Galilean invariance. (Kibble was mostly concerned in Ref. [18] with an interesting suggestion on how to formulate relativistic non-linear quantum-mechanical models.)

The present paper seeks to establish a general framework for considering non-linear generalizations of quantum mechanics, not just the addition of non-linear terms to a particular Schrödinger equation. As discussed earlier in this section, for particles with internal quantum numbers like spin, it is possible and even natural

within this framework to include homogeneous non-linear terms in the Schrödinger equation, in such a way as to allow a proper treatment of separated systems and without needing to introduce extra potentials like  $\mathbf{A}$  or to violate Galilean invariance. By concentrating on the dynamics of these internal degrees of freedom, we are led to propose experiments (in Sections 4 and 6) that can subject quantum mechanics to unprecedentedly stringent tests.

### 3. EIGENVALUES

In quantum mechanics, the state represented by a wave function  $\psi_k$  has a definite value  $\alpha$  for the observable represented by a matrix  $A_{kl}$  if  $\psi_k$  is an eigenvector of  $A_{kl}$  with eigenvalue  $\alpha$ . We generalize this here, by postulating that a state represented by  $\psi_k$  has a definite value  $\alpha$  for the observable represented by a non-bilinear function  $a(\psi, \psi^*)$ , if and only if

$$\frac{\partial a}{\partial \psi_k^*} = \alpha \psi_k \tag{3.1}$$

and

$$\frac{\partial a}{\partial \psi_k} = \alpha \psi_k^*. \tag{3.2}$$

(Where  $a$  is a bilinear  $\psi_k^* A_{kl} \psi_l$ , this is just the usual eigenvalue condition of quantum mechanics.) Note that if  $\psi_k$  satisfies (3.1) or (3.2), then so does  $Z\psi_k$ , where  $Z$  is an arbitrary complex number. Also, contracting (3.1) with  $\psi_k^*$  and using (2.1), allows us to express  $\alpha$  in terms of  $a$  itself

$$\alpha = \frac{a(\psi, \psi^*)}{n(\psi, \psi^*)}, \tag{3.3}$$

where  $\psi$  is taken as the eigenvector satisfying (3.1). This shows, in particular, that  $\alpha$  is real if  $a$  is. In this case, (3.2) just follows from (3.1) by taking the complex conjugate. However, for general complex functions, (3.2) must be taken as a separate condition.

Where  $a$  is a symmetry generator, Eq. (3.1) just tells us that

$$\delta_a \psi_k = -i\alpha \psi_k.$$

That is,  $\psi$  is invariant up to a phase under the transformation generated by  $a$ . In particular, a state with a definite value  $E$  for the energy has a time dependence given by (2.12) and (3.1) as  $\psi \propto \exp(-iEt)$ , just as in ordinary quantum mechanics.

In quantum mechanics, the eigenvalues of an observable represented by a matrix  $A$  are the stationary values of the function  $\psi_k^* A_{kl} \psi_l / \psi_k^* \psi_k$ . The corresponding

variational principle holds here as well: the eigenvectors of an observable represented by a function  $a(\psi, \psi^*)$  are the stationary points of the function

$$\bar{a}(\psi, \psi^*) = \frac{a(\psi, \psi^*)}{n(\psi, \psi^*)} \quad (3.4)$$

and the eigenvalues are the values of  $\bar{a}$  at these stationary points. (Differentiating (3.4) with respect to  $\psi_k$  and  $\psi_k^*$  yields

$$\begin{aligned} \frac{\partial \bar{a}}{\partial \psi_k^*} &= \frac{1}{n} \frac{\partial a}{\partial \psi_k^*} - \frac{a}{n^2} \psi_k \\ \frac{\partial \bar{a}}{\partial \psi_k} &= \frac{1}{n} \frac{\partial a}{\partial \psi_k} - \frac{a}{n^2} \psi_k^*. \end{aligned}$$

If these both vanish then  $\psi$  is an eigenvector of  $a$  with eigenvalue  $a/n$ . On the other hand, if  $\psi$  is an eigenvector of  $a$  with eigenvalue  $\alpha$  then (3.3) shows that  $\alpha = a/n$ , so  $\partial \bar{a} / \partial \psi_k^* = \partial \bar{a} / \partial \psi_k = 0$ .) We will see in Section 5 that the function (3.4) is the expectation value of the observable represented by  $a(\psi, \psi^*)$  in the state with wave function  $\psi$ .

The variational principle allows us to reach useful conclusions about the number of eigenvectors of a real function  $a(\psi, \psi^*)$ . We note that the function  $\bar{a}$  defined by (3.4) is invariant under multiplication of  $\psi$  by an arbitrary complex number  $Z$ ,

$$\bar{a}(Z\psi, Z^*\psi^*) = \bar{a}(\psi, \psi^*). \quad (3.5)$$

Therefore, for an  $N$ -component wave function  $\psi_k$ , we should think of  $\bar{a}$  as being defined not on the space of  $N$  complex variables, but on the projective space  $CP^{N-1}$ , a compact space.

A well-known result of Morse theory [19] tells us that there are at least  $N$  stationary points of any smooth function on  $CP^{N-1}$ . Hence, the observable represented by  $a(\psi, \psi^*)$  has at least  $N$  eigenvectors. Where  $a(\psi, \psi^*)$  is the bilinear  $\psi_k^* A_{kl} \psi_l$ , the eigenvectors of the function are just the eigenvectors of the Hermitian  $N \times N$  matrix  $A_{kl}$ , and so there are precisely  $N$  of them. By continuity, it follows that for a weakly non-bilinear function, where  $a(\psi, \psi^*)$  is close to a bilinear  $\psi_k^* A_{kl} \psi_l$ , there are also just  $N$  eigenvectors.

We may have to consider observables constructed as functions of other observables. Now, it is trivially obvious that if a state represented by a wave function  $\psi_k$  has definite values  $\alpha$  and  $\beta$  for observables represented by functions  $a$  and  $b$ , then it has a definite value  $\xi\alpha + \eta\beta$  for the observable represented by  $\xi a + \eta b$ , with  $\xi$  and  $\eta$  any complex numbers. It is a little less obvious that it also has a definite value  $\alpha\beta$  for the observable represented by  $a * b$ . Here is the proof. First note that

$$\frac{\partial}{\partial \psi_i^*} (a * b) = \frac{\partial^2 a}{\partial \psi_i^* \partial \psi_k} \frac{\partial b}{\partial \psi_k^*} + \frac{\partial a}{\partial \psi_k} \frac{\partial^2 b}{\partial \psi_k^* \partial \psi_i^*}.$$

Next use the eigenvalue condition (3.1) for  $a$ , and the corresponding condition (3.2) for  $b$ :

$$\frac{\partial}{\partial \psi_l^*} (a * b) = \beta \frac{\partial^2 a}{\partial \psi_l^* \partial \psi_k} \psi_k + \alpha \frac{\partial^2 b}{\partial \psi_k^* \partial \psi_l^*} \psi_k^*.$$

Then note that by differentiating (2.1) with respect to  $\psi_l^*$ ,

$$\psi_k \frac{\partial^2 a}{\partial \psi_k \partial \psi_l^*} = \psi_k^* \frac{\partial^2 a}{\partial \psi_k^* \partial \psi_l^*} + \frac{\partial a}{\partial \psi_l^*} = \frac{\partial a}{\partial \psi_l^*}.$$

Using the eigenvalue condition again gives then

$$\psi_k \frac{\partial^2 a}{\partial \psi_k \partial \psi_l^*} = \alpha \psi_l, \quad \psi_k^* \frac{\partial^2 a}{\partial \psi_k^* \partial \psi_l^*} = 0, \tag{3.6}$$

and likewise for  $b$ . We have then the desired result,

$$\frac{\partial}{\partial \psi_l^*} (a * b) = \alpha \beta \psi_l. \tag{3.7}$$

In the same way, we can verify the other half of the eigenvalue condition

$$\frac{\partial}{\partial \psi_l} (a * b) = \alpha \beta \psi_l^*. \tag{3.8}$$

We are tacitly assuming throughout this work that departures from quantum mechanics are generally very small. In particular, it should be possible to write any observable  $a$  (like the Hamiltonian  $h$ ) in the form

$$a = a_0 + a_1,$$

where  $a_0$  is the bilinear  $\psi_k^* A_{kl} \psi_l$  of ordinary quantum mechanics, and  $a_1$  is a small non-bilinear perturbation. It is natural then to ask, how much does the perturbation  $a_1$  shift the eigenvalues of  $a$ ? More generally, we would like to know how to calculate shifts in eigenvalues due to small perturbations of observables, whether or not  $a_0$  is the bilinear of ordinary quantum mechanics.

Suppose that  $\psi_0$  is an eigenvector of  $a_0$  with eigenvalue  $\alpha_0$ ,

$$\frac{\partial a_0(\psi_0, \psi_0^*)}{\partial \psi_{0k}^*} = \alpha_0 \psi_{0k}.$$

Then to first order in perturbations, the condition that  $\psi_0 + \psi_1$  should be an eigenvector of  $a_0 + a_1$  with eigenvalue  $\alpha_0 + \alpha_1$  is that

$$\begin{aligned} & \frac{\partial a_1(\psi_0, \psi_0^*)}{\partial \psi_{0k}^*} + \frac{\partial^2 a_0(\psi_0, \psi_0^*)}{\partial \psi_{0k}^* \partial \psi_{0l}} \psi_{1l} + \frac{\partial^2 a_0(\psi_0, \psi_0^*)}{\partial \psi_{0k}^* \partial \psi_{0l}^*} \psi_{1l}^* \\ & = \alpha_0 \psi_{1k} + \alpha_1 \psi_{0k}. \end{aligned}$$

Multiplying with  $\psi_{\delta k}^*$  and using Eq. (3.6), we have then

$$\alpha_1 \psi_{\delta k}^* \psi_{\delta k} = \psi_{\delta k}^* \frac{\partial a_1(\psi_0, \psi_0^*)}{\partial \psi_{\delta k}^*} = a_1(\psi_0, \psi_0^*)$$

or, in other words,

$$\alpha_1 = \bar{a}_1(\psi_0, \psi_0^*). \quad (3.9)$$

This is pretty much the same as the first Born approximation of ordinary quantum mechanics.

For a simple example, consider the 2-component system with Hamiltonian function (2.49). The energy eigenstates may be found by requiring that all components (2.52) oscillate with the same time-dependent factor  $\exp(-iEt)$ . For all forms of the function  $\bar{h}(a)$ , there are at least two solutions:

$$A: \quad \psi_1 \neq 0, \quad \psi_2 = 0, \quad E = \bar{h}(0) \quad (3.10)$$

$$B: \quad \psi_1 = 0, \quad \psi_2 \neq 0, \quad E = \bar{h}(1). \quad (3.11)$$

In addition, for sufficiently strong non-linearities the function  $\bar{h}(a)$  may have a stationary point at some  $a_0$  in the range  $0 < a_0 < 1$ , and in this case there is a third class of solutions,

$$C: \quad a = a_0, \quad E = \bar{h}(a_0). \quad (3.12)$$

These form a degenerate family of physically inequivalent energy eigenstates, because there is nothing to fix the *relative* phase of  $\psi_1$  and  $\psi_2$ . Note that in accordance with the general remarks above there are just  $N=2$  energy eigenstates for sufficiently weak non-bilinearities. For instance, if we give  $\bar{h}(a)$  a small quadratic term

$$\bar{h}(a) = (1-a)\bar{h}(0) + a\bar{h}(1) + \varepsilon a(1-a),$$

then the third class of solutions appears only for

$$|\varepsilon| > |\bar{h}(1) - \bar{h}(0)|.$$

#### 4. SPINNING PARTICLES IN EXTERNAL FIELDS

We shall now consider what may be the simplest physically interesting sort of system that can show departures from ordinary quantum mechanics: a spinning particle (with spatial motion ignored) in various sorts of weak external field. We shall first examine a spin  $\frac{1}{2}$  particle in a uniform magnetic field or quadrupole electric field, and then consider a particle of spin 1 or  $\frac{3}{2}$  in a weak uniform magnetic field. As we shall see, all of these are integrable systems, with general time-

dependence described by Eq. (2.15). We shall calculate not only the energy eigenvalues, but also the characteristic precession frequencies appearing in Eq. (2.15).

These calculations were originally carried out for purposes of illustration, and because as discussed in Section 6, the characteristic frequencies play a key role in prediction of the system's spectrum for absorption of radiation. However, after the publication of Ref. [4], both N. Fortson [20] and T. Chupp [21] independently pointed out to me that it was possible to monitor the precession of spinning nuclei coherently over rather long times and measure the characteristic frequencies (or frequency differences) with great precision. Small departures from the linearity of quantum mechanics would show up as a weak dependence of the characteristic frequencies on the initial state of the nuclear spin. Both Fortson and Chupp are considering the measurement of these precision frequencies, for  $\text{Hg}^{201}$  and  $\text{Ne}^{21}$ , respectively. These nuclei both have spin  $\frac{3}{2}$ , and although the time-dependent Schrödinger equation for spin  $\frac{3}{2}$  is integrable for a free particle or in a weak magnetic field, it is not integrable for the sort of external field that would be present in such experiments. Treating the non-bilinear part of  $h$  as a small perturbation, we shall calculate the precession frequencies for this case by using the method of averaging, and find an interesting dependence on the initial wave function.

*Spin 1/2*

Here the wave function has just two components, so as remarked in Section 2, the time-dependent Schrödinger equation is automatically integrable, whatever external fields may be present.

Now let us consider the case of a spin  $\frac{1}{2}$  particle in an external field with the 3-dimensional rotation properties of a symmetric traceless tensor of rank  $r = 1, 2, \dots$ . The only term in the Hamiltonian function with the correct rotation-invariance properties and homogeneity properties takes the form of a sum of products of  $r$  factors of  $\psi_m$  and  $r$  factors of  $\psi_m^*$ , times a factor  $n^{1-r}$ . (To see this, note that a product of  $r$  factors of  $\psi$ 's transforms uniquely like a quantity of spin  $r/2$ , and likewise for a product of  $r$  factors of  $\psi^*$ 's, so the product of  $r$  factors of  $\psi$ 's and  $r$  factors of  $\psi^*$ 's transforms as a sum of spins  $r, r-1, \dots, 0$ . Now, an obvious way of forming spins  $r-1, r-2, \dots, 0$  out of  $r\psi$ 's and  $r\psi^*$ 's is to form then from a product of  $r-1 \psi$ 's and  $r-1 \psi^*$ 's, and multiply with  $n$ . Since each spin in the product of  $r \psi$ 's and  $r \psi^*$ 's occurs only once, the terms in this product of spin  $r-1, r-2, \dots, 0$  must be of this form. Thus a symmetric traceless tensor of rank  $r$ , which transforms as spin  $r$ , appears as a new tensor only in the product of  $r \psi$ 's and  $r \psi^*$ 's.)

For a homogeneous magnetic field  $r=1$ , so the most general Hamiltonian function is bilinear

$$h = E_0 n + \mu \psi_m^* (\boldsymbol{\sigma})_{ml} \psi_l \cdot \mathbf{B} \tag{4.1}$$

with  $\boldsymbol{\sigma}$  the usual Pauli matrices. There is no difference here between ordinary quantum mechanics and our generalized theory. This is important, because it shows that

no non-linear effects in the internal structure of particles of spin  $\frac{1}{2}$  like the proton, neutron, or electron or in their interaction with weak uniform magnetic fields can show up when these can be considered as point particles.

On the other hand, a quadrupole electric field has  $r=2$ . In ordinary quantum mechanics a spin  $\frac{1}{2}$  particle cannot have an interaction of first order with such a field, but now it can, uniquely of the form

$$h = E_0 n + Q n^{-1} \mathcal{E}_{ij} \psi^\dagger \sigma_i \psi \psi^\dagger \sigma_j \psi, \quad (4.2)$$

where  $Q$  is a constant, and

$$\mathcal{E}_{ij} \equiv \frac{\partial E_i}{\partial x_j} + \frac{\partial E_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \mathbf{V} \cdot \mathbf{E}. \quad (4.3)$$

It is not hard to show that in the generic case, where all eigenvalues of  $\mathcal{E}$  are different, the energy eigenvalues are just equal to  $E_0 + Q\lambda$ , where  $\lambda$  runs over the three eigenvalues of  $\mathcal{E}$ . For each energy eigenvalue there are two eigenstates, namely, the two eigenstates of the component of  $\sigma$  in the direction of the corresponding eigenvector of  $\mathcal{E}$ . For each energy eigenvalue there are two eigenstates, namely, the two eigenstates of the component of  $\sigma$  in the direction of the corresponding eigenvector of  $\mathcal{E}$ . Thus a free spin  $\frac{1}{2}$  particle in a generic quadrupole electric field would have three distinct energy values, which would be split by a very weak magnetic field into six energy values. If the strength of the magnetic field is increased so that the magnetic dipole term in  $h$  is much larger than the electric quadrupole, then the six energy eigenstates will merge into the two expected in ordinary quantum mechanics.

The time-dependent Schrödinger equation is integrable here, but the general solution is a bit complicated. It simplifies greatly in the case of axial symmetry, where two of the eigenvalues of the quadrupole field tensor  $\mathcal{E}_{ij}$  are equal. Taking the 3-axis in the direction of the eigenvector for the other eigenvalue, we easily see that the Hamiltonian function takes the form (2.49), with

$$\hbar(a) = Q\lambda_{\parallel}(1-2a)^2 + 4Q\lambda_{\perp}a(1-a), \quad (4.4)$$

where  $\lambda_{\parallel}$ ,  $\lambda_{\perp}$ ,  $\lambda_{\perp}$  are the three eigenvalues of  $\mathcal{E}_{ij}$ , and  $a$  is given by (2.51), with  $\psi_1 \equiv \psi_{+1/2}$ ,  $\psi_2 \equiv \psi_{-1/2}$ . The time-dependence is given by (2.52), with frequencies (2.53) and (2.54). As we have seen, in this case we find energy eigenstates of all three types, (3.10), (3.11), and (3.12).

### Spin 1

The three components of  $\psi$  can be assembled into a complex 3-vector,  $\Psi$ . The angular momentum is taken to have the bilinear form of ordinary quantum mechanics

$$\mathbf{j} = -i\Psi^* \times \Psi.$$

As long as the external fields are axially symmetric,  $j_z$  as well as  $h$  and  $n$  will be conserved and the time-dependent Schrödinger equation will be integrable. In the absence of external fields, the most general unperturbed Hamiltonian function with the appropriate reality and homogeneity properties and that commutes with  $\mathbf{j}$  takes the form

$$h_0 = ng(|\alpha|^2), \tag{4.5}$$

where  $g$  is a real function and  $\alpha$  is the only scalar in our problem, apart from  $n$ :

$$\alpha \equiv \boldsymbol{\psi} \cdot \boldsymbol{\psi}/n \tag{4.6}$$

with  $n$  the usual norm

$$n \equiv \boldsymbol{\psi}^* \cdot \boldsymbol{\psi}. \tag{4.7}$$

We turn on a weak uniform magnetic field  $\mathbf{B}$ . Rotational invariance requires that the term in the Hamiltonian function of first order in  $\mathbf{B}$  takes the form

$$h_1 = -\boldsymbol{\mu} \cdot \mathbf{B}, \tag{4.8}$$

where  $\boldsymbol{\mu}$  transforms as a 3-vector, in the sense that

$$[j_a, \mu_b] = i\epsilon_{abc}\mu_c.$$

The most general  $\boldsymbol{\mu}$  with the required reality, homogeneity, and isotropy properties takes the form

$$\boldsymbol{\mu} = -i(\boldsymbol{\psi}^* \times \boldsymbol{\psi}) \mu(|\alpha|^2) \tag{4.9}$$

with  $\mu(\alpha)$  another real function of  $|\alpha|^2$ .

Now that we have our Hamiltonian, let us find its energy eigenstates. The energy eigenvalue problem here takes the form

$$\begin{aligned} E\boldsymbol{\psi} &= \frac{\partial(h_0 + h_1)}{\partial\boldsymbol{\psi}^*} = \boldsymbol{\psi}g - i(\mathbf{B} \times \boldsymbol{\psi})\mu \\ &+ 2(\alpha\boldsymbol{\psi}^* - |\alpha|^2\boldsymbol{\psi})(g' + i\mathbf{B} \cdot (\boldsymbol{\psi}^* \times \boldsymbol{\psi})\mu'/n). \end{aligned} \tag{4.10}$$

(Primes denote differentiation with respect to  $|\alpha|^2$ .) It is straightforward to find the general solutions. For arbitrary magnetic field (in the  $z$ -direction), we have a triplet of eigensolutions: one pair,

$$E = g(0) \pm B\mu(0) \tag{4.11}$$

$$\boldsymbol{\psi} \propto (1, \mp i, 0) \tag{4.12}$$

and also

$$E = g(1) \tag{4.13}$$

$$\Psi \propto (0, 0, 1). \tag{4.14}$$

Also, for a sufficiently small magnetic field we have a continuum of additional degenerate solutions

$$E = g(|\alpha|^2) + \frac{2(1 - |\alpha|^2) g'(|\alpha|^2) \mu(|\alpha|^2)}{\mu(|\alpha|^2) - \mu'(|\alpha|^2)(1 - |\alpha|^2)} \tag{4.15}$$

$$\Psi = ([\mu(|\alpha|^2) - \mu'(|\alpha|^2)(1 - |\alpha|^2)] B, \quad 2i(1 - \alpha) g'(|\alpha|^2), \quad 0), \tag{4.16}$$

where  $|\alpha|$  is determined in terms of the magnetic field by

$$B^2 = \frac{4(1 - |\alpha|^2) g'^2(|\alpha|^2)}{(\mu(|\alpha|^2) - \mu'(|\alpha|^2)(1 - |\alpha|^2))^2} \tag{4.17}$$

and the phase of  $\alpha$  is arbitrary. These extra solutions disappear if  $B^2$  is greater than the maximum value of the right-hand side of Eq. (4.17) (presumably reached at  $\alpha = 0$ ). This is quite consistent with the general remarks of Section 3: we always have at least  $N = 3$  energy eigenstates, and we have precisely three eigenstates for sufficiently small departures from bilinearity. For instance, in Ref. [2] we briefly considered the case  $g(|\alpha|^2) = \varepsilon |\alpha|^2$ ,  $\mu(|\alpha|^2) = \mu$ . Equation (4.17) here gives

$$|\alpha|^2 = 1 - \mu^2 B^2 / 4\varepsilon^2,$$

so this solution disappears for  $|\mu B| > 2 |\varepsilon|$ . In this case, (4.15) gives the energy

$$E = \varepsilon + \frac{\mu^2 B^2}{4\varepsilon},$$

a result quoted in Ref. [2]. Note that at the critical field,  $E = 2\varepsilon$ , so this energy eigenvalue merges with one of the eigenvalues (4.11) at this field.

It is interesting that even for a large magnetic field, the energy eigenvalues (4.11), (4.13) do not satisfy the equal-spacing rule of ordinary quantum mechanics; the energy differences between adjacent levels are  $g(0) - g(1) + B\mu(0)$  and  $g(1) - g(0) + B\mu(0)$ . We will find a similar result below for spin  $\frac{3}{2}$ . This can happen here because the derivation of the equal-spacing rule relies not only on rotational invariance but also on the linearity of ordinary quantum mechanics. This example suggests one general approach to developing high-precision tests of quantum mechanics: one may search for small departures from those consequences of exact symmetry principles in ordinary quantum mechanics that depend on linearity as well as on symmetry.

In fact, the equal spacing rule has already been subjected to high-precision tests in some of the experiments [22–26] designed to search for possible failures of

rotation invariance arising from an anisotropy of inertia. However, as discussed in Section 6, in the presence of non-linearities the quantities measured in resonance absorption experiments, as well as in spin precession experiments, are not necessarily simply related to differences of energy eigenvalues, but depend instead on the details of the experiment. To analyze such experiments, we need to know not just the energy eigenvalues, but the characteristic frequencies  $\omega$ , for arbitrary initial conditions. A somewhat tedious but straightforward calculation shows that the general time-dependent solution is

$$\psi_1 = e^{-i\omega_A t} [\chi_1 \cos \omega_B t + \chi_2 \sin \omega_B t] \tag{4.18}$$

$$\psi_2 = e^{-i\omega_A t} [-\chi_1 \sin \omega_B t + \chi_2 \cos \omega_B t] \tag{4.19}$$

$$\psi_3 = e^{-i\omega_A t} \chi_3, \tag{4.20}$$

where

$$\chi = \alpha_0 \mathbf{c} e^{-i\omega_C t} + (1 + \sqrt{1 - |\alpha_0|^2}) \mathbf{c}^* e^{+i\omega_C t}. \tag{4.21}$$

Here  $\alpha_0$  and  $\mathbf{c}$  are constants depending on initial data, and the characteristic frequencies are

$$\omega_A = g(|\alpha_0|^2) + 2(1 - |\alpha_0|^2) g'(\alpha_0^2) \tag{4.22}$$

$$\omega_B = \mu B \tag{4.23}$$

$$\omega_C = 2g'(|\alpha_0|^2) \sqrt{1 - |\alpha_0|^2}. \tag{4.24}$$

(For simplicity, it is assumed here that  $\mu(|\alpha|^2)$  is independent of its argument, which should be a good approximation in the usual case where the magnetic energy is a weak perturbation.) The solutions (4.11)–(4.17) of the energy eigenvalue problem may easily be recovered from the condition that the whole wave function should oscillate with a single frequency  $E$ . (The eigensolution (4.11), (4.12) has  $\alpha_0 = 0$ ,  $E = \omega_A - \omega_C \pm \omega_B$ ; the eigensolution (4.13), (4.14) has  $\alpha_0 = 1$ ,  $E = \omega_A$ ,  $\omega_C = 0$ ; and the eigensolution (4.15)–(4.17) has  $\omega = \omega_A$ ,  $\omega_B = \pm \omega_C$ .)

The frequency  $\omega_A$  is not directly observable, as the factor  $\exp(-i\omega_A t)$  appears as a common factor in all components of the wave function. The frequency  $\omega_B$  is observable, but it represents just the same precession in a magnetic field that we would expect in ordinary quantum mechanics. In fact, this precession frequency can be derived (for arbitrary spin) just as in quantum mechanics: Equation (2.14) and (2.16) show that

$$\begin{aligned} \frac{d\mathbf{j}}{dt} &= -i[\mathbf{j}, h] = -i\mu[\mathbf{j}, \mathbf{j} \cdot \mathbf{B}] \\ &= \mu \mathbf{B} \times \mathbf{j}. \end{aligned}$$

The really interesting new phenomenon produced here by the non-linear corrections to quantum mechanics is the oscillation at frequency  $\omega_C$ , a frequency that

vanishes in ordinary quantum mechanics, where  $g(\alpha)$  is a constant. If we somehow prepare a spin 1 particle so that  $\text{Re } \psi$  and  $\text{Im } \psi$  are initially neither parallel nor perpendicular, then even in the absence of a magnetic field the magnitude of each component of  $\psi$  will oscillate at a frequency  $\omega_C$ , dependent on the angle between  $\text{Re } \psi$  and  $\text{Im } \psi$ . However, the spin expectation value  $\mathbf{j}/n$  simply precesses at the frequency  $\omega_B$ , just as in ordinary quantum mechanics.

### Spin 3/2

According to the general rules discussed in Section 2, for integrability here we need  $2j+1=4$  conserved quantities, all commuting ("in involution") with each other. This condition is satisfied for a spin  $\frac{3}{2}$  particle that is free, or in a weak uniform external magnetic field, say in the  $z$ -direction. In such cases the Hamiltonian function takes the form

$$h = h_0 + \mu B j_z, \quad (4.25)$$

where  $h_0$  is rotationally invariant, and  $\mathbf{j}$  is the angular momentum function

$$\mathbf{j} = \psi^\dagger \mathbf{S} \psi \quad (4.26)$$

with  $\mathbf{S}$  the usual  $4 \times 4$  spin matrix for  $j = \frac{3}{2}$ . (Non-bilinear terms in the magnetic moment are dropped here, because this term is assumed to be already very small.) Both  $h_0$  and  $j_z$  "commute" with each other and with  $n$ , and also with  $\mathbf{j}^2$ , which provides our fourth conserved quantity. (By  $\mathbf{j}^2$  is meant the sum of the ordinary squares of the functions  $j_x, j_y, j_z$ , not the function  $\mathbf{j} \cdot \mathbf{j} = \psi^\dagger \mathbf{S}^2 \psi$ , which is here just equal to  $15n/4$ .) We write the wave function here as  $\psi_m$ , with  $m$  running over the values  $\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}$ , and  $-\frac{3}{2}$ . This is somewhat more complicated than earlier examples, so instead of working with the most general non-bilinear Hamiltonian function, we will limit ourselves to homogeneous functions of degree one of the simplest non-bilinear form, consisting of a product of two  $\psi$ 's, two  $\psi^*$ 's, and a factor  $n^{-1}$ . The product of two identical  $j = \frac{3}{2}$   $\psi$ 's can only have spins  $j=3$  or  $j=1$ , and likewise for two  $\psi^*$ 's, so there are just two ways of constructing a rotationally invariant function of this form: by combining  $(\psi\psi)_{j=3}$  with  $(\psi^*\psi^*)_{j=3}$ , or  $(\psi\psi)_{j=1}$  with  $(\psi^*\psi^*)_{j=1}$ . However, one linear combination of these must be proportional to  $n^2/n$ , which is a rotational invariant of the same general form. This term amounts to an additive constant appearing in the energy of all states, and hence can be disregarded here. Another independent linear combination must be proportional to  $\mathbf{j}^2/n$ , which is also a rotational invariant of the same general form. We therefore take

$$h_0 = -\varepsilon \mathbf{j}^2/n \quad (4.27)$$

with  $\varepsilon$  a small energy that may be either positive or negative. This is essentially the same (as it must be) as the Hamiltonian function given in Ref. [4], aside from an additive constant. That is, (4.27) may be written

$$\begin{aligned}
 h_0 = & -\frac{9\epsilon n}{4} + \frac{\epsilon}{n} \{ 2 |\sqrt{3} \psi_{3/2} \psi_{-1/2} - \psi_{1/2} \psi_{1/2}|^2 \\
 & + |3\psi_{3/2} \psi_{-3/2} - \psi_{1/2} \psi_{-1/2}|^2 \\
 & + 2 |\sqrt{3} \psi_{-3/2} \psi_{1/2} - \psi_{-1/2} \psi_{-1/2}|^2 \}. \tag{4.28}
 \end{aligned}$$

(The factor 3 appeared outside the absolute value sign in the second term of Eq. (12) in Ref. [4] through a typographical error.) Even though we take  $h_0$  proportional to  $\mathbf{j}^2$ , the model is still integrable; for the fourth conserved quantity we can take any function of the  $8 - 3 = 5$  rotational invariants that can be formed from a complex  $j = \frac{3}{2}$  wave function.

For a Hamiltonian function given by (4.25) and (4.27), the time-dependent Schrödinger equation yields

$$i\dot{\psi} = -2\epsilon \mathbf{j} \cdot \mathbf{S} \psi / n + \epsilon \mathbf{j}^2 \psi / n^2 + \mu B S_z \psi. \tag{4.29}$$

As usual, we can eliminate the magnetic field by transforming to a rotating frame of reference. Define

$$\psi_m = \exp(-i\mu B m t) \varphi_m. \tag{4.30}$$

Then (4.29) takes the form

$$i\dot{\varphi} = -2\epsilon \mathbf{j}_0 \cdot \mathbf{S} \varphi + \epsilon \mathbf{j}_0^2 \varphi, \tag{4.31}$$

where

$$\mathbf{j}_0 \equiv \varphi^\dagger \mathbf{S} \varphi / \varphi^\dagger \varphi. \tag{4.32}$$

The vector  $\mathbf{j}_0$  is time-independent, so the general solution of (4.31) takes the form

$$\varphi_m = \sum_l \varphi_m^{(l)} \exp(-i\omega_l t),$$

where  $\varphi^{(l)}$  denotes the eigenvectors of  $\mathbf{j}_0 \cdot \mathbf{S}$ ,

$$\mathbf{j}_0 \cdot \mathbf{S} \varphi^{(l)} = l |\mathbf{j}_0| \varphi^{(l)} \tag{4.33}$$

with  $l = -\frac{3}{2}, -\frac{1}{2}, +\frac{1}{2}, +\frac{3}{2}$ , and

$$\omega_l = -2\epsilon l |\mathbf{j}_0| + \epsilon |\mathbf{j}_0|^2. \tag{4.34}$$

This is a quasi-periodic time-dependence, with just three independent characteristic frequencies

$$\frac{1}{2} \mu B, \quad \epsilon |\mathbf{j}_0|, \quad \epsilon |\mathbf{j}_0|^2 \tag{4.35}$$

which appear in the components of  $\psi_m$  in just 16 integer linear combinations. (The

quasi-periodic behavior for  $B=0$  was discovered in numerical calculations by M. Marder, before it was realized that this is actually an integrable system.) The constant  $|\mathbf{j}_0|$  can take any value in the range from 0 to  $\frac{3}{2}$ , depending on the initial conditions. The appearance of two additional frequencies in the time-dependence of the wave function, which unlike  $\mu B/2$  depend on initial conditions, would be a sign of departures from the linearity of quantum mechanics.

The energy eigenvalues here are found as usual by looking for time-dependent solutions that oscillate with just a single frequency  $E$ . These are of two types:

1.  $|\varepsilon \mathbf{j}_0| \neq |\mu B|$ . In order to avoid a multiplicity of frequencies, the energy eigenstate here must have just a single non-vanishing component  $\psi_m$ , so that  $\mathbf{j}_0$  is along the  $z$ -axis, with  $z$ -component  $m$ . Equation (4.33) shows that here  $l = |m|$ , so the energy eigenvalue is

$$E = \mu B M - \varepsilon m^2. \quad (4.36)$$

We again find a departure from the equal-spacing rule of ordinary quantum mechanics.

2.  $|\varepsilon \mathbf{j}_0| = |\mu B|$ . Here we can have an arbitrary mixture of components  $\psi_m$ , by arranging that the  $l$ -dependent term in (4.34) cancel the time-dependence in (4.30), by restricting  $l$  to  $\pm m$ , where  $\pm$  is the sign of  $\mu B/\varepsilon$ . This leaves us with a common oscillation frequency

$$E = \varepsilon |\mathbf{j}_0|^2 = \mu^2 B^2 / 4\varepsilon. \quad (4.37)$$

The wave function here is restricted only by the condition that  $\mathbf{j}_0$  should be in the  $z$ -direction, with  $|\mathbf{j}_0| = |\mu B/2\varepsilon|$ . To make the  $x$ - and  $y$ -components of  $\mathbf{j}_0$  vanish, we need only arrange that there are no components of the wave function with  $m$ -values differing by only one unit; the non-vanishing components can have  $m = -\frac{3}{2}$  and  $m = \frac{1}{2}$ , or  $m = -\frac{3}{2}$  and  $m = \frac{3}{2}$ , or  $m = -\frac{1}{2}$  and  $m = +\frac{3}{2}$ . We can then satisfy the condition that  $2\varepsilon j_{0z} = \mp \mu B$  by an appropriate choice of the relative magnitude of the two non-vanishing components of the wave function (the relative phase is arbitrary), provided that  $|\mu B/2\varepsilon| < \frac{3}{2}$ , i.e., provided that

$$|\varepsilon| > |\mu B|/3. \quad (4.38)$$

(To have non-vanishing components with  $m = \pm \frac{3}{2}$  and  $m = \mp \frac{1}{2}$ , we need the more stringent condition that  $|\varepsilon| > |\mu B|$ .) These results are typical of such eigenvalue problems; for small non-linearities there are just the four eigenvalues (4.36), but for  $|\varepsilon|$  above the threshold (4.38), there appears an additional highly degenerate energy eigenvalue (4.37).

In the proposed spin-precession experiments [20, 21] as in earlier work [25, 26], the nucleus is subject to additional fields, due in part to interaction with the vessel

walls. As long as the environment (including any magnetic field) is axially symmetric, the extra terms in the Hamiltonian function may be approximated by

$$\Delta h = \sum_m |\psi_m|^2 \Omega_m. \tag{4.39}$$

(Typically,  $\Omega_m$  is of the form  $\Omega_m \simeq bm + cm^2$ .) Both (4.39) and (4.27) separately define integrable systems, but the sum  $h_0 + \Delta h$  yields a system that is not integrable. Although  $\Delta h$  is small in absolute terms, it is presumably much larger than the non-bilinear term  $h_0$ , so we may take  $\Delta h$  as our “unperturbed” Hamiltonian function and treat  $h_0$  as a small perturbation. However, this is not by itself enough for us to be able to use the theorems of Refs. [6, 7] to justify treating the system as if it were integrable. Strictly speaking, as discussed in the Appendix, for this purpose we would also need the initial wave function to be close to an eigenvector of  $\Delta h$ , i.e., to have one component  $\psi_m$  much larger from the others. Nevertheless, computer experiments [27] with  $h_0/\Delta h \approx 0.01$  indicate a behavior that seems very close to quasi-periodic for randomly chosen initial conditions—the Fourier transform of  $\psi_m(t)$  shows just a few very sharp peaks.

With this encouragement, we can try supposing that the inclusion of  $h_0$  in the Hamiltonian function leaves the behavior of the wave function essentially quasi-periodic and use the averaging method described in Section 2(d) to estimate the shift in the frequencies  $\Omega_m$ . Equation (4.39) is an integrable Hamiltonian function, with action and angle variables

$$J_m = |\psi_m|^2, \quad \Theta_m = \text{Arg } \psi_m \tag{4.40}$$

and frequencies

$$\omega_m \equiv \frac{\partial \Delta h}{\partial J_m} = \Omega_m. \tag{4.41}$$

Inserting the zeroth order solution  $\psi_m \propto \exp(-i\Omega_m t)$  in (4.28) and averaging over time gives

$$\begin{aligned} \langle h_0 \rangle = & -\frac{9\epsilon n}{4} + \frac{\epsilon}{n} \{ 6J_{3/2}J_{-1/2} + 2J_{1/2}^2 + 9J_{3/2}J_{-3/2} \\ & + J_{1/2}J_{-1/2} + 6J_{-3/2}J_{1/2} + 2J_{-1/2}^2 \} \end{aligned} \tag{4.42}$$

with

$$n = J_{-3/2} + J_{-1/2} + J_{1/2} + J_{3/2}. \tag{4.43}$$

This shifts the frequencies  $\omega_m$  by

$$\delta\omega_m = \frac{\partial \langle h_0 \rangle}{\partial J_m}. \tag{4.44}$$

Even the precession of the spin expectation value is affected by these frequency shifts, because although (4.28) is rotationally invariant, its average (4.42) is not [20]. Their dependence on the  $J$ 's is a symptom of a departure from the linearity of quantum mechanics.

## 5. PROBABILITIES AND EXPECTATION VALUES

In the historical development of quantum mechanics, the interpretation of the wave function in terms of probabilities emerged somewhat later than the interpretation of eigenvalues as the possible values of observables. In Section 3 we have already discussed the eigenvalue problem and its physical interpretation in our generalized version of quantum mechanics. As in quantum mechanics, it will take some extra work to establish the probabilistic interpretation of the wave function.

There are various ways of formulating the interpretative postulates of ordinary quantum mechanics. In one common version, it is assumed that the measurement of an observable represented by an operator  $A$  must give a result equal to one of the eigenvalues of  $A$ , with a probability equal to the absolute value squared of the component of the wave function along the corresponding eigenvector. This assumption will not work in our generalized version of quantum mechanics; the analysis below of the measurement process will show that the possible values obtained when we measure some observable in a general state depend on the wave function of that state.

However, there is another way of stating the interpretative postulates of quantum mechanics, actually used by Dirac in his classic book [28] and occasionally in textbooks [29] since then, that carries over very nicely to the generalized version of quantum mechanics studied here. In ordinary quantum mechanics, one can postulate that in a state represented by a wave function  $\psi_k$ , any measurement of a set of observables  $A, B, \dots$  represented by commuting Hermitian operators (in matrix notation)  $A_{kl}, B_{kl}, \dots$ , will yield as average values, the expectation values  $\psi_k^* A_{kl} \psi_l / \psi_k^* \psi_k, \psi_k^* B_{kl} \psi_l / \psi_k^* \psi_k$ , etc. As a natural generalization, we shall assume here that in a state represented by a wave function  $\psi_k$ , the measurement of a set of observables  $A, B, \dots$  represented by commuting real homogeneous functions  $a(\psi, \psi^*), b(\psi, \psi^*), \dots$  (commuting in the sense that  $a * b = b * a$ , etc.) will yield as average values, the expectation values

$$\langle a \rangle_\psi = \frac{a(\psi, \psi^*)}{n(\psi, \psi^*)}, \quad \langle b \rangle_\psi = \frac{b(\psi, \psi^*)}{n(\psi, \psi^*)}, \dots \quad (5.1)$$

This is just a postulate, but it has a number of physically plausible features:

(a) Because the functions  $a(\psi, \psi^*), b(\psi, \psi^*)$ , etc. as well as  $n(\psi, \psi^*)$  are all homogeneous of degree one in both  $\psi_k$  and  $\psi_k^*$ , the expectation values (5.1) are the same for wave functions  $\psi_k$  and  $Z\psi_k$  for arbitrary complex  $Z$ , as required if  $\psi_k$  and  $Z\psi_k$  are to represent the same physical state.

(b) As discussed in Section 2(g), an observable which receives independent additive contributions from two separated systems is represented by a function of the form

$$a_{I+II}(\Psi, \Psi^*) = \sum_l a_I(\psi^{(l)}, \psi^{(l)*}) + \sum_k a_{II}(\varphi^{(k)}, \varphi^{(k)*}), \quad (5.2)$$

where  $\Psi_{kl}$  is the joint wave function (the first and second indices describing degrees of freedom of systems I and II, respectively) and

$$\psi_k^{(l)} \equiv \varphi_l^{(k)} \equiv \Psi_{kl}. \quad (5.3)$$

If there is no correlation between the subsystems, then  $\Psi_{kl}$  takes the separable form

$$\Psi_{kl} = \psi_k \varphi_l. \quad (5.4)$$

Using the homogeneity of these functions, we see that for a wave function of this form, Eq. (5.2) has the value

$$\begin{aligned} a_{I+II}(\Psi, \Psi^*) &= \left( \sum_l |\varphi_l|^2 \right) a_I(\psi, \psi^*) \\ &+ \left( \sum_k |\psi_k|^2 \right) a_{II}(\varphi, \varphi^*). \end{aligned} \quad (5.5)$$

Also, the norm here is

$$n(\Psi, \Psi^*) = \sum_{kl} |\Psi_{kl}|^2 = \left( \sum_l |\varphi_l|^2 \right) \left( \sum_k |\psi_k|^2 \right), \quad (5.6)$$

so the expectation value is simply a sum of the expectation values for the separated systems

$$\langle a_{I+II} \rangle_{\Psi} = \langle a_I \rangle_{\psi} + \langle a_{II} \rangle_{\varphi} \quad (5.7)$$

as physically required if these systems are really independent.

(c) Suppose a state is represented by a wave function  $\psi_k$  that satisfies the eigenvalue condition

$$\frac{\partial a(\psi, \psi^*)}{\partial \psi_k^*} = \alpha \psi_k.$$

Using the homogeneity of  $a(\psi, \psi^*)$ , we have then

$$a(\psi, \psi^*) \psi_k = \psi_k^* \frac{\partial a}{\partial \psi_k^*} = \alpha \psi_k^* \psi_k$$

and so the expectation value of this observable is

$$\langle a \rangle_\psi = \alpha \quad (5.8)$$

in accord with our earlier assumption that the observable has definite value  $\alpha$  in this state.

(d) We saw in Section 3 that the stationary points of the function  $\bar{a}(\psi, \psi^*) \equiv a(\psi, \psi^*)/n(\psi, \psi^*)$  on the compact manifold  $CP^{N-1}$  are the eigenvalues of the function  $a(\psi, \psi^*)$ . The expectation value  $\langle a \rangle_\psi = \bar{a}(\psi, \psi^*)$  in a state represented by an arbitrary wave function  $\psi_k$  must lie between the maximum and minimum of  $\bar{a}(\psi, \psi^*)$  on this manifold, and hence between the largest and smallest eigenvalues

$$\alpha_{\min} \leq \langle a \rangle_\psi \leq \alpha_{\max}. \quad (5.9)$$

In ordinary quantum mechanics, we could use the postulated formula for expectation values to infer the probability distribution for values of any observable in any given state. This is because any matrix  $A_{kl}$  commutes with all its powers, so all powers of the observable can be simultaneously measured. The usual result for the probability distribution (that the observable takes values equal to its various eigenvalues, with probabilities equal to the absolute value squared of the component of the normalized wave function along the eigenvector) can then be justified by noting that it gives the right expectation values for all powers of the observable. In particular, if a state is represented by an eigenvector of some observable, then the expectation value of any power of the observable is equal to that power of the eigenvalue, from which we can infer that in such a state the observable takes a unique value, equal to that eigenvalue, without needing this as a separate postulate.

In contrast, in our generalized version of quantum mechanics, because of the failure of associativity, a function  $a(\psi, \psi^*)$  will not in general commute with its "powers"  $a * a$ ,  $a * (a * a)$ , etc. Indeed, although the square of  $a$  is unambiguously  $a * a$ , and its cube can be fixed by a reality requirement as

$$\frac{1}{2}[a * (a * a) + (a * a) * a],$$

there are three independent candidates for a real fourth power

$$(a * a) * (a * a)$$

$$\frac{1}{2}\{a * [(a * a) * a] + [a * (a * a)] * a\}$$

$$\frac{1}{2}\{a * [a * (a * a)] + [(a * a) * a] * a\},$$

so there is no unique way to deduce a probability distribution from the expectation values of the powers of the observable. (A similar problem arises at a different level in ordinary quantum mechanics; although it is possible to infer the probability distribution for an observable from the expectation values of its powers, it is not possible to deduce the *joint* probability distribution of two observables from the

expectation values of the products of their powers, unless the operators representing these observables commute.)

Fortunately, even in our generalized quantum mechanics there are some observables that commute with all their powers; these are the observables which symmetry principles require to be represented (up to a homogeneous canonical transformation) by bilinear functions

$$a(\psi, \psi^*) = \psi_k^* A_{kl} \psi_l.$$

We guessed in Section 2 that observables like momentum and angular momentum are in this class. The usual arguments of quantum mechanics apply to such observables and show that a measurement of such an observable must yield one of the eigenvalues of  $A_{kl}$  as the measured value, with a probability equal to the absolute value of the square of the component of the normalized wave function  $\psi_k$  along the corresponding eigenvector. We can exploit this fact to work out the probability distribution for an arbitrary observable, if it is measured by allowing it to interact with one of these bilinearly represented observables.

An example is provided by experiments of the Stern–Gerlach variety. Suppose that we wish to measure some observable  $A$  (like a component of the magnetic moment; see Section 4) that is represented by a *non-bilinear* function  $a(\psi, \psi^*)$ , with  $\psi$  an  $N$ -component wave function. Send a beam of the systems (e.g., atoms or nuclei) that carry this observable through an external field, so that a term is added to the system’s Hamiltonian function

$$\Delta h(\psi, \psi^*) = gx a(\psi, \psi^*), \tag{5.10}$$

where  $x$  is the coordinate in some direction transverse to the beam and  $g$  is a coupling parameter. Take the constant  $g$  large enough so that  $\Delta h$  dominates over other terms in the Hamiltonian function. The time-dependence of the wave function is then given by

$$i \frac{d\psi_k}{dt} = gx \frac{\partial a}{\partial \psi_k^*}. \tag{5.11}$$

As discussed in Section 2, the solution of Eq. (5.11) is the same as if the equation were integrable for a wide range of initial conditions. That is, we typically find

$$\psi_k = \sum_{n_1 n_2 \dots} c_k(n_1 n_2 \dots) \exp\left(-igt \sum_v n_v \alpha_v\right) \tag{5.12}$$

(The  $\alpha_v$  are the values of the frequency  $\omega_v$  in (2.15) if the  $a(\psi, \psi^*)$  were the Hamiltonian function.) We see that the beam breaks up into an infinite number of sub-beams, one for each set of the  $N$  integers  $n_v$ , each sub-beam with a transverse momentum which after a time  $t$  has the value

$$\Delta p_x = -gt \sum_v n_v \alpha_v \tag{5.13}$$

with probability

$$P(n_1 n_2 \cdots) = \sum_k |c_k(n_1 n_2 \cdots)|^2. \quad (5.14)$$

Interpreting this as a measurement of the observable  $A$ , we would conclude that the possible values are

$$A_n = \sum_v n_v \alpha_v, \quad (5.15)$$

each with probability (5.14).

We can now check that the probability distribution we have deduced for results of the Stern–Gerlach experiment are consistent with our general assumptions about expectations values. First, note that the norm  $n$  is

$$\begin{aligned} n(\psi, \psi^*) &\equiv \sum_k |\psi_k|^2 = \sum_k \sum_{n_1 n_2 \cdots} \sum_{n'_1 n'_2 \cdots} c_k(n_1 n_2 \cdots) c_k^*(n'_1 n'_2 \cdots) \\ &\times \exp\left(-igxt \sum_v (n_v - n'_v) \alpha_v\right). \end{aligned} \quad (5.16)$$

Also, using the homogeneity condition, the function  $a(\psi, \psi^*)$  takes the value

$$\begin{aligned} a(\psi, \psi^*) &= \sum_k \psi_k^* \frac{\partial a}{\partial \psi_k^*} = \frac{i}{gx} \sum_k \psi_k^* \frac{d\psi_k}{dt} \\ &= \sum_k \sum_{n_1 n_2 \cdots} \sum_{n'_1 n'_2 \cdots} c_k(n_1 n_2 \cdots) c_k^*(n'_1 n'_2 \cdots) \\ &\times \left(\sum_v n_v \alpha_v\right) \exp\left(-igxt \sum_v (n_v - n'_v) \alpha_v\right). \end{aligned} \quad (5.17)$$

But on general grounds, both  $n$  and  $a$  (which here plays the role of the Hamiltonian) must be time-independent. As long as we avoid resonant initial conditions, for which these solutions are not valid anyway, there are no non-zero integers  $m_v$  for which  $\sum_v m_v \alpha_v$  vanishes, so the only time-independent terms in (5.16) and (5.17) which could contribute to  $n$  and  $a$  are those with  $n'_v = n_v$  for all  $v$ . We thus have

$$n = \sum_k \sum_{n_1 n_2 \cdots} |c_k(n_1 n_2 \cdots)|^2 \quad (5.18)$$

$$a = \sum_k \sum_{n_1 n_2 \cdots} |c_k(n_1 n_2 \cdots)|^2 \sum_v n_v \alpha_v. \quad (5.19)$$

The expectation value of  $A$  is then given by (5.15) and (5.14) as

$$\begin{aligned} & \sum_{n_1 n_2 \dots} P(n_1 n_2 \dots) \sum_{n_1 n_2 \dots} n_v \alpha_v \bigg/ \sum_{n_1 n_2 \dots} P(n_1 n_2 \dots) \\ &= \frac{\sum_k \sum_{n_1 n_2 \dots} |c_k(n_1 n_2 \dots)|^2 \sum_v n_v \alpha_v}{\sum_k \sum_{n_1 n_2 \dots} |c_k(n_1 n_2 \dots)|^2} \\ &= a/n \end{aligned} \tag{5.20}$$

in agreement with our earlier assumptions.

To check our previous assumptions a bit more thoroughly, we should also consider how to measure several commuting observables. Consider two observables, represented by functions  $a(\psi, \psi^*)$  and  $b(\psi, \psi^*)$ , that commute in the sense that  $a * b = b * a$ . We couple these to external fields in such a way that the Hamiltonian function is dominated by an interaction term

$$\Delta h = gxa + hyb, \tag{5.21}$$

where  $x$  and  $y$  are coordinates in different directions transverse to the beam, and  $g$  and  $h$  are coupling parameters. We again treat the system as if it were integrable. Because  $a$  and  $b$  commute (that is, "in involution") it is possible to find a set of action and angle variables  $j_v$  and  $\theta_v$ , that are related by a canonical transformation to  $\sqrt{2} \text{Im } \psi_k$  and  $\sqrt{2} \text{Re } \psi_k$ , respectively, such that both  $a$  and  $b$  depend only on  $j_v$ , not  $\theta_v$ . The wave junction is given in terms of the  $j_v$  and  $\theta_v$  by an expression of form

$$\psi_k(j, \theta) = \sum_{n_1 n_2 \dots} c_k(j_1 j_2 \dots; n_1 n_2 \dots) \exp\left(-i \sum_v n_v \theta_v\right).$$

If the Hamiltonian were dominated by  $a$  or  $b$  alone, then the time-dependence of  $\theta_v$  would be given by  $\dot{\theta}_v = \partial a / \partial j_v \equiv \alpha_v$  or  $\dot{\theta}_v = \partial b / \partial j_v \equiv \beta_v$ , respectively. With the Hamiltonian dominated by (5.21), the time-dependence is

$$\begin{aligned} \frac{d\theta_v}{dt} &= \frac{\partial}{\partial j_v} (gxa(j) + hyb(j)) \\ &= gx\alpha_v(j) + hy\beta_v(j). \end{aligned}$$

The beam thus breaks up into an infinite number of sub-beams, one for each set of the  $N$  integers  $n_v$ , each sub-beam with a transverse momentum which after a time  $t$  has the components

$$\begin{aligned} \Delta p_x &= -g \sum_v n_v \alpha_v \\ \Delta p_y &= -h \sum_v n_v \beta_v \end{aligned}$$

with probability (5.14). We can now repeat the previous analysis and confirm that

the mean values of the observables represented by  $a$  and  $b$ , as inferred from measurements of  $\Delta p_x$  and  $\Delta p_y$ , are just  $a/n$  and  $b/n$ , respectively, as assumed earlier.

## 6. SPECTRAL LINES

There are various ways of measuring the spectrum of an atomic or nuclear system. One may observe the resonant absorption or scattering of individual photons by the system, or, at an opposite extreme, one may observe the effects on the system of a macroscopic oscillating electromagnetic field, containing unlimited numbers of photons. Also, the atomic or nuclear system may be at zero temperature, and hence in the lowest energy eigenstate, or at a high temperature, in which case the individual atoms or nuclei are almost uniformly distributed among all possible superpositions of states, with only a small excess of lower energy states. Of course, the distribution may also be non-thermal, as in the measurement of emission spectra, and here also individual atoms or nuclei may or may not be initially in pure energy eigenstates.

In ordinary quantum mechanics, none of these experimental details make much difference in the interpretation of observed spectral frequencies: we always get absorption or emission lines at frequencies simply equal to differences of energy eigenvalues. As we shall see here, matters are very different in our non-linear generalization of quantum mechanics.

To explore the various possibilities, let us consider the simplest realistic case. We assume that the atomic or nuclear system may be described by a 2-component wave function  $\psi_k$ , with  $k = 1$  or  $2$ . (This should be a good approximation even in more complicated systems, if only two components of  $\psi$  are appreciably excited.) Also, we will assume that only one normal mode of the electromagnetic field is excited, with frequency  $\omega$ . The wave function of the system of matter plus radiation is then  $\Psi_{k,n}$ , where  $k = 1$  or  $2$ , and  $n = 0, 1, 2, \dots$  gives the number of photons in the normal mode in question. In accordance with the general considerations of Section 2, the Hamiltonian is taken in the form

$$h_{\text{TOT}}(\Psi, \Psi^*) = \sum_n h(\Psi_{1,n}, \Psi_{2,n}, \Psi_{1,n}^*, \Psi_{2,n}^*) + \sum_{k,n} n\omega \Psi_{k,n}^* \Psi_{k,n} + h_{\text{INT}}(\Psi, \Psi^*). \quad (6.1)$$

Here  $h(\psi_1, \psi_1^*, \psi_2, \psi_2^*)$  is the Hamiltonian function in the absence of radiation, and  $h_{\text{INT}}$  represents the effects of the interaction of radiation and matter. Since this interaction is assumed here to be a weak perturbation, we take it to have the form it would have in ordinary quantum mechanics

$$h_{\text{INT}}(\Psi, \Psi^*) = \sum_{nkl} d_{kl} \Psi_{k,n-1}^* \Psi_{l,n} \sqrt{n} + \text{c.c.} \quad (6.2)$$

with  $d_{ki}$  a constant  $2 \times 2$  matrix characterizing the interaction between matter and radiation.

For simplicity, let us also assume an angular-momentum selection rule, which requires the matter Hamiltonian function  $h$  to depend only on  $|\Psi_{1,n}|^2$  and  $|\Psi_{2,n}|^2$ , and allows single-photon absorption or emission only in transitions  $1 \rightarrow 2$  and  $2 \rightarrow 1$ , respectively, so that the only non-vanishing component of  $d_{ki}$  is  $d_{21} \equiv d$ . (For instance, we might suppose that in the single normal mode of our problem, photons have a pure helicity,  $+1$  or  $-1$ , and that components  $k = 1, 2$  refer to states of the matter system whose angular momentum in the photon direction differs by one unit.) The time-dependence of the system is then governed by the equations

$$i \frac{d\Psi_{1,n}}{dt} = \Psi_{1,n} h_{,1}(|\Psi_{1,n}|^2, |\Psi_{2,n}|^2) + n\omega \Psi_{1,n} + d^* \sqrt{n} \Psi_{2,n-1} \tag{6.3}$$

$$i \frac{d\Psi_{2,n}}{dt} = \Psi_{2,n} h_{,2}(|\Psi_{1,n}|^2, |\Psi_{2,n}|^2) + n\omega \Psi_{2,n} + d \sqrt{n+1} \Psi_{1,n+1}, \tag{6.4}$$

where

$$h_{,k}(|\psi_1|^2, |\psi_2|^2) \equiv \frac{\partial h(|\psi_1|^2, |\psi_2|^2)}{\partial |\psi_k|^2}.$$

In the absence of radiation, the matter system would have energy eigenvalues

$$h_{,1}(|\psi_1|^2, 0) = E_1, \quad h_{,2}(0, |\psi_2|^2) = E_2. \tag{6.5}$$

(Homogeneity makes these derivatives independent of their arguments.)

Let us first consider the simplest interesting case, in which initially there is just one photon, and the matter system is in its  $k = 1$  energy eigenstate. The only non-zero component of the wave function is initially just  $\Psi_{1,1}$ , and the only other component that becomes excited is  $\Psi_{2,0}$ , representing photon absorption. With only these two components non-zero, the time-dependence equations (6.3), (6.4) become

$$i \frac{d\Psi_{1,1}}{dt} = (E_1 + \omega) \Psi_{1,1} + d^* \Psi_{2,0} \tag{6.6}$$

$$i \frac{d\Psi_{2,0}}{dt} = E_2 \Psi_{2,0} + d \Psi_{1,1}. \tag{6.7}$$

These are just the same equation that would apply in ordinary quantum mechanics. As is well known, for an arbitrary weak coupling  $d$ , the matter system is driven by

photon absorption from  $k = 1$  to  $k = 2$ , provided we wait a time of order  $1/|d|$  and provided that  $\omega$  is chosen to have the value

$$\omega = E_2 - E_1 \quad (6.8)$$

to within a width of order  $|d|$ .

Matters are much more complicated if we assume that the wave function in the initial one-photon state is a mixture of  $k = 1$  and  $k = 2$  components,  $\Psi_{1,1}$  and  $\Psi_{2,1}$ . The growth of the no-photon component  $\Psi_{2,0}$  is still given by Eq. (6.7), but the decay of this state back to  $\Psi_{1,1}$  is now governed by

$$i \frac{d\Psi_{1,1}}{dt} = [h_{,1}(|\Psi_{1,1}|^2, |\Psi_{2,1}|^2) + \omega] \Psi_{1,1} + d^* \Psi_{2,0}. \quad (6.9)$$

To evaluate the right-hand side we need to calculate the change in the other one-photon amplitude  $\Psi_{2,1}$ , which is governed by its coupling to the two-photon amplitude  $\Psi_{2,2}$ .

Instead of pursuing this further now, we shall consider a different sort of complication, of greater experimental interest: the initial state may contain many photons. Consider a range of photon numbers  $n$  that are so large that  $d\sqrt{n}$  and  $d\sqrt{n-1}$  may both be taken as approximately equal to an  $n$ -independent constant

$$d\sqrt{n} \simeq d\sqrt{n-1} \simeq g. \quad (6.10)$$

Equations (6.3) and (6.4) then have solutions of the form

$$\Psi_{k,n} = \psi_k \exp(-i\omega t) \quad (6.11)$$

with  $\psi_k$  independent of  $n$  and satisfying the equations

$$i \frac{d\psi_1}{dt} = \psi_1 h_{,1}(|\psi_1|^2, |\psi_2|^2) + g^* e^{i\omega t} \psi_2 \quad (6.12)$$

$$i \frac{d\psi_2}{dt} = \psi_2 h_{,2}(|\psi_1|^2, |\psi_2|^2) + g e^{-i\omega t} \psi_1. \quad (6.13)$$

These are just the equations we would derive from an effective Hamiltonian function

$$h_{\text{EFF}} = h(|\psi_1|^2, |\psi_2|^2) + g e^{-i\omega t} \psi_2^* \psi_1 + \text{c.c.} \quad (6.14)$$

corresponding to a matter system with Hamiltonian function  $h$  in a classical external field that oscillates with frequency  $\omega$ .

In the absence of the time-varying perturbation term in (6.14), the components  $\psi_k$  would simply oscillate as  $\exp(-i\omega_k t)$ , with characteristic frequencies given by

$$\omega_k = h_{,k}(|\psi_1|^2, |\psi_2|^2). \quad (6.15)$$

The time-varying perturbation term in (6.14) will therefore have a time dependence given approximately by

$$ge^{-i\omega t}\psi_2^*\psi_1 \propto \exp(-i(\omega - \omega_1 + \omega_1)t).$$

Even if  $g$  is very small, the effects of this perturbation term will continue to grow over a time  $T$ , provided that this factor does not average to zero over this time, i.e., provided that

$$|\omega - \omega_2 + \omega_1| \lesssim 1/T. \tag{6.16}$$

Because of the non-linearities in the unperturbed Hamiltonian function  $h(|\psi_1|^2, |\psi_2|^2)$ , the frequency difference  $\omega_2 - \omega_1$  varies by a small amount  $\Delta\omega$  as  $|\psi_1|^2/|\psi_2|^2$  varies. We can try to observe this variation, as a sign of departure from quantum mechanics, either by measuring resonant absorption in a sample of atoms or nuclei with differing initial values of the ratio  $|\psi_1|^2/|\psi_2|^2$ , or by allowing the transition to continue so long that this ratio changes (or both). Either way, if want to observe the variation of the frequency difference  $\omega_2 - \omega_1$  over a range  $\Delta\omega$ , we need to observe the system for a time  $T$  at least as long as  $1/\Delta\omega$ ,

$$T \gtrsim 1/\Delta\omega. \tag{6.17}$$

Before considering the details of these different sorts of experiment, this is a good place to pause and comment on the relation between the frequency difference  $\omega_2 - \omega_1$  and the difference  $E_2 - E_1$  of energy eigenvalues. It will be convenient from now on to make the homogeneity of the Hamiltonian function manifest, writing

$$h = n\bar{h}(a), \tag{6.18}$$

where, as in Section 2,

$$n \equiv |\psi_1|^2 + |\psi_2|^2 \tag{6.19}$$

$$a \equiv |\psi_2|^2/n. \tag{6.20}$$

From (2.53) and (2.54), we see that the frequency difference is

$$\omega_2(a) - \omega_1(a) = \bar{h}'(a). \tag{6.21}$$

On the other hand, (3.10) and (3.11) give the energy difference as

$$E_2 - E_1 = \bar{h}(1) - \bar{h}(0). \tag{6.22}$$

We see that energy difference is the *average* frequency difference

$$E_2 - E_1 = \int_0^1 [\omega_2(a) - \omega_1(a)] da, \tag{6.23}$$

but the quantity measured in individual experiments may have nothing directly to do with the energy difference. Even if we start in a pure energy eigenstate, with  $a=0$  or  $a=1$ , the resonant frequency (6.21) is not generally exactly equal to the energy difference (6.22).

It may appear as a violation of the conservation of energy that the resonant photon absorption frequency is in general not precisely equal to the energy difference  $E_2 - E_1$ , but it should be kept in mind that the initial and final states of the radiation field are assumed here to not have a definite energy, since they contain an infinite number of photons. We say earlier that when the radiation field initially contains just one photon, the photon energy  $\omega$  required to drive a transition from one energy eigenstate to the other is precisely  $E_2 - E_1$ .

Now let us consider experiments designed to detect the changes in the characteristic frequency difference due either to (A) changes in the initial wave function or (B) changes in the wave function as it evolves.

#### (A) Variations in Initial Characteristic Frequency

Suppose that we try to detect the dependence of the characteristic frequency on the *initial* wave function by exposing our system to the time-varying perturbation in (6.14) for a time  $T$ , which is long enough to detect a spread in the resonant frequency  $\omega_2 - \omega_1$  of order  $\Delta\omega$ , and yet short enough so that the action parameter  $a$  on which  $\omega_2 - \omega_1$  depends does not change very much during this period. That is, we want

$$(\Delta\omega)^{-1} \ll T \ll |g|^{-1}. \quad (6.24)$$

We can then apply perturbation theory to the solution of Eqs. (6.12) and (6.13). These equations may be rewritten in the form

$$\frac{da}{dt} = f(a) \sin \alpha \quad (6.25)$$

$$\frac{d\alpha}{dt} = \hbar'(a) - \omega + f'(a) \cos \alpha, \quad (6.26)$$

where  $a$  is the variable (6.20),  $\alpha$  is the angle,

$$\alpha = \text{Arg}[g\psi_1/\psi_2] - \omega t, \quad (6.27)$$

and  $f(a)$  is the function,

$$f(a) = 2 |g| \sqrt{a(1-a)}. \quad (6.28)$$

(The function  $f(a)$  would be different if we allowed non-bilinear terms in the interaction (6.2) between matter and radiation, but as we shall see, the most important features of the solution would be unaffected.)

To zeroth order in  $g$ , the solution is simply

$$a(t) = a_0 \tag{6.29}$$

$$\alpha(t) = \alpha_0 - \Omega_0 t \equiv \phi(t), \tag{6.30}$$

where

$$\Omega_0 \equiv \omega - \bar{h}'(a_0) \tag{6.31}$$

and  $a_0$  and  $\alpha_0$  are the initial values for  $a(t)$  and  $\alpha(t)$ . Inserting these on the right-hand side of (6.25) and (6.26), integrating, and then repeating the process, we obtain  $a(t)$  and  $\alpha(t)$  as power series' in  $g$ :

$$\begin{aligned} a(t) = & a_0 + f(a_0)[\cos \phi(t) - \cos \alpha_0]/\Omega_0 \\ & + f(a_0) f'(a_0)(1 - \cos \Omega_0 t)/\Omega_0^2 \\ & + f^2(a_0) \bar{h}''(a_0) \left[ \frac{(\sin \phi(t) - \sin \alpha_0)^2}{2\Omega_0^3} \right. \\ & \left. + \frac{\cos \alpha_0(\cos \alpha_0 - \cos \phi(t))}{\Omega_0^3} + \frac{t \cos \alpha_0 \sin \phi(t)}{\Omega_0^2} \right] \\ & + O(g^3) \end{aligned} \tag{6.32}$$

$$\begin{aligned} \alpha(t) = & \phi(t) + \left[ -\frac{f'(a_0)}{\Omega_0} + \frac{f(a_0) \bar{h}''(a_0)}{\Omega_0^2} \right] (\sin \phi(t) - \sin \alpha_0) \\ & + O(g^2). \end{aligned} \tag{6.33}$$

The absorption of radiation can be calculated from the change with time of the energy expectation value  $\bar{h}(a(t))$ , which is easily calculated as a power series in  $g$  by use of Eq. (6.32). The result is complicated and not especially illuminating, but in most cases of experimental interest the individual atoms or molecules will have randomly distributed phases of the initial wave function components  $\psi_k$ , and hence randomly distributed values of  $\alpha_0$ . Averaging  $\bar{h}(a(t))$  over  $\alpha_0$ , we obtain the fairly simple result

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \bar{h}(a(t)) d\alpha_0 & \equiv \langle \bar{h}(a(t)) \rangle \\ & = \bar{h}(a_0) + \frac{d}{da_0} \left[ \frac{f^2(a_0) \bar{h}'(a_0)(1 - \cos \Omega_0 t)}{2\Omega_0^2} \right]. \end{aligned} \tag{6.34}$$

At very high temperatures we would expect the atoms or molecules to be initially distributed uniformly in "phase space" on the surface of fixed norm:

$$\begin{aligned}
 dN &\propto d \operatorname{Re} \psi_1(0) d \operatorname{Im} \psi_1(0) d \operatorname{Re} \psi_2(0) \\
 &\quad \times d \operatorname{Im} \psi_2(0) \delta(|\psi_1(0)|^2 + |\psi_2(0)|^2 - 1) \\
 &\propto da_0 d\alpha_0
 \end{aligned} \tag{6.35}$$

We have already averaged over  $\alpha_0$ , and if we now also average over  $a_0$ , we find that the second term in (6.34) drops out (because  $f(a_0)$  vanishes at the end points  $a_0 = 0$  and  $a_0 = 1$ ), so

$$\langle\langle \hbar(a(t)) \rangle\rangle \equiv \int_0^1 da_0 \langle \hbar(a(t)) \rangle = \langle\langle \hbar(a(0)) \rangle\rangle. \tag{6.36}$$

That is, just as in ordinary quantum mechanics, the material system absorbs no energy from radiation at high temperature, because induced emission cancels absorption. There is a net absorption of energy at finite temperatures, where  $dN$  contains a Boltzmann factor. In this case, the average energy of the atoms or nuclei is

$$\begin{aligned}
 \langle\langle \hbar(a(t)) \rangle\rangle &\equiv \int_0^1 da_0 e^{-\beta \hbar(a_0)} \langle \hbar(a(t)) \rangle \Big/ \int_0^1 da_0 e^{-\beta \hbar(a_0)} \\
 &= \langle\langle \hbar(a(0)) \rangle\rangle + \beta \int_0^1 da_0 \\
 &\quad \times \frac{f^2(a_0) \hbar'^2(a_0) [1 - \cos \Omega_0 t]}{2\Omega_0^2} + O(\beta^2).
 \end{aligned} \tag{6.37}$$

Note that the second term is always positive, because at finite temperatures absorption exceeds induced emission.

Now let us give  $t$  a value  $T$  that is much larger than the reciprocal of the range  $\Delta\omega$  over which  $\hbar'(a_0)$  and hence  $\Omega_0$  varies as  $a_0$  runs from 0 to 1. Then effectively the factor  $(1 - \cos \Omega_0 t)/\Omega_0^2$  in (6.37) is proportional to a delta function

$$(1 - \cos \Omega_0 T)/\Omega_0^2 \rightarrow \pi T \delta(\Omega_0). \tag{6.38}$$

The rate at which radiation energy is absorbed is then

$$\begin{aligned}
 \Gamma &\equiv \frac{d}{dT} \langle\langle \hbar(a(T)) \rangle\rangle \\
 &= \frac{1}{2} \beta \pi \int_0^1 da_0 f^2(a_0) \hbar'^2(a_0) \delta(\Omega_0) \\
 &= \frac{1}{2} \beta \pi f^2(a_\omega) \hbar'^2(a_\omega) / |\hbar''(a_\omega)| \\
 &= 2\beta \pi |g|^2 a_\omega (1 - a_\omega) \hbar'^2(a_\omega) / |\hbar''(a_\omega)|,
 \end{aligned} \tag{6.39}$$

where  $a_\omega$  is defined by

$$\bar{h}'(a_\omega) \equiv \omega. \tag{6.40}$$

For values of  $\omega$  at which Eq. (6.40) has no solutions with  $0 < a_\omega < 1$ , the absorption rate vanishes.

We see that the absorption line extends over the frequency range from  $\bar{h}'(0)$  to  $\bar{h}'(1)$ . The precise line shape depends on the function  $\bar{h}(a)$ . For instance, for the simplest possible departure from quantum mechanics, we would have  $\bar{h}(a)$  quadratic,

$$\begin{aligned} \bar{h}(a) &= \bar{h}(0) + (E_2 - E_1)a + \varepsilon a(1 - a) \\ |\varepsilon| &\ll E_2 - E_1. \end{aligned} \tag{6.41}$$

(Recall that  $E_2 - E_1 = \bar{h}(1) - \bar{h}(0)$ .) Then (6.39) and (6.40) would give

$$\begin{aligned} \Gamma_\omega &\simeq 2\beta\pi |g|^2 (E_2 - E_1)^2 a_\omega(1 - a_\omega)/|\varepsilon| \\ &\propto 1 - (\omega - E_2 + E_1)^2/\varepsilon^2. \end{aligned} \tag{6.42}$$

In this case the line has width  $|\varepsilon|$  and is centered on the energy difference  $E_2 - E_1$ . This broadening of absorption lines for systems in a distribution of initial states is a general symptom of the presence of non-linear corrections to quantum mechanics. No such broadening was seen in the experiment of Ref. [22], within the natural width of the line. The interpretation of this experiment is complicated, because the nucleus has spin  $\frac{3}{2}$ , so there are three absorption transitions going on at the same time and also because the experiment searched for a line splitting that would vary with sidereal time. However, from the absence of anomalous broadening, we can presumably infer that the nonlinearity parameter here must be less than about  $10^{-20}$  MeV. It should be possible to do even better than this in an experiment designed for the purpose.

(B) *Variation of Characteristic Frequency with Time: Detuning*

We now suppose that the individual atoms or nuclei all start in the same initial state and, hence, with the same characteristic frequency difference  $\omega_2 - \omega_1$ , and we examine the effects of changes in this frequency as the wave function evolves. To study this evolution, we return to Eqs. (6.25) and (6.26) and note that these equations can be integrated to give the phase  $\alpha$  as a function of  $a$ ,

$$f(a) \cos \alpha = \omega a - \bar{h}(a) + C, \tag{6.43}$$

where  $C$  is an integration constant. Combining this with (6.25) gives

$$\left(\frac{da}{dt}\right)^2 = f^2(a) - (\omega a - \bar{h}(a) + C)^2. \tag{6.44}$$

For definiteness, suppose we start our system in the energy eigenstate  $a = 0$ , with  $E = E_1$ . Then, since  $f(0) = 0$ , Eq. (6.44) forces us to take the integration constant as

$$C = \hbar(0). \quad (6.45)$$

We now must adopt a positive sign for  $da/dt$ , so (6.44) yields

$$\frac{da}{dt} = + [f^2(a) - (\omega a - \hbar(a) + \hbar(0))^2]^{1/2}. \quad (6.46)$$

The transition will proceed all the way to the other energy eigenstate  $a = 1$ , with  $E = E_2$ , provided that two conditions are met. First, since  $f(1) = 0$ , it is necessary for the second term in the square root in (6.46) also to vanish when  $a = 1$ :

$$\omega = \hbar(1) - \hbar(0) = E_2 - E_1. \quad (6.47)$$

This condition would also be required in ordinary quantum mechanics. Here it is further necessary that the argument of the square root in (6.46) not change sign during the transition:

$$|f(a)| \geq |\omega a - \hbar(a) + \hbar(0)| \quad \text{for } 0 < a < 1. \quad (6.48)$$

This condition is automatically satisfied in quantum mechanics as a consequence of (6.47), because in quantum mechanics  $\hbar(a)$  is linear, and so the right-hand side of (6.48) vanishes. In our generalized version of quantum mechanics  $\hbar(a)$  is non-linear, so the right-hand side of (6.48) does not vanish, and Eq. (6.48) sets a lower bound on the strength of the radiation field that will drive the transition from one energy eigenstate to the other. With  $f(a)$  given (as usual) by (6.28), Eq. (6.48) requires that

$$|g| \geq [|\omega a - \hbar(a) + \hbar(0)| / 2 \sqrt{a(1-a)}]_{\max}. \quad (6.49)$$

For instance, if we give  $\hbar(a)$  the simple non-linear form (6.41), this condition is

$$|g| \geq [ |(\omega - E_2 + E_1)a + \varepsilon a(1-a)| / 2 \sqrt{a(1-a)} ]_{\max}. \quad (6.50)$$

With  $\omega$  tuned to the value (6.47), the right-hand side of (6.50) reaches its maximum at  $a = \frac{1}{2}$ , so this condition reads

$$|g| \geq |\varepsilon|/4. \quad (6.51)$$

If this condition is not satisfied, then  $a(t)$  cannot reach  $a = \frac{1}{2}$ , much less  $a = 1$ . Observing a transition go from near one energy eigenstate to near another thus sets an upper bound in the non-linearity parameter  $\varepsilon$ . Observations of this sort [3] have already been used [4] to set an upper bound on the non-linearity parameter  $|\varepsilon|$  of the  $\text{Be}^9$  nucleus of order  $10^{-15}$  eV, a result that has very recently been improved [30] to  $|\varepsilon| < 2.4 \times 10^{-20}$  eV. As far as I know, this is the most stringent current limit on nonlinear corrections to quantum mechanics.

Even where the condition (6.49) is satisfied, the non-linearities in  $\bar{h}(a)$  can have a conspicuous effect on the line shape. Suppose (as in Ref. [3]) we monitor the transition  $1 \rightarrow 2$  by inducing a resonant transition to a third state, which can only be reached from the  $k=2$  component of  $\psi_k$ . The intensity of the  $2 \rightarrow 3$  transition is proportional to the value of  $a$  that has been reached. For instance, assume again that  $\bar{h}(a)$  is given by Eq. (6.41), but now suppose that (6.47) is *not* satisfied. The maximum value reached by  $a(t)$  is given by the smallest value of  $a > 0$  at which the inequality (6.50) becomes an equality, i.e., by the smallest root of the cubic equation

$$a[\omega - E_2 + E_1 + \varepsilon(1 - a)]^2 - 4|g|^2(1 - a) = 0.$$

If (6.51) is satisfied the smallest root reaches  $a = 1$  at  $\omega = E_2 - E_1$ , but the points where  $a = 1/2$  are at  $E_2 - E_1 - (\varepsilon/2) \pm \sqrt{2}|g|$ . (When (6.51) is not satisfied, the smallest root rises monotonically to a value  $a > \frac{1}{2}$  as  $\omega$  increases from  $-\infty$ , and then at a critical value  $\omega_c$  drops discontinuously to a value  $a < \frac{1}{2}$ , after which it falls monotonically as  $\omega$  increases to  $+\infty$ . The critical frequency where  $a$  reaches its maximum is not at  $E_2 - E_1$ ; rather,  $\omega_c - E_2 + E_1$  has a sign opposite to that of  $\varepsilon$ .) Observation of this highly asymmetric line shape would be a sign of nonlinear departures from quantum mechanics.

#### APPENDIX: A USEFUL TRANSFORMATION

This Appendix will describe a transformation of the  $N$  complex components  $\psi_k$  of the wave function that replaces them with the norm  $n$ , an over-all phase  $\alpha$ , and  $N - 1$  complex functions  $z_l$  of the ratios of the  $\psi_k$ . This transformation is useful in solving specific problems, and more generally, because it allows an immediate application of the KAM theorem to our nonlinear version of quantum mechanics.

Consider any one value of the wave function  $\psi_k$  and apply a unitary (and hence canonical) transformation, to make all components vanish except  $\psi_1$ . In some neighborhood around this point,  $\psi_1$  will be non-vanishing, and hence have a well-defined phase  $\alpha$ . Within this neighborhood, we introduce our transformation by writing

$$\psi_k = n^{1/2} e^{i\alpha} z_k \tag{A.1}$$

Since  $n \equiv \sum_k |\psi_k|^2$ , the  $z_k$  are subject to the constraint  $\sum_k |z_k|^2 = 1$ . But  $\psi_1$  has phase  $\alpha$ , so  $z_1$  is real and positive and can therefore be expressed in terms of the other  $z_l$  as

$$z_1 = + \left( 1 - \sum_{l \geq 2} |z_l|^2 \right)^{1/2}. \tag{A.2}$$

The independent dynamical variables here are taken as  $n$ ,  $\alpha$ , and the  $z_l$  with  $l \geq 2$ . These  $z_l$  may easily be expressed directly in terms of the  $\psi_k$ :

$$z_l = \frac{\psi_l / \psi_1}{\sqrt{1 + \sum_{m \geq 2} |\psi_m / \psi_1|^2}}. \tag{A.3}$$

The great advantage of using these  $z_l$  as variables (instead of, say, the ratios  $\psi_l/\psi_1$  themselves) lies in the simplicity of the equations that govern their time-dependence. The homogeneity of the Hamiltonian functions allows it to be written as

$$h = n\bar{h}(z, z^*). \quad (\text{A.4})$$

Using (A.4) and the time-dependent Schrödinger equation (2.12), we easily find that the  $z_l$  with  $l \geq 2$  satisfy the time-dependence equation

$$i \frac{dz_l}{dt} = \frac{\partial \bar{h}(z, z^*)}{\partial z_l^*} \quad (\text{for } l \geq 2). \quad (\text{A.5})$$

These equations are again of the Hamiltonian form, with canonically conjugate variables  $\sqrt{2} \operatorname{Re} z_l$  and  $\sqrt{2} \operatorname{Im} z_l$ .

The energy eigenstates are represented by solutions of (2.12) for which all components  $\psi_k$  have the same time dependence, given by a factor  $\exp(-iEt)$ . In this case the  $z_k$  are all *constant*, so (A.5) shows that energy eigenstates are given by stationary points of  $\bar{h} \equiv h/n$ , as already seen for general observables in Section 3.

This transformation has a useful application to the case where the reduced Schrödinger equation (A.5) (and hence the original equation (2.12)) is integrable. We shall show that in this case it is possible by a homogenous canonical transformation to introduce new wave-function components  $\tilde{\psi}_k$  such that the Hamiltonian function depends only on the  $|\tilde{\psi}_k|^2$ . Integrability allows us to express  $z_l$  and  $z_l^*$  in terms of  $N-1$  angle and action variables,  $j_l$  and  $\theta_l$ , which are canonically conjugate in the sense that

$$\begin{aligned} [j_l, j_m] &= 0, & [\theta_l, \theta_m] &= 0 \\ [\theta_l, j_m] &= i \delta_{lm}, \end{aligned} \quad (\text{A.6})$$

where, now, for any functions of  $z$  and  $z^*$

$$[a, b] \equiv \frac{i}{2} \left( \frac{\partial a}{\partial \operatorname{Re} z_l} \frac{\partial b}{\partial \operatorname{Im} z_l} - \frac{\partial b}{\partial \operatorname{Re} z_l} \frac{\partial a}{\partial \operatorname{Im} z_l} \right) \quad (\text{A.7})$$

and with  $\bar{h}$  dependent only on the  $j$ 's, not the  $\theta$ 's. The time-dependence of these variables is then

$$\frac{dj_l}{dt} = - \frac{\partial \bar{h}}{\partial \theta_l} = 0 \quad (\text{A.8})$$

$$\frac{d\theta_l}{dt} = \frac{\partial \bar{h}}{\partial j_l}. \quad (\text{A.9})$$

We may therefore introduce a new set of  $z$ -variables

$$\tilde{z}_l \equiv j_l^{1/2} \exp(-i\theta_l) \quad (\text{A.10})$$

with a time-dependence governed again by the reduced Schrödinger equation (A.5),

$$i \frac{d\tilde{z}_l}{dt} = j_l^{1/2} \exp(-i\theta_l) \frac{\partial \bar{h}(j)}{\partial j_l} = \tilde{z}_l \frac{\partial \bar{h}(|\tilde{z}|^2)}{\partial |\tilde{z}_l|^2} = \frac{\partial \bar{h}}{\partial \tilde{z}_l^*} \quad (\text{A.11})$$

but with  $\bar{h}$  now a function only of the  $|\tilde{z}_l|^2$ . Finally, we can return to the  $N$ -component formalism, defining

$$\tilde{\psi}_1 = \sqrt{n} e^{-i\tilde{\alpha}} \left( 1 - \sum_{l \geq 2} |\tilde{z}_l|^2 \right)^{1/2} \quad (\text{A.12})$$

$$\tilde{\psi}_l = \sqrt{n} e^{-i\tilde{\alpha}} \tilde{z}_l \quad (l \geq 2), \quad (\text{A.13})$$

where  $n$  is the constant  $\sum_k |\psi_k|^2$ , and the phase  $\tilde{\alpha}$  satisfies

$$\frac{d\tilde{\alpha}}{dt} = \bar{h}(|\tilde{z}|^2) - \sum_{l \geq 2} |\tilde{z}_l|^2 \frac{\partial \bar{h}(|z|^2)}{\partial |\tilde{z}_l|^2}. \quad (\text{A.14})$$

With this definition of the components of the wave function, we have the usual time-dependence equation

$$i \frac{d\tilde{\psi}_k}{dt} = \frac{\partial h}{\partial \tilde{\psi}_k^*} \quad (\text{A.15})$$

but with a Hamiltonian function that now depends only on the moduli  $|\tilde{\psi}_k|$ ,

$$h = n\bar{h}(|\tilde{\psi}|^2/n), \quad (\text{A.16})$$

where now we write

$$n = \sum_k |\tilde{\psi}_k|^2. \quad (\text{A.17})$$

Note that  $h$  is now homogeneous of degree one in the  $\tilde{\psi}_k$  and  $\tilde{\psi}_k^*$ , just as it was in the  $\psi_k$  and  $\psi_k^*$ . Not only  $h$ , but any observable that is homogeneous of degree one in  $\psi_k$  and  $\psi_k^*$  may be written as the norm  $n$  times a function of the  $z_l$  and  $z_l^*$ , and hence as the norm  $n$  times a function of the  $\tilde{z}_l$  and  $\tilde{z}_l^*$ , and is therefore homogeneous of degree one in the  $\tilde{\psi}_k$  and  $\tilde{\psi}_k^*$ . (It was in order to achieve this homogeneity property that we needed to introduce the  $z$ -variables; otherwise it would have been easy to convert the Hamiltonian function into a function only of moduli  $|\tilde{\psi}_k|^2$  by writing  $\tilde{\psi}_k = J_k^{1/2} \exp(-i\Theta_k)$ , where  $J_k$  and  $\Theta_k$  are the action and angle variables of the  $N$ -component system.)

This result applies for integrable systems, and only for integrable systems. However, Birkhoff's theory of normal forms [31] allows us to write a formal power series for variables  $\tilde{\psi}_k$  in terms of the original  $\psi_k$  in such a way that the Hamiltonian function is formally a function only of the  $|\tilde{\psi}_k|^2$  in a neighborhood of any energy eigenstate. Suppose that we take arbitrary wave function with which we

started, which by a unitary transformation we gave a value with only  $\psi_1$  non-zero, to be at one of the stationary points of  $h/n$ . At this stationary point, the  $z_l$  with  $l \geq 2$  all vanish, so the leading terms in a series expansion are, aside from a constant, of second order in the  $z_l$  and  $z_l^*$ ,

$$\begin{aligned} \bar{h} = \bar{h}(0) + a_{lm} z_l^* z_m + \frac{1}{2} b_{lm} z_l z_m + \frac{1}{2} b_{lm}^* z_l^* z_m^* \\ + O(z^3, z^2 z^*, z z^{*2}, z^{*3}) \end{aligned} \quad (\text{A.18})$$

with  $a_{lm}$  Hermitian. Hence, infinitesimally close to the energy eigenstate, the time-dependent Schrödinger equation (and its adjoint) takes the form

$$\begin{aligned} i \frac{d}{dt} \begin{bmatrix} z \\ z^* \end{bmatrix} = M \begin{bmatrix} z \\ z^* \end{bmatrix} \\ M \equiv \begin{bmatrix} a & b \\ -b^* & -a^* \end{bmatrix}. \end{aligned} \quad (\text{A.19})$$

It is easy to see that if  $\lambda$  is an eigenvalue of  $M$  then so are  $-\lambda$ ,  $\lambda^*$ , and  $-\lambda^*$ . (This is because  $-M^*$  and  $-M^T$  are related to  $M$  by similarity transformations with the unitary supermatrices  $\sigma_1$  and  $\sigma_2$ , respectively.) Hence the eigenvalues come either in complex quartets, or real doublets, or imaginary doublets. In ordinary quantum mechanics,  $h$  is bilinear in  $\psi$  and  $\psi^*$ , so  $\bar{h}$  is bilinear in  $z$  and  $z^*$ , so  $b_{lm} = 0$ ; and therefore the eigenvalues all come in real doublets,  $\omega_l$  and  $-\omega_l$ . We assume that our theory is close to a generic quantum mechanical theory, with all  $\omega_l$  different and non-zero. By continuity then the eigenvalues must still come in real doublets when we add very small non-bilinear terms to the Hamiltonian function. (Any real doublet of eigenvalues that developed small imaginary parts would have to become a quartet, yielding two too many different eigenvalues.) With all eigenvalues of  $M$  real and different, it follows that it is possible by a linear canonical  $\text{Sp}(2N-2)$  transformation  $z_l \rightarrow z'_l$  to bring the quadratic terms in  $\bar{h}$  to a bilinear diagonal form

$$\bar{h} = \bar{h}(0) + \sum_{l=2}^N \omega_l |z'_l|^2 + O(z'^3, z'^2 z'^*, z' z'^{*2}, z'^{*3}) \quad (\text{A.20})$$

with real  $\omega_l$ .

We can make use of Birkhoff's results. The frequencies  $\omega_l$  are said to satisfy a resonance relation of order  $K$  if there exist integers  $v_l$  such that

$$\sum_l v_l \omega_l = 0 \quad (\text{A.21})$$

$$\sum_l |v_l| = K. \quad (\text{A.22})$$

If the frequencies  $\omega_l$  do not satisfy any resonance relation of order 1, 2, ...,  $K$ , then

we can make a non-linear canonical transformation  $z'_i \rightarrow z''_i$  to bring the whole Hamiltonian function to the form

$$h = P_K(|z''|^2) + R_K(z'', z''^*), \tag{A.23}$$

where  $P_K$  is a polynomial of order  $[K/2]$ , and  $R_K$  is of total order in  $z''$  and  $z''^*$  at least  $[K/2] + 1$ . Generically, the frequencies will not satisfy a resonance relation of any order, and so at least formally we can let  $K$  go to infinity and write  $\bar{h}$  as a power series in the moduli  $|z''_i|^2$  of  $N - 1$  canonical variables  $z''_i$  or, equivalently, write  $h$  as a power series in the moduli  $|\bar{\psi}_k|^2$  of  $N$  canonical variables  $\bar{\psi}_k$ .

The trouble is, of course, with the convergence of this procedure. A given set of frequencies may not satisfy a resonance relation of any finite order, but by taking large integers  $\nu_i$  we can satisfy (A.21) increasingly closely. (The exception is for  $N - 1 = 1$ , where (A.21) can never be satisfied. In this case (A.5) is integrable, and therefore as discussed earlier, it is possible by a canonical transformation to write  $\bar{h}$  as a function only of the modulus  $|z|^2$  of a canonical variable  $z$ .)

Fortunately, the KAM theorem [6] provides a rigorous alternative to this construction of a dubiously convergent power series. This theorem says in effect that a nearly integrable system behaves as if integrable for “most” initial conditions. The difficult question is to specify the volume of phase space within which the KAM theorem applies. Unfortunately, in our case it is not enough to say that the Hamiltonian function  $h(\psi, \psi^*)$  or  $\bar{h}(z, z^*)$  is close to the bilinear and hence integrable Hamiltonian function of ordinary quantum mechanics. This is because in order to apply the KAM theorem to, say, (A.5), it is necessary that the Hamiltonian function should take the form

$$\bar{h}(z, z^*) = \bar{h}_0(j) + \varepsilon \bar{h}_1(j, \theta), \tag{A.24}$$

where  $\varepsilon$  is sufficiently small,  $j_i$  and  $\theta_i$  are action and angle variables, and  $\bar{h}_0(j)$  is not only integrable but also non-degenerate, in the sense that

$$\text{Det} \left( \frac{\partial^2 \bar{h}_0(j)}{\partial j_i \partial j_m} \right) \neq 0. \tag{A.25}$$

Unfortunately, we cannot just take  $\bar{h}_0(j)$  as the Hamiltonian function of ordinary quantum mechanics, because this is bilinear in  $z_i$  and  $z_i^*$  and, therefore, linear in the action variables  $|z'_i|^2$  (where  $z'_i$  are unitarity related to the  $z_i$ ).

There is a version of the KAM theorem due to J. Moser [32], that allows us to apply it to a Hamiltonian system in a sufficiently small neighborhood of an elliptic fixed point, without having to split the Hamiltonian function into a non-degenerate integrable term and a small perturbation. By an “elliptic fixed point” is meant a point in phase space where the first derivatives of the Hamiltonian all vanish, and the eigenvalues of the matrix of second derivatives are all real. As we have seen, the transformation of coordinates from  $N\psi$ 's and  $\psi^*$ 's to  $N - 1z$ 's and  $z^*$ 's has the effect (when the non-bilinear terms in the Hamiltonian function are sufficiently

small) of converting an energy eigenstate of  $h(\psi, \psi^*)$  to an elliptic fixed point of  $\bar{h}(z, z^*)$ . Thus by means of this, transformation, we can apply this version of the KAM theorem to our problem. To summarize the analysis very briefly, one must make use of the Birkhoff result (A.23) and move part of the non-bilinear terms in  $\bar{h}(\psi, \psi^*)$  into the "unperturbed" Hamiltonian  $\bar{h}_0$ . Assuming that the bilinear quantum-mechanical Hamiltonian function  $\bar{h}_{\text{QM}}$  of our system does not satisfy a resonance relation of order 1, 2, 3, or 4, in the neighborhood of an energy eigenstate we can define canonical variables  $z_i''$  such that

$$\bar{h} = \bar{h}(0) + \sum_i \omega_i |z_i''|^2 + \frac{1}{2} \sum_{lm} \omega_{lm} |z_l''|^2 |z_m''|^2 + R_4(z'', z''^*), \quad (\text{A.26})$$

where  $R_4$  is at least of total order 5 in the  $z_i''$  and  $z_i''^*$ . Generically,  $\omega_{lm}$  has a non-vanishing determinant, so we can take the first three terms of (A.26) as our integrable non-degenerate unperturbed Hamiltonian, with action variables  $j_i = |z_i''|^2$ , and use  $R_4$  as the non-integrable perturbation  $\bar{h}_1$ . However, we cannot rely on the smallness of  $R_4$  to justify the application of the KAM theorem, because the determinant (A.25), though non-zero, is equally small for general  $z_i$ . Instead, we may limit ourselves to a very small region around the assumed energy eigenstate, taken here to have the coordinates  $z_i'' = 0$ , and use the smallness of  $|z_i''|$  to justify the treatment of  $\bar{h}_1 = R_4$  as a small perturbation.

As far as I know, even the Hamiltonian consists of an integrable term and a small perturbation, in cases where the integrable term is degenerate there are no theorems that justify the application of the KAM results, except in small neighborhoods around elliptic fixed points. Thus, even though we assume here that the departures from quantum mechanics are very small, we cannot use the KAM theorem to justify treating these theories as if they were integrable, except in correspondingly small neighborhoods of energy eigenvectors. However, experience shows that degenerate integrable systems with weak non-integrable perturbations actually do behave as if integrable, as long as the frequencies in the unperturbed Hamiltonian are sufficiently incommensurate. An example is provided by the numerical calculations of M. Marder, described in Section 4. One may conjecture that although the non-degeneracy condition is really needed to prove that the chaotic regions around resonant tori occupy a small volume of phase space or action-variable space, it is not needed if one is content to show that they occupy a small volume of frequency space.

*Note added in proof.* (i) After this paper was submitted for publication, it was pointed out to me that the interpretation of the time-dependent Schrödinger equation of ordinary quantum mechanics as a particular case of classical Hamiltonian systems has also been discussed by A. Heslot, *Phys. Rev. D* **31** (1985), 1341. (ii) J. Polchinski [private communication] has raised the question of whether in non-linear generalizations of quantum mechanics it is possible to violate causality in experiments of the Einstein-Podolsky-Rosen type. This appears to be a serious problem, because with the treatment of separated systems given in Section 2(g) of this paper, functions representing observables associated with separated systems (such as the two terms in Eq. (2.35)) do not necessarily commute. Work is in progress by Polchinski and myself to give a treatment of separated systems that would avoid this problem.

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