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Timelike entanglement entropy Revisited

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We present an operator-algebraic definition for timelike entanglement entropy in CFT. This rigorously defined timelike entanglement entropy is real-valued due to the timelike tube theorem. We further demonstrate why the timelike entanglement entropy should be real-valued from both path integral argument and holography perspective.

INTRODUCTION

As an intrinsic ingredient in quantum theory, entanglement entropy is increasingly playing a central role in quantum field theory (QFT) and quantum gravity. The standard definition of entanglement entropy in a pure state is the von Neumann entropy $S_{\mathrm{vN}} = -\mathrm{Tr}\rho_A \log \rho_A$ for a reduced density matrix $\rho_A = \mathrm{Tr}_{A^c}\rho$ which is obtained after tracing out the complement of A. Usually, the entanglement entropy in QFT is defined for a spacelike region. Recently, entanglement entropy for "timelike regions" has been introduced from different theoretical motivations [1–3]. An intrinsic definition of timelike entanglement entropy, however, is still unclear. In particular, two key questions remain unanswered: What exactly is the timelike subsystem in QFT? And how to define entanglement entropy of the timelike subsystem?

A local field operator $\phi(x)$ in QFT cannot be considered a true observable, as $\phi(x)$ takes states out of the Hilbert space $\mathcal{H}$ when acting on them. This is manifested by the singularity appears in the operator product expansion (OPE) of $\phi(x)\phi(y)$. To obtain a well-defined Hilbert space operator, one might smear $\phi(x)$ over an open region $\mathcal{O}$, yielding $\Phi_f = \int_{\mathcal{O}} \mathrm{d}\mu\, f(x)\phi(x)$, where f is a smooth test function. However, smearing field operators over a spatial region V does not always produce a true operator. For instance, in QCD, smearing in space fails due to the non-integrable singularities in the OPE [4]. Instead, Borchers proved [5] that it suffices to smear out the field operator in *real time*,

$$\Phi_f(\vec{x}_0) = \int_{\mathcal{T}} \mathrm{d}t\, f(t)\phi(x), \tag{1}$$

where the test-function $f(t)$ is supported on a timelike interval,

$$\mathcal{T} = \left\{ x = (t, \vec{x}); |t| < \frac{T}{2}, \vec{x} = \vec{x}_0 \right\}. \tag{2}$$

Smearing in real time always converts $\phi(t, \vec{x})$ into true operators, enabling the definition of an algebra $\mathcal{A}$ generated by bounded functions of operators supported on $\mathcal{T}$. In the framework of algebraic QFT [6, 7], each open region in Minkowski spacetime is associated with an algebra $\mathcal{A}$ of observables, constituting a subsystem. We can thus view the algebra $\mathcal{A}(\mathcal{T})$, supported on the timelike interval $\mathcal{T}$, as a proper timelike subsystem in QFT across general dimensions. Such a timelike subsystem is physically meaningful, as it is generated by observables along an observer's worldline. In other words, a timelike subsystem represents what an observer can theoretically measure.

Defining the density matrix and von Neumann entropy in QFT is not straightforward. Unlike the Type I von Neumann algebra in quantum mechanics, the algebra $\mathcal{A}(\mathcal{O})$ associated with a causal diamond (or double cone) $\mathcal{O}$ in QFT is typically not isomorphic to $\mathfrak{B}(\mathcal{H}_{\mathcal{O}})$, the algebra of all bounded linear operators on some Hilbert space. For a causal diamond $\mathcal{O}$ and its causal complement $\mathcal{O}'$, the factorization of the vacuum Hilbert space $\mathcal{H} = \mathcal{H}_{\mathcal{O}} \otimes \mathcal{H}_{\mathcal{O}'}$ does not exist. Notwithstanding these challenges, for two centric causal diamonds $\mathcal{O}$ and $\widetilde{\mathcal{O}}$, the split property [7] guarantees the existence of an intermediate Type I factor $\mathcal{N}$ such that

$$\mathcal{A}(\mathcal{O}) \subset \mathcal{N} \subset \mathcal{A}(\widetilde{\mathcal{O}}), \quad \mathcal{O} \subset \widetilde{\mathcal{O}}. \tag{3}$$

The vacuum Hilbert space $\mathcal{H}$ can then be factorized as $\mathcal{H} = \mathcal{H}_{\mathcal{N}} \otimes \mathcal{H}_{\mathcal{N}'}$, where $\mathcal{N}'$ is the commutant of $\mathcal{N}$, defined as the set of all bounded operators on $\mathcal{H}$ that commute with $\mathcal{N}$. It has been suggested [8] that the vacuum von Neumann entropy of $\mathcal{A}$ with respect to $\mathcal{O}$ and $\widetilde{\mathcal{O}}$ could be defined as

$$S_{\mathcal{A}}(\mathcal{O}, \widetilde{\mathcal{O}}) = -\mathrm{Tr}\rho_{\mathcal{N}} \log \rho_{\mathcal{N}}, \tag{4}$$

where $\mathcal{N}$ is chosen as a canonical intermediate Type I factor [9], and $\rho_{\mathcal{N}}$ is the vacuum density matrix corresponding to $\mathcal{N}$. In our analysis, it is sufficient to consider the limit,

$$S(\mathcal{O}) := \lim_{\widetilde{\mathcal{O}} \to \mathcal{O}} S_{\mathcal{A}}(\mathcal{O}, \widetilde{\mathcal{O}}). \tag{5}$$

Here, $S(\mathcal{O})$ would capture the universal UV divergence when $\widetilde{\mathcal{O}}$ approaches to $\mathcal{O}$. Therefore, in QFT, $S(\mathcal{O})$ can be interpreted as the entanglement entropy for the algebra $\mathcal{A}(\mathcal{O})$ of operators in a causal diamond $\mathcal{O}$.

TIMELIKE TUBE THEOREM AND TIMELIKE ENTANGLEMENT ENTROPY

To define the entanglement entropy for the algebra $\mathcal{A}(\mathcal{T})$ of operators on a timelike interval $\mathcal{T}$, we first investigate the relationship between the algebra $\mathcal{A}(\mathcal{T})$ and the algebra $\mathcal{A}(\mathcal{O})$ associated with a causal diamond $\mathcal{O}$. In Minkowski spacetime, this question has been addressed by the timelike tube theorem [10, 11]. Starting with the timelike interval $\mathcal{T}$, its timelike envelope $\mathcal{E}_{\mathcal{T}}$ consists of all points reachable by deforming $\mathcal{T}$ along a family of timelike curves while keeping it fixed near its endpoints. In the following discussion, we restrict our analysis to favorable cases where the timelike envelope $\mathcal{E}_{\mathcal{T}}$ coincides with a causal diamond $\mathcal{O}$, which is defined as the intersection of the past and future of $\mathcal{T}$,

$$\mathcal{E}_{\mathcal{T}} = \mathcal{O} = J^{+}(\mathcal{T}) \cap J^{-}(\mathcal{T}), \tag{6}$$

as illustrated in Figure 2.

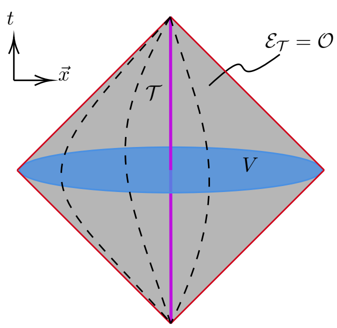

Figure 1. The timelike envelope $\mathcal{E}_{\mathcal{T}}$ (the gray shaded region) consists of all points that can be reached by deforming the timelike interval $\mathcal{T}$ (the purple line) to a family of timelike curves (black dashed lines), which is equivalent to a causal diamond $\mathcal{O}$. All points at $t = 0$ constitutes a spacelike ball V (the blue region).

The timelike tube theorem asserts that the algebra of operators on a timelike interval $\mathcal{T}$ is identical to the algebra of operators in its timelike envelope $\mathcal{E}_{\mathcal{T}}$, i.e.,

$$\mathcal{A}(\mathcal{T}) = \mathcal{A}(\mathcal{E}_{\mathcal{T}}). \tag{7}$$

This relation indicates that the timelike entanglement entropy can be defined as

$$S(\mathcal{T}) := S(\mathcal{E}_{\mathcal{T}}), \tag{8}$$

where $S(\mathcal{E}_{\mathcal{T}})$ is the von Neumann entropy defined in (5). This definition of timelike entanglement entropy relies on the principle that identical algebras should yield identical entanglement entropies. We emphasize that the timelike entanglement entropy depends only on the timelike interval $\mathcal{T}$, since the timelike envelope $\mathcal{E}_{\mathcal{T}}$ is uniquely determined by $\mathcal{T}$. Notice that $\mathcal{E}_{\mathcal{T}}$ is the domain of dependence of a spacelike ball V at $t = 0$, see Figure 2. Consequently, the algebra $\mathcal{A}(\mathcal{E}_{\mathcal{T}})$ is equivalent to the algebra $\mathcal{A}(V)$ of operators in an arbitrarily small spacetime neighborhood of V, up to unitary time evolution. Several known results for $S(V)$, or equivalently $S(\mathcal{E}_{\mathcal{T}})$, have been derived using the Euclidean path integral approach [12] and holography [13, 14].

It is ready to calculate the timelike entanglement entropy in CFT for a timelike interval,

$$\mathcal{T} = \left\{ x = (t, \vec{x}); |t| < \frac{T}{2}, \vec{x} = \vec{x}_0 \right\}. \tag{9}$$

In particular for a $(1+1)$-dimensional zero-temperature CFT on the spacetime $\mathbb{R}^2$, the spacelike ball V reduces to a spacelike interval of the length T. The entanglement entropy of V is known as [15, 16]:

$$S(V) = \frac{c}{3} \log \frac{T}{\epsilon}, \tag{10}$$

where c is the central charge and ϵ is the UV cutoff. Following the definition (8), the timelike entanglement entropy is then given by

$$S(\mathcal{T}) = S(V) = \frac{c}{3} \log \frac{T}{\epsilon}, \tag{11}$$

in agreement with earlier results in [2]. This result can be straightforwardly extended to finite temperatures:

$$S(\mathcal{T}) = \frac{c}{3} \log \left(\frac{\beta}{\pi \epsilon} \sinh \frac{\pi T}{\beta} \right), \tag{12}$$

where β is the inverse temperature. For a finite-size CFT_2 on $\mathbb{R} \times S^1$, the timelike entanglement entropy becomes

$$S(\mathcal{T}) = \frac{c}{3} \log \left(\frac{L}{\pi \epsilon} \sin \frac{\pi T}{L} \right), \tag{13}$$

with restriction $T < L$, where L is the total length of the spatial circle S^1. However, caution is required when $T \geq L$, as the timelike envelope $\mathcal{E}_{\mathcal{T}}$ may not always correspond to a causal diamond. More generally, for QFT_{d+1}, V is a d-dimensional spacelike ball with radius $R = T/2$, we still have

$$S(\mathcal{T}) = S(V), \tag{14}$$

since the timelike tube theorem (7) holds in general dimensions.

WHY SHOULD TIMELIKE ENTANGLEMENT ENTROPY BE REAL-VALUED?

Recent literature has predominantly considered a complex-valued timelike entanglement entropy, while our proposal yields a real-valued result. It is therefore important to resolve this contradiction. We work this out from both path integral argument and holography perspective.

A. Path integral argument

Let us consider the density matrix shown in Figure 2, on which the entanglement entropy between disjoint symmetric spacelike intervals A and B is computed [17, 18],

$$S_{\mathrm{vN}}(A:B)=\frac{c}{6}\log\frac{R_2}{R_1}. \tag{15}$$

The usual divergent entanglement entropy between adjacent regions is a simple adjacent limit of this disjoint system. It is evident that the path integrals on this single annulus Z_1 and replica $n-$folded annulus Z_n are both independent of specific intervals A and B. Consequently, the entanglement entropy between any two disjoint segments on this annulus is always equation (15). The asymmetric configurations are obtained by conformal transformations which do not give rise to complex numbers. *Thus, for any disjoint pairwise combination of timelike, spacelike, and lightlike regions, the entanglement entropy $S_{\mathrm{vN}}(A:B)$ is always real-valued.* The half-space entanglement entropy is achieved by setting $R_1 \to \epsilon$ and taking $R_2 \to \xi$ as the IR cutoff in equation (15). Note the usual divergent adjacent entanglement entropy $S_{\mathrm{vN}} = c/3 \log \ell/\epsilon$ for a finite length ℓ is obtained from $S_{\mathrm{vN}}(C:D)$ in the proper limits [17, 18]. From this picture, it is clear that if one only Wick rotates R_2 (i.e. ℓ) but not simultaneously rotates R_1 (i.e. ϵ), a spurious complex number appears.

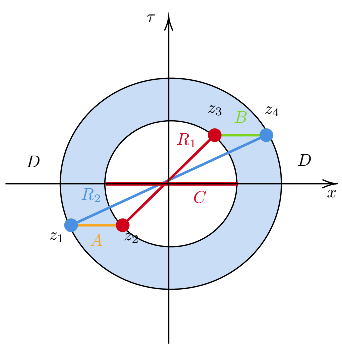

Figure 2. The density matrix for disjoint symmetric segments A and B. The usual adjacent configurations with divergent entanglement entropies $c/3\log\ell/\epsilon$ or $c/6\log\xi/\epsilon$ are obtained by taking simple limits.

B. Holography perspective

Without loss of generality, consider the entanglement entropy of a single timelike interval $\mathcal{T}_1 = \{0<t<1, x=0\}$ in zero-temperature CFT$_2$. This interval can be mapped onto the temporal half-axis $\mathcal{T}_{\mathrm{h}} = \{t>0, x=0\}$ via a special conformal transformation,

$$x^\mu \to \frac{x^\mu - b^\mu x^2}{1-2(b\cdot x)+b^2x^2}, \tag{16}$$

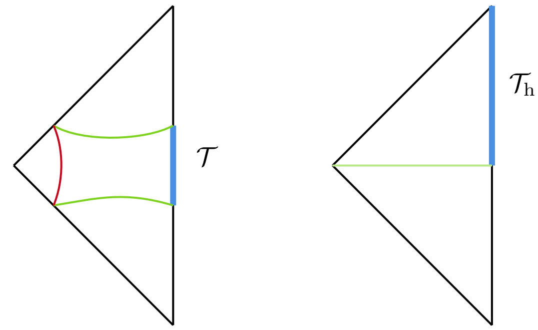

Figure 3. Penrose diagrams of Poincaré patch of AdS$_3$ spacetime and holographic duals of $S(\mathcal{T})$, $S(\mathcal{T}_{\mathrm{h}})$. In the left panel, the blue line represents a finite timelike interval $\mathcal{T}$ on the conformal boundary of AdS$_3$, the red line denotes a timelike geodesic, and the two green lines denote spacelike geodesics. In the right panel, the blue line represents the temporal half-axis $\mathcal{T}_{\mathrm{h}}$, and the green line denotes a spacelike geodesic.

with a timelike vector $b^\mu = (1,0)$. Therefore, the entanglement entropy of any finite timelike interval $\mathcal{T}$ equals that of the temporal half-axis $\mathcal{T}_{\mathrm{h}}$ due to conformal symmetry:

$$S(\mathcal{T}) = S(\mathcal{T}_{\mathrm{h}}). \tag{17}$$

It has been conjectured that the holographic dual of $S(\mathcal{T})$ consists of two spacelike geodesics and one timelike geodesic connecting them in AdS$_3$ spacetime, with the metric

$$\mathrm{d}s^2 = \frac{\mathrm{d}z^2 + \mathrm{d}x^\mu \mathrm{d}x_\mu}{z^2}, \tag{18}$$

as illustrated in Figure 3. The timelike geodesic is argued to contribute an imaginary part to $S(\mathcal{T})$.

However, this conjecture fails for $S(\mathcal{T}_{\mathrm{h}})$, whose holographic dual is a single spacelike geodesic from $(z,x^\mu) = (0,0)$ to $(z,x^\mu) = (+\infty, 0)$, containing no timelike geodesics. Thus, from the holographic perspective, $S(\mathcal{T}_{\mathrm{h}})$ is purely real-valued, and by the equivalence in (17), $S(\mathcal{T})$ must also be real-valued.

ENTANGLEMENT ACROSS TIME

It is natural to ask whether the timelike entanglement entropy represents entanglement across time. The answer appears to be affirmative. A fundamental example of entanglement in QFT is the case of the left and right Rindler wedges, originally analyzed by Bisognano and Wichmann [19]. In fact, for a massless and non-interacting QFT, it has been shown [20, 21] that entanglement exists between the operators within the future and past light-cones, analogous to the entanglement between the left and right Rindler wedges. This arises because, in such theories, every influence propagates along only lightlike intervals, leading to both spacelike commutativity and timelike commutativity. However, this is not generally true in massive, interacting QFTs. Nevertheless, we can conclude that entanglement across time does exist in certain special cases.

By the timelike tube theorem, the future and past light cones are associated with distinct segments of an observer's worldline. This provides a more physical interpretation of entanglement across time: if an observer measures only massless, non-interacting quantum fields, the observables measured in their future will be entangled with those measured in their past. To explore entanglement across time in the real world, a promising candidate is the pure Maxwell field, which is closely related to Wheeler's delayed-choice experiment with photons.

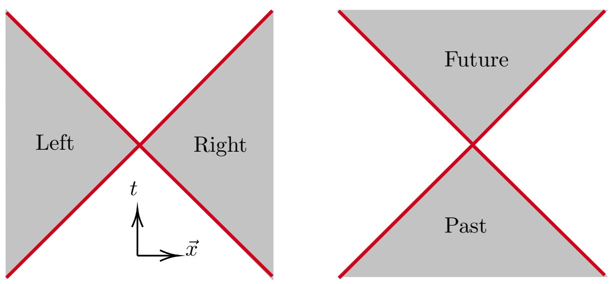

Figure 4. The left/right Rindler wedges and the future/past light-cones are illustrated, respectively.

DISCUSSIONS

In this letter, using known results in algebraic QFT, we proposed a rigorous definition for timelike entanglement entropy in QFT of general dimensions. The key insight is that timelike entanglement entropy depends only on the timelike interval and is real-valued.

The algebra of observables has attracted significant attention recently in quantum gravity and holography [22–25]. In particular, the algebra of an observer's worldline plays a central role in recent developments in quantum gravity [4, 26, 27]. The timelike tube theorem has been generalized to QFTs in curved spacetime [28–30], which, in principle, allows us to define the timelike entanglement entropy in the presence of gravity.

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