Violating Bell's inequalities in vacuum

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We employ an approach wherein the ground state entanglement of a relativistic free scalar field is directly probed in a controlled manner. The approach consists of having a pair of initially nonentangled detectors locally interact with the vacuum for a finite duration T, such that the two detectors remain causally disconnected, and then analyzing the resulting detector mixed state. We show that the correlations between arbitrarily far-apart regions of the vacuum cannot be reproduced by a local hidden-variable model, and that as a function of the distance L between the regions, the entanglement decreases at a slower rate than $\sim \exp[-(L/cT)^3]$.

It is known that the vacuum state of a relativistic free field is entangled. For two complementary regions of space-time, such as $x < 0$ and $x > 0$, this entanglement is closely related to the Unruh acceleration radiation effect [1], and gives rise to a violation of Bell's inequalities [2,3]. For two fully separated regions, entanglement persists [4], although, it is not known in this case whether Bell's inequalities are violated, and how entanglement decays with the increase of separation, as compared to correlations. Similar questions concerning entanglement have been addressed in the case of discrete models [5–7].

In this paper we shall study this problem by probing the field's entanglement with a pair of localized two-level detectors [8]. This is done as follows. A state is prepared in which the two detectors are not entangled with one another, or the field. We then have each of the detectors locally interact with the field for a finite duration, such that the detectors remain causally disconnected throughout the process (Fig. 1). Since entanglement cannot be produced locally [9], the net entanglement between the detectors, once the interaction has been switched off, must necessarily have its origin in vacuum correlations. The interaction thus serves as a means of redistributing entanglement between the field and the detectors. We shall show that for arbitrarily far-apart regions, the detectors' final mixed state, after filtering, violates Bell's inequalities, and in the process obtain a lower bound on the amount of vacuum entanglement.

To setup the model, we shall assume that the detectors are localized within a region of a typical scale of R, and are separated by a much larger distance $L \gg R$. Consistency with relativity requires us to use detectors of a rest mass M, for which $R \gg \lambda_{\text{Compton}} = \hbar/Mc$. In this limit, the effects of both detector pair creation, and the "leakage" of each detector's wave function to the outside of its localization region, become exponentially small, of the order of $\sim \exp(-2cMR/\hbar)$ [10,11]. Note that this ensures that the overlap between the detectors' wave functions is negligible. Under these conditions, in their rest frame, the detectors can be described as nonrelativistic quantum-mechanical systems. Finally, we shall assume that, by means of an external coupler, each detector's degrees of freedom can be coupled "at will" to the field. Since the coupler need not be of the same type as the studied field, we shall make the additional assumption that it can be described classically, and therefore does not generate entanglement.

There have been several proposals for detector models which can satisfy the above requirements; notably, the Unruh-Wald "particle in a box" detector [12] and the DeWitt monopole detector model [13]. In both models the detector Hamiltonian is $(\Omega/2)\sigma_z$, with Ω being the energy gap between the two levels and σ_z a Pauli matrix. The field-detector interaction Hamiltonian is

$$H_{\text{int}} = \epsilon(t) \int d^3x \, \psi(\vec{x})(e^{+i\Omega t}\sigma^+ + e^{-i\Omega t}\sigma^-)\phi(\vec{x},t). \quad (1)$$

$\phi(\vec{x},t)$ is a relativistic free scalar field in three spatial dimensions, the $\sigma^{\pm}$ are the detector's ladder operators, and $\epsilon(t)$ governs the strength and duration of the interaction. $\psi(\vec{x})$ is a function of the detector's spatial degrees of freedom, and is determined by the model employed [14,15].

Consider now a pair of DeWitt monopole detectors, A and B, that are localized about the coordinates $\vec{x}_A$ and $\vec{x}_B$, respectively. These detectors interact with the field through $H_{\text{int}} = H_A + H_B$, where H_A and H_B are interaction Hamiltonians of the form of Eq. (1). The window functions $\epsilon_A(t)$ and $\epsilon_B(t)$ are chosen to vanish except for a finite duration T, such that $cT \ll L = |\vec{x}_B - \vec{x}_A|$, ensuring that the detectors remain causally disconnected throughout the interaction. In the following we shall work in the Dirac interaction representation and employ "natural" units ($\hbar = c = 1$).

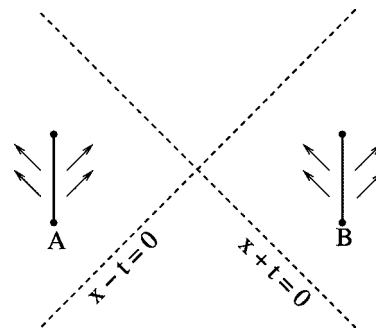Since the interaction takes place in two causally discon-

FIG. 1. The world lines of detectors A and B are shown for the duration of the interaction. The horizontal and vertical axes are space and time, respectively. The arrows denote the emitted radiation. Notice that the radiation emitted by detector $A(B)$ does not affect detector $B(A)$, since for $t > T$ the interaction is switched off.
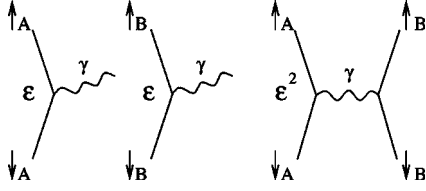
FIG. 2. Emission and exchange processes. The exchange amplitude, portrayed on the right, is dominated by a single off-shell emission followed by an on-shell absorption, while the double emission amplitude, portrayed on the left, consists of two off-resonance processes. Thus, as $\Omega_{A,B}$ increases, the emission term $\|E_A\|\|E_B\|$ decreases more rapidly than the exchange term $|\langle 0|X_{AB}\rangle|$.

nected regions, the Hamiltonians H_A and H_B commute. The evolution operator U for the whole system thus factors to a product of local unitary transformations

$$U = \hat{T}[e^{-i\int H_A(t)dt} \times e^{-i\int H_B(t')dt'}], \qquad (2)$$

where $\hat{T}$ denotes time ordering. This guarantees that U does not change the net entanglement between the regions.

We take the initial state of the detectors and the field to be $|\Psi_i\rangle = |\downarrow_A\rangle|\downarrow_B\rangle|0\rangle$, where $|\downarrow\rangle$ and $|0\rangle$ denote detector and field ground states, respectively. In the weak-coupling limit $\epsilon_i(t) \ll 1$ $(i=A,B)$, expanding to the second order we get

$$\begin{aligned}|\Psi_f\rangle = [(\mathbf{1}-C)|\downarrow\downarrow\rangle - \Phi_A^+\Phi_B^+|\uparrow\uparrow\rangle \\ - i\Phi_A^+\mathbf{1}_B|\uparrow\downarrow\rangle - i\mathbf{1}_A\Phi_B^+|\downarrow\uparrow\rangle]|0\rangle + O(\epsilon^3), \qquad (3)\end{aligned}$$

where $\Phi_i^+ = \int dt\, \epsilon_i(t)e^{\pm i\Omega_i t}\int d^3x\, \psi_i(\vec{x})\phi(\vec{x},t)$ and $C = \frac{1}{2}\int\int dt dt'\, \hat{T}[H_A(t)H_A(t')] + (A\leftrightarrow B)$. We observe that in the first term above, the state of the detectors is unchanged, while in the second term both detectors are excited and the final state of the field is $|X_{AB}\rangle \equiv \Phi_A^+\Phi_B^+|0\rangle$. Since $|X_{AB}\rangle$ contains either two photons or none, it describes, respectively, an emission of a photon by each of the detectors or an exchange of a single virtual-photon between them (Fig. 2). Finally, the

last couple of terms describe an emission of a single photon by either detector A or B. In this case the final state of the field is, respectively, $|E_A\rangle \equiv \Phi_A^+|0\rangle$, or $|E_B\rangle \equiv \Phi_B^+|0\rangle$.

Tracing over the field degrees of freedom, when working in the basis $\{|i\rangle\}_{i=0}^3 = \{\uparrow\uparrow,\uparrow\downarrow,\downarrow\uparrow,\downarrow\downarrow\}$ and employing the notation $\|X_{AB}\|^2 = \langle X_{AB}|X_{AB}\rangle$, we obtain the detectors' reduced density matrix [16]

$$\rho = \begin{pmatrix} \|X_{AB}\|^2 & 0 & 0 & -\langle 0|X_{AB}\rangle \\ 0 & \|E_A\|^2 & \langle E_B|E_A\rangle & 0 \\ 0 & \langle E_A|E_B\rangle & \|E_B\|^2 & 0 \\ -\langle X_{AB}|0\rangle & 0 & 0 & 1-\|E_A\|^2-\|E_B\|^2 \end{pmatrix}$$
$$+ O(\epsilon^4). \qquad (4)$$

Note the two types of off-diagonal terms. The amplitude $\langle 0|X_{AB}\rangle$ acts to maintain coherence between $|\downarrow_A\downarrow_B\rangle$ and $|\uparrow_A\uparrow_B\rangle$, while the amplitude $\langle E_A|E_B\rangle$ acts to maintain coherence between $|\downarrow_A\uparrow_B\rangle$ and $|\uparrow_A\downarrow_B\rangle$. It is the relative magnitude of these off-diagonal terms, as compared to the diagonal decoherence terms, that determines whether the density matrix is entangled.

A density matrix is said to be inseparable or entangled iff it cannot be expressed as a convex sum of local density matrices [17]. In the present case of a 2×2 system, a necessary [18] and sufficient [19] condition for inseparability is that the negativity [20] be positive. We shall therefore use the negativity as a measure of entanglement. The following expression is obtained for the negativity:

$$\mathcal{N}(\rho) \approx |\langle 0|X_{AB}\rangle| - \|E_A\|\|E_B\| > 0. \qquad (5)$$

Physically speaking, the inequality above is satisfied if the single virtual-photon exchange process is more probable than the off-resonance emissions of a single photon by each of the detectors. The main contribution to the entanglement then arises from states of the form $\alpha|\downarrow_A\downarrow_B\rangle + \beta|\uparrow_A\uparrow_B\rangle$.

The inequality (5) can be reexpressed as

$$\int_0^\infty \frac{d\omega}{L}\sin(\omega L)e^{-\omega^2 R^2}\widetilde{\epsilon}_A(\Omega_A+\omega)\widetilde{\epsilon}_B(\Omega_B-\omega) > \sqrt{\int_0^\infty \omega\, d\omega\, e^{-\omega^2 R^2}|\widetilde{\epsilon}_A(\Omega_A+\omega)|^2 \int_0^\infty \omega\, d\omega\, e^{-\omega^2 R^2}|\widetilde{\epsilon}_B(\Omega_B+\omega)|^2}, \qquad (6)$$

where the factor $e^{-\omega^2 R^2}$ accounts for the "smearing" of the detectors and $\widetilde{\epsilon}_i(\omega)$ denotes the Fourier transform of $\epsilon_i(t)$. (Note that for a massive field a factor of $\omega/\sqrt{\omega^2+m^2}$ must be added to each integral.) The term $\sin(\omega L)$ on the left-hand side can be interpreted as an effective window function that characterizes the field's response to the detectors. When integrating over ω, this function governs the overall sign of each mode's contribution, and thus acts to reduce the exchange amplitude. This destructive interference effect can be minimized by employing a window function $\widetilde{\epsilon}_A$ for which $\sin(\omega L)\widetilde{\epsilon}_A(\Omega_A+\omega)$ remains positive over a finite integration

regime in the limit of large L. A superoscillating function meets this requirement [21,22]. In particular the function [23]

$$\widetilde{\epsilon}_A(\omega) = f(\omega)J_0\{\sqrt{(\omega T)^2 - N^2[(L/T)^2-1]}\}, \qquad (7)$$

where $f(\omega)$ is any function that converges faster than $1/\omega$ and has finite temporal support. We observe that $\widetilde{\epsilon}_A(\omega)$ is bounded in time as required, and oscillates as $\sin(\omega L)$ about $\omega = \omega_s \pm \sqrt{N}/2L$, where $T\omega_s = NL/T$, approximately $\sqrt{N}$ times, before gradually resuming "normal" slow oscillations for larger values of ω. The use of superoscillations, however, is

not without a price. For $\omega T < N\sqrt{(L/T)^2 - 1}$ the function $\widetilde{\epsilon}_A(\omega)$ decays exponentially, rendering the exchange term, and hence the negativity, exponentially small in L. The second window function $\widetilde{\epsilon}_B$ is a fixed hat function, convolved k times with itself. In ω space it assumes the form $[\sin(\omega T/k)/\omega T/k]^k$. For given values of L and T we choose the energy gaps $\Omega_A = (N/T)\sqrt{(L/T)^2 - 1}$ and $\Omega_B = \omega_s - \Omega_A \approx N/2L$. This choice of Ω_B fixes the center of the window function $\widetilde{\epsilon}_B(\Omega_B - \omega)$ in the region of superoscillations. For large values of L, the inequality [Eq. (6)] can then be approximated by $\sqrt{N}(T/L)^2(1/\Omega_A T \Omega_B T)[\widetilde{\epsilon}_B(0)/\widetilde{\epsilon}_B(\Omega_B)] > 1$. This ratio can be made arbitrarily large, at the expense of reducing the negativity, by increasing N. To get a lower bound, we take $N = (L/T)^2$. Substituting the expressions for Ω_A and Ω_B into the above approximation, it takes on the form $(T/L)^5(L/2T)^k > 1$. For $k > 5$ this ratio increases with L. We thus get a lower bound on the negativity

$$\mathcal{N}(\rho) \geq e^{-(L/cT)^3}. \tag{8}$$

Numerical computations show that this bound can be further improved. Taking $N \geq L/T$, we get $\mathcal{N}(\rho) \geq e^{-(L/T)^2}$ [24]. The leakage of each detector's wave function to the outside of their localization regions introduces a correction of the order of e^{-2MR} to the above expression. However, this correction can be made arbitrarily small by setting $e^{-(L/T)^2} \gg e^{-2MR}$. Note that we are free to do this, since the mass scale M and the distance scale L are independent.

In passing, we would like to point out that in the case of the free electromagnetic field the analysis can easily be repeated. Similar results are obtained in the case of the finite duration coupling of a detector's magentic or electric dipole to the field. For a massive field, in the limit of large L, the above result remains unchanged, because then the contribution to the integrals arises from the range of frequencies $\omega \gg m$ for which the field effectively behaves as if massless.

The reduced density matrix derived in the previous section is entangled. The question arises as to whether these vacuum correlations admit a local hidden-variable (LHV) description [3]. Applying the Horodecki theorem [25] to Eq. (4), we find that the detectors' final state does not violate the CHSH inequality [26]. We shall now demonstrate that by using local filters [27], a violation of the CHSH inequality can be achieved for every separation distance L. Hence, the density matrix (4) reveals a "hidden" nonlocality [28], as in the case of Werner states [29].

To show this we follow Gisin [27]. Once the interaction with the field has been switched off we have each detector pass through the filter

$$f_{A,B} = \begin{pmatrix} 1 & 0 \\ 0 & \eta \end{pmatrix}. \tag{9}$$

The density matrix is thus transformed according to $\rho \to \rho_f = (f_A \otimes f_B)\rho(f_A \otimes f_B)$, so that the (non-normalized) filtered density matrix is given by

$$\begin{pmatrix} \|X_{AB}\|^2 & 0 & 0 & -\eta^2\langle 0|X_{AB}\rangle \\ 0 & \eta^2\|E_A\|^2 & \eta^2\langle E_B|E_A\rangle & 0 \\ 0 & \eta^2\langle E_A|E_B\rangle & \eta^2\|E_B\|^2 & 0 \\ -\eta^2\langle X_{AB}|0\rangle & 0 & 0 & \eta^4(1 - \|E_B\|^2 - \|E_A\|^2) \end{pmatrix}. \tag{10}$$

Consider now the choice $\eta^2 \approx \langle X_{AB}|0\rangle$. We note that the 00, 33, 03, 30 terms [Eq. (10)] are now nearly equal, and of the order of $|\langle X_{AB}|0\rangle|^2$. Ideally, in the absence of decoherence terms in the inner block, these terms come close to reproducing a maximally entangled state $|\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle$. Notice, however, that the decoherence terms are of the order of $|\langle X_{AB}|0\rangle|\|E_{A,B}\|^2$. Previously we have shown that the ratio $|\langle X_{AB}|0\rangle|/(\|E_A\|\|E_B\|)$ can be made, by a suitable choice of window functions, arbitrarily large. Therefore in this extreme limit the relative strength of the decoherence terms, as compared to the entangling terms, is greatly reduced, and ρ_f can be brought as close as we like to a pure, maximally entangled state. This implies a maximal violation of Bell's inequalities for the final state of the detectors, and since initially the detectors are not entangled (thus admitting a LHV description), it follows that correlations between arbitrarily separated regions of the vacuum cannot be ascribed to a LHV model.

Maximal violation can be achieved at the price of reducing the detectors' entanglement (negativity), which grows smaller in the above limit (Fig. 3). We now wish to quantify the more general case, for which the final state is more entangled (larger negativity), but gives rise to a weaker violation of Bell's inequalities. Applying yet again the Horodecki theorem [25], we find that the CHSH inequality is violated if

$$\frac{|\langle 0|X_{AB}\rangle|}{\|E_A\|\|E_B\|} > 4\frac{\|X_{AB}\|}{|\langle 0|X_{AB}\rangle|}. \tag{11}$$

Note that for far-apart regions, this inequality is only slightly stronger than the condition for entanglement.

Interestingly, if we repeat our process many times, the resulting ensemble of entangled pairs can be reduced to a smaller one of higher quality entangled pairs. This process is known as distillation of entanglement, and is feasible for any inseparable 2×2 mixed state [30]. Furthermore, since Bell's inequalities are violated in our example, the two detectors
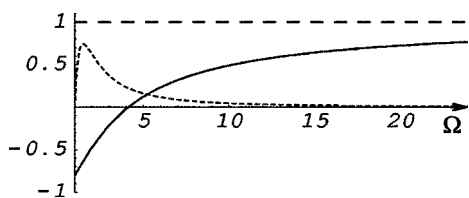
FIG. 3. Violation of the CHSH inequality. The dashed line represents the negativity of the detectors before passing through the filter, while the solid line represents the quantity $M(\rho)-1$, which is calculated after the detectors have passed through the filter. Since the CHSH inequality can be written in the form $M(\rho)<1$ [25], we see that for $\Omega \to \infty$, the CHSH inequality is maximally violated.

could be used directly for teleportation tasks, without having to distill them first [31].

In conclusion, we have presented a physical effect of vacuum fluctuations which is associated with quantum non-locality. This effect stands in marked contrast to other vacuum phenomena, such as the Lamb shift or the Casimir effect, which to some extent can be "mimicked" by classical stochastic local noise [32].

[1] W. G. Unruh, Phys. Rev. D **14**, 870 (1976).

[2] J. S. Bell, Physics (Long Island City, N.Y.) **1**, 195 (1964).

[3] S. J. Summers and R. Werner, Phys. Lett. **110A**, 257 (1985); Commun. Math. Phys. **100**, 247 (1987).

[4] H. Halvorson and R. Clifton, J. Math. Phys. **41**, 1711 (2000).

[5] T. J. Osborne and M. A. Nielsen, Phys. Rev. A **66**, 032110 (2002).

[6] A. Osterloh, L. Amico, G. Falci, and R. Fazio, Nature (London) **416**, 608 (2002).

[7] G. Vidal, J. I. Latorre, E. Rico, and A. Kitaev, Phys. Rev. Lett. **90**, 227902 (2003).

[8] B. Reznik, Found. Phys. **33**, 167 (2003); e-print quant-ph/0008006.

[9] C. H. Bennett *et al.*, Phys. Rev. Lett. **76**, 722 (1996).

[10] G. C. Hegerfeldt, Phys. Rev. Lett. **54**, 2395 (1985).

[11] A. Peres and D. R. Terno, Rev. Mod. Phys. **76**, 93 (2004).

[12] W. G. Unruh and R. M. Wald, Phys. Rev. D **29**, 1047 (1984).

[13] B. S. DeWitt, in *General Relativity: An Einsten Centenary Survey*, edited by S. W. Hawking and W. Israel (Cambridge University Press, Cambridge, UK, 1979).

[14] In the Unruh-Wald model, the local interaction $\delta(\vec{x}-\vec{Q})\phi(x)$ gives rise to a dipolelike coupling, where the $\sigma^{\pm}$ are ladder operators switching between "spatial" eigenstates. In the De-Witt model, the field is coupled to the detector's internal degrees of freedom. Note that we employ a trivially modified version of the DeWitt detector that accounts for its spatial "smearing."

[15] This Hamiltonian is reminiscent of the standard phonon field-ion interaction employed in cavity QED and ion-traps models: J. I. Cirac and P. Zoller, Phys. Rev. Lett. **74**, 4091 (1995). However, we do not make a rotating wave approximation, in order for it to be possible to identify vacuum entanglement.

[16] In general, the reduced density matrix has components $\rho_{i=\pm 1, j=\pm 1}(\vec{x}_A, \vec{x}_B; \vec{y}_A, \vec{y}_B)$. However, under the implicit assumption that the spatial and internal degrees of freedom of each

detector are initially not entangled, and that the back-reaction on the spatial degrees of freedom can be ignored ($\Delta P \gg \Omega$), the entanglement in the reduced density matrix (4), will correspond to virtually all of the entanglement between the detectors.

[17] R. F. Werner, Phys. Rev. A **40**, 4277 (1989).

[18] A. Peres, Phys. Rev. Lett. **77**, 1413 (1996).

[19] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Lett. A **223**, 1 (1996).

[20] G. Vidal and R. F. Werner, Phys. Rev. A **65**, 032314 (2002).

[21] Y. Aharonov, D. Z. Albert, and L. Vaidman, Phys. Rev. Lett. **60**, 1351 (1988).

[22] M. V. Berry, J. Phys. A **27**, L391 (1994).

[23] B. Reznik, Phys. Rev. D **55**, 2152 (1997).

[24] Under a Lorentz transformation the entanglement between the internal degrees of freedom may decrease (see Ref. [11], and references therein). However, the net entanglement between the detectors obviously remains unchanged when each detector's spatial degrees of freedom are taken into account. Our results are thus Lorentz invariant.

[25] R. Horodecki, P. Horodecki, and M. Horodecki, Phys. Lett. A **200**, 340 (1995).

[26] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Phys. Rev. Lett. **23**, 880 (1969).

[27] N. Gisin, Phys. Lett. A **210**, 151 (1996).

[28] P. G. Kwiat, S. Barraza-Lopez, A. Stefanov, and N. Gisin, Nature (London) **409**, 1014 (2001).

[29] S. Popescu, Phys. Rev. Lett. **74**, 2619 (1995).

[30] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. Lett. **78**, 574 (1997).

[31] R. Horodecki, M. Horodecki, and P. Horodecki, Phys. Lett. A **222**, 21 (1996).

[32] P. W. Milonni, *The Quantum Vacuum* (Academic Press, New York, 1994).