Doctopic: Quantum physics ARTICLE IN PRESS PLA:25040

Physics Letters A ••• (••••) •••–•••

Contents lists available at ScienceDirect

Physics Letters A

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Relativistic entanglement

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ARTICLE INFO

Article history:
Received 16 January 2018
Accepted 3 April 2018
Available online xxxx
Communicated by A. Eisfeld

Keywords:
Entanglement
Relativistic spin
Relativistic quantum theory

ABSTRACT

The relativistic quantum theory of Stueckelberg, Horwitz and Piron (SHP) describes in a simple way the experiment on interference in time of an electron emitted by femtosecond laser pulses carried out by Lindner et al. In this paper, we show that, in a way similar to our study of the Lindner et al. experiment (with some additional discussion of the covariant quantum mechanical description of spin and angular momentum), the experiment proposed by Palacios et al. to demonstrate entanglement of a two electron state, where the electrons are separated in time of emission, has a consistent interpretation in terms of the SHP theory. We find, after a simple calculation, results in essential agreement with those of Palacios et al.; but with the observed times as values of proper quantum observables.

1. Introduction

Palacios, Rescigno and McCurdy [1] have described a proposed experiment which could show entanglement of a two electron system in which each electron is emitted at a slightly different time. Although the anticipation of this effect is very reasonable, it does not have a theoretical justification in the framework of the standard nonrelativistic quantum theory, since in the nonrelativistic theory, both electrons must be prepared in states at precisely equal times. As for the Lindner et al. [2] experiment showing interference in time for the wave function of a particle, for which extensive calculations were done using the nonrelativistic Schrödinger evolution of the electron, wave functions at different times (corresponding to elements of different Hilbert spaces [3]) are incoherent in the nonrelativistic quantum theory. The direct product states corresponding to the basis for many body systems must, in the same way, be constructed from states in the same Hilbert space. Therefore, the same conclusion can be reached for the entanglement of the spins of a two body system. In actual practice, in fact, it would not be possible experimentally to generate two body states at precisely equal times, so that it is important to construct a theoretical basis, as we shall do below, in which effects of the type we expect to see (and are seen, for example, in the experiment of Lindner et al. [2]) can be consistently described.

The nonrelativistic theory of the two body state with spin is constructed from linear combinations of direct product wave functions taken at equal time [4]. One could argue intuitively from the vector model, in which the result $\mathbf{J}^2 = j(j+1)$ (for $\mathbf{J}$ the angular momentum operator, and j the integer or half-integer eigenvalue), that it appears that the physical angular momentum is not precisely along the "direction" of the vector $\mathbf{J}$, but can be thought of as precessing around it. The entangled spin zero state of two spin 1/2 systems therefore would be the result of an exact synchronization of these oppositely oriented precessing spins so that the total angular momentum is zero. At slightly different times, this synchronization would be, in principle, lost. Under nonrelativistic Schrödinger evolution the superposition of two-body states at different times would therefore be ineffective. Stated more rigorously, states are not coherent [3] at nonequal times and linear superposition is not defined in the nonrelativistic theory.

As for the Lindner et al. experiment [2], an explanation can be given in terms of the relativistic quantum theory of Stueckelberg, Horwitz and Piron (to be called SHP) [5]. The computation in terms of the SHP [6] was in precise agreement with the experimental result (actually predicted in 1976 [7], when the technology was not available for verification). In this paper, we apply a similar reasoning to the entangled two body state.

We start with a review of the basic SHP theory [5] and a discussion of how the Wigner theory of induced representations for relativistic spin is applied in this framework. We then argue that the proposed experiment of Palacios et al. should yield well-defined

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https://doi.org/10.1016/j.physleta.2018.04.004
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entanglement for the constituent particles at not precisely equal times.

Stueckelberg [5], in 1941, imagined that a particle world line would be straight for no interaction, but that interaction could bend the world line so that it would turn to propagate in the negative direction of time. To describe such a picture, he introduced an invariant parameter along the world line, which he called τ, and interpreted the backward in time evolving branch of the line as an antiparticle. Horwitz and Piron [5] then generalized this idea in the sense that the parameter τ was to be considered as a universal invariant time, as for the original postulate of Newton, in order to formulate the many body problem in this framework, as we discuss below.

As a model for the structure of the dynamical laws that might be considered, Stueckelberg proposed a Lorentz invariant Hamiltonian for free motion of the form

$$K = \frac{p^\mu p_\mu}{2M}, \tag{1.1}$$

where M is considered a parameter, with dimension mass, associated with the particle being described, but is not necessarily its measured mass. In fact, the numerator (with metric $-+++$; we generally take $c=1$),

$$p^\mu p_\mu = -m^2, \tag{1.2}$$

corresponds to the actual observed mass (according to the Einstein relation $E^2 = \mathbf{p}^2 + m^2$), where, in this context, m^2 is a dynamical variable.

The Hamilton equations, generalized covariantly to four dimensions, are then

$$\begin{aligned} \dot{x}^\mu &\equiv \frac{dx^\mu}{d\tau} = \frac{\partial K}{\partial p_\mu} \\ \dot{p}_\mu &\equiv \frac{dp_\mu}{d\tau} = -\frac{\partial K}{\partial x^\mu}. \end{aligned} \tag{1.3}$$

These equations are postulated to hold for any Hamiltonian model, such as with additive potentials or gauge fields. A Poisson bracket may be then defined in the same way as for the nonrelativistic theory. The construction is as follows. Consider the τ derivative of a function $F(x,p)$, *i.e.*,

$$\begin{aligned} \frac{dF}{d\tau} &= \frac{\partial F}{\partial x^\mu}\frac{dx^\mu}{d\tau} + \frac{\partial F}{\partial p^\mu}\frac{dp^\mu}{d\tau} \\ &= \frac{\partial F}{\partial x^\mu}\frac{\partial K}{\partial p_\mu} - \frac{\partial F}{\partial p^\mu}\frac{\partial K}{\partial x_\mu} \\ &= \{F, K\}, \end{aligned} \tag{1.4}$$

thus defining a Poisson bracket $\{F, G\}$ quite generally. The arguments of the nonrelativistic theory then apply, *i.e.*, that functions which obey the Poisson algebra isomorphic to their group algebras will have vanishing Poisson bracket with the Hamiltonian which has the symmetry of that group, and are thus conserved quantities, and the (τ independent) Hamiltonian itself is then (identically) a conserved quantity.

It follows from the Hamilton equations that for the free particle case

$$\dot{x}^\mu = \frac{p^\mu}{M} \tag{1.5}$$

and therefore, dividing the space components by the time components, cancelling the $d\tau$'s ($p^0 = E$ and $x^0 = t$),

$$\frac{d\mathbf{x}}{dt} = \frac{\mathbf{p}}{E}, \tag{1.6}$$

the Einstein relation for the observed velocity. Furthermore, we see that

$$\dot{x}^\mu \dot{x}_\mu = \frac{p^\mu p_\mu}{M^2}; \tag{1.7}$$

with the definition of the invariant

$$ds^2 = -dx^\mu dx_\mu, \tag{1.8}$$

corresponding to proper time squared (for a timelike interval), this becomes

$$\frac{ds^2}{d\tau^2} = \frac{m^2}{M^2}. \tag{1.9}$$

Therefore, the proper time interval Δs of a particle along a trajectory parametrized by τ is equal to the corresponding interval $\Delta\tau$ only if $m^2 = M^2$, a condition we shall call "on mass shell".

Stueckelberg [5] formulated the quantized version of this theory by postulating the commutation relations

$$[x^\mu, p^\nu] = i\hbar g^{\mu\nu}, \tag{1.10}$$

where $g^{\mu\nu}$ is the Lorentz metric given above, and a Schrödinger type equation (we shall take $\hbar = 1$ in the following)

$$i\frac{\partial}{\partial\tau}\psi_\tau(x) = K\psi_\tau(x), \tag{1.11}$$

where $\psi(x)$ is an element of a Hilbert space on R^4 satisfying

$$\int |\psi(x)|^2 d^4x = 1, \tag{1.12}$$

and satisfies the required Hilbert space property of linear superposition. With the generalization of Horwitz and Piron [5], Eq. (1.11) can be written for any number N of particles as

$$i\frac{\partial}{\partial\tau}\psi_\tau(x_1, x_2 \dots x_N)) = K\psi_\tau(x_1, x_2 \dots x_N), \tag{1.13}$$

where K could have, for example, the form

$$K = \Sigma_i^N \frac{{p_i}^\mu p_{i\mu}}{2M_i} + V(x_1, x_2 \dots x_N), \tag{1.14}$$

and $V(x_1, x_2 \dots x_N)$ is assumed, for our present purposes, to be Poincaré invariant.

The basis of the Hilbert space describing such states is provided by the direct product of one particle wave functions *taken at equal* τ (as for equal time t in the nonrelativistic theory [4]). In the following, we apply this structure to the description of two particles with spin.

2. Relativistic spin and the Dirac representation

We shall discuss in this section the basic idea of a relativistic particle with spin, based on Wigner's seminal work [8]. The theory is adapted here to be applicable to relativistic quantum theory; in this form, Wigner's theory, together with the requirements imposed by the observed correlation between spin and statistics in nature for identical particle systems, makes it possible to define the total spin of a state of a relativistic many body system.

The spin of a particle in a nonrelativistic framework corresponds to the lowest dimensional nontrivial representation of the rotation group; the generators are the Pauli matrices σ_i divided by two, the generators of the fundamental representation of the double covering of $SO(3)$. The self-adjoint operators that are the generators of this group measure angular momentum and are associated with magnetic moments. Such a description is not relativistically covariant, but Wigner [8] has shown how to describe this dynamical property of a particle in a covariant way. The method developed by Wigner provided the foundation for what is now known as the theory of induced representations [9], with very wide applications, including a very powerful approach to finding the representations of noncompact groups [9].

In the nonrelativistic quantum theory, the spin states of a two or more particle system are defined by combining the spins of these particles at equal time using appropriate Clebsch–Gordan coefficients [4][10] at each value of the time. The restriction to equal time follows from the tensor product form of the representation of the quantum states for a many body problem [4]. For two spin 1/2 (Fermi–Dirac) particles, for example, an antisymmetric space distribution would correspond to a symmetric combination of the spin factors, *i.e.* a spin one state, and a symmetric space distribution would correspond to an antisymmetric spin combination, a spin zero state. This correlation is the source of the famous Einstein–Podolsky–Rosen discussion [11]. The experiment proposed by Palacios et al. [1] suggests that spin entanglement could occur for two particles at non-equal times; the spin carried by wave functions of SHP type would naturally carry such correlations over the width in t of the wave packets, and therefore would provide a simple and rigorous prediction for this experiment.

Wigner's formulation [8], however, was not appropriate for application to a consistent relativistic quantum theory, since it does not preserve, as we shall explain below, the covariance of the expectation value of coordinate operators [5]. Before constructing a generalization of Wigner's method which is useful in relativistic quantum theory we first review Wigner's method in its original form, and show how the difficulties arise.

To establish some notation and the basic method, we start with the basic principle of relativistic covariance for a scalar quantum wave function $\psi(p)$. In a new Lorentz frame described by the parameters Λ of the Lorentz group, for which $p'^{\mu} = \Lambda^{\mu}_{\nu} p^{\nu}$ (we work in momentum space here for convenience), the same physical point in momentum space described in different coordinates, by arguing that the probability density must be the same,

$$\psi'(p') = \psi(p) \tag{2.1}$$

up to a phase, which we take to be unity. It then follows that as a function of p,

$$\psi'(p) = \psi(\Lambda^{-1} p). \tag{2.2}$$

Since, in Dirac's notation,

$$\psi'(p) \equiv < p|\psi' >, \tag{2.3}$$

Eq. (2.2) follows equivalently by writing

$$|\psi' > = U(\Lambda)|\psi > \tag{2.4}$$

so that

$$\begin{aligned}\psi'(p) = < p|\psi' > &= < p|U(\Lambda)|\psi > \\ &= < \Lambda^{-1} p|\psi > \\ &= \psi(\Lambda^{-1} p),\end{aligned} \tag{2.5}$$

where we have used

$$U(\Lambda)^{\dagger}|p > = U(\Lambda^{-1})|p > = |\Lambda^{-1} p > . \tag{2.6}$$

To discuss the transformation properties of the representation of a relativistic particle with spin, Wigner proposed that we consider a special frame in which $p_0^{\mu} = (m, 0, 0, 0)$; the subgroup of the Lorentz group that leaves this vector invariant is clearly $O(3)$, the rotations in the three space in which $\mathbf{p} = 0$, or its covering $SU(2)$. Under a Lorentz boost, transforming the system to its representation in a moving inertial frame, the rest momentum appears as $p_0^{\mu} \to p^{\mu}$, but under this unitary transformation, the subgroup that leaves p_0^{μ} invariant is carried to a form which leaves p^{μ} invariant, and the group remains $SU(2)$. The 2×2 matrices representing this group are altered by the Lorentz transformation, and are functions of the momentum p^{μ}. The resulting state then transforms by a further change in p^{μ} and an $SU(2)$ transformation compensating for this change. This additional transformation is called the "little group" of Wigner. The family of values of p^{μ} generated by Lorentz transformations on p_0^{μ} is called the "orbit" of the induced representation. This $SU(2)$, in its lowest dimensional representation, parametrized by p^{μ} and the additional Lorentz transformation Λ, corresponds to Wigner's covariant relativistic definition of the spin of a relativistic particle [8].

We now apply this method to review Wigner's construction based on a representation induced on the momentum p^{μ}. Let us define the momentum-spin ket

$$|p, \sigma > \equiv U(L(p))|p_0, \sigma >, \tag{2.7}$$

where $U(L(p))$ is the unitary operator inducing a Lorentz transformation of the timelike $p_0 = (m, 0, 0, 0)$ (rest frame momentum) to the general timelike vector p^{μ}. The effect of a further Lorentz transformation parameterized by Λ, induced by $U(\Lambda^{-1})$, can be written as

$$\begin{aligned}&U(\Lambda^{-1})|p, \sigma > \\ &\quad = U(L(\Lambda^{-1} p))U^{-1}(L(\Lambda^{-1} p))U(\Lambda^{-1})U(L(p))|p_0, \sigma >\end{aligned} \tag{2.8}$$

The product of the last three unitary factors

$$U^{-1}(L(\Lambda^{-1} p))U(\Lambda^{-1})U(L(p)) \tag{2.9}$$

has the property that under this combined unitary transformation, the ket is transformed so that $p_0 \to p_0$, and thus corresponds to just a rotation (called the Wigner rotation), the stability subgroup of the vector p_0. This rotation can be represented by a 2×2 matrix acting on the index σ, *i.e.*, so that

$$\begin{aligned}U(\Lambda^{-1})|p, \sigma > &= U(L(\Lambda^{-1} p))|p_0, \sigma' > D_{\sigma,\sigma'}(\Lambda, p) \\ &= |\Lambda^{-1} p, \sigma' > D_{\sigma,\sigma'}(\Lambda, p),\end{aligned} \tag{2.10}$$

where, as a representation of rotations, D is unitary. Therefore, taking the complex conjugate of

$$< \psi|U(\Lambda^{-1})|p, \sigma > = < \psi|\Lambda^{-1} p, \sigma' > D_{\sigma,\sigma'}(\Lambda, p),$$

one obtains

$$< p, \sigma|U(\Lambda)\psi > = < \Lambda^{-1} p, \sigma'|\psi > D_{\sigma',\sigma}(\Lambda, p), \tag{2.11}$$

where, in this construction,

$$D_{\sigma',\sigma}(\Lambda, p) = \left((L(p)^{-1} \Lambda L(\Lambda^{-1} p))\right)_{\sigma',\sigma} , \tag{2.12}$$

expressed in terms of the $SL(2, C)$ matrices corresponding to the unitary transformation (2.9). The result (2.11) can be written as

$$\psi'(p, \sigma) = \psi(\Lambda^{-1} p, \sigma') D_{\sigma',\sigma}(\Lambda, p). \tag{2.13}$$

The algebra of the 2×2 matrices of the fundamental representation of the group $SL(2, C)$ are isomorphic to that of the Lorentz group, and the product of the corresponding matrices provide the 2×2 matrix representation of $D_{\sigma',\sigma}(\Lambda, p)$; we may therefore have

$$D_{\sigma',\sigma}(\Lambda, p) = \left(L^{-1}(p) \Lambda L(\Lambda^{-1} p)\right)_{\sigma',\sigma}, \tag{2.14}$$

where L and Λ are the 2×2 matrices of $SL(2, C)$.

Please cite this article in press as: L. Horwitz, R.I. Arshansky, Relativistic entanglement, Phys. Lett. A (2018), https://doi.org/10.1016/j.physleta.2018.04.004

As we have mentioned above, the presence of the p-dependent matrices representating the spin of a relativistic particle in the transformation law of the wave function destroys the covariance, in a relativistic quantum theory, of the expectation value of the coordinate operators. To see this, consider the expectation value of the dynamical variable x^μ, *i.e.*

$$< x^\mu > = \Sigma_\sigma \int d^4 p \psi(p,\sigma)^\dagger i \frac{\partial}{\partial p_\mu} \psi(p,\sigma). \tag{2.15}$$

A Lorentz transformation would introduce the p-dependent 2×2 unitary transformation on the function $\psi(p)$, and the derivative with respect to momentum would destroy the covariance property that we would wish to see of the expectation value $< x^\mu >$.

It is also not possible, in this framework, to form wave packets of definite spin by integrating over the momentum variable, since this would add functions over different parts of the orbit, with a different $SU(2)$ at each point.

As we describe in the following, these problems can be solved by inducing a representation of the spin on a timelike unit vector n^μ in place of the four-momentum, using a representation induced on a timelike vector, say, n^μ, which is independent of x^μ or p^μ [12][13]. This solution also permits the linear superposition of momentum states to form wave packets of definite spin, and admits the construction of definite spin states for many body relativistic systems. In the following, we show how such a representation can be constructed.

Let us define, as in (2.7),

$$|n, \sigma, x > \equiv U(L(n))|n_0, \sigma, x >, \tag{2.16}$$

where we may admit a dependence on x (or, through Fourier transform, on p). Here, we distinguish the action of $U(L(n))$ from the general Lorentz transformation $U(\Lambda)$; $U(L(n))$ acts only on the vector space of the n^μ. Its infinitesimal generators are given by

$$M_n^{\mu\nu} = -i(n^\mu \frac{\partial}{\partial n_\nu} - n^\nu \frac{\partial}{\partial n_\mu}), \tag{2.17}$$

while the generators of the transformations $U(\Lambda)$ act on the full vector space of both the n^μ and the x^μ (as well as p^μ). In terms of the canonical variables,

$$M^{\mu\nu} = M_n^{\mu\nu} + (x^\mu p^\nu - x^\nu p^\mu). \tag{2.18}$$

The operator (2.17) is self-adjoint in the full Hilbert space norm defined by the integral of the norm (in the sheets of the foliation defined by n^μ) to be defined in (2.25) over $d^4 n \delta(n^\mu n_\mu + 1) d^4 x = \frac{d^3\mathbf{n}}{n_0} d^4 x$. The two terms of the full generator commute. Following the method outlined above, we now investigate the properties of a total Lorentz transformation, *i.e.*

$$\begin{aligned} &U(\Lambda^{-1})|n, \sigma, x > \\ &\quad = U(L(\Lambda^{-1}n)\big(U^{-1}(L(\Lambda^{-1}n))U(\Lambda^{-1})U(L(n))\big)\big)|n_0, \sigma, x > . \end{aligned} \tag{2.19}$$

Now, consider the conjugate of (2.19),

$$\begin{aligned} &< n, \sigma, x|U(\Lambda) \\ &\quad = < n_0, \sigma, x|\big(U(L^{-1}(n))U(\Lambda)U(L(\Lambda^{-1}n))\big)U^{-1}(L(\Lambda^{-1}n)). \end{aligned} \tag{2.20}$$

The operator in the first factor (in parentheses) preserves n_0, and therefore, as before, contains an element of the little group associated with n^μ which may be represented by the matrices of $SL(2,C)$. It also acts, due to the factor $U(\Lambda)$ (for which the generators are those of the Lorentz group acting both on n and x (or p), as in (2.18)), taking $x \to \Lambda^{-1} x$ in the conjugate ket on the left. Taking the product on both sides with $|\psi >$, we obtain

$$< n, \sigma, x|\psi >' = < \Lambda^{-1}n, \sigma', \Lambda^{-1}x|\psi > D_{\sigma',\sigma}(\Lambda, n), \tag{2.21}$$

or

$$\psi'_{n,\sigma}(x) = \psi_{\Lambda^{-1}n,\sigma'}(\Lambda^{-1}x) D_{\sigma',\sigma}(\Lambda, n), \tag{2.22}$$

where

$$D(\Lambda, n) = L^{-1}(n)\Lambda L(\Lambda^{-1}n), \tag{2.23}$$

with Λ and $L(n)$ the corresponding 2×2 matrices of $SL(2,C)$ (Λ and $L(n)$ are the corresponding 2×2 matrices of $SL(2,C)$).

With this transformation law, one may take the Fourier transform to obtain the wave function in momentum space, and conversely. The matrix D is an element of $SU(2)$, and therefore linear superpositions over momenta or coordinates maintain the definition of the particle spin, and interference phenomena for relativistic particles with spin may be studied consistently. Furthermore, if two or more particles with spin are represented in representations induced on n^μ, at a given value of n^μ on their respective orbits, their spins can be added by the standard methods with the use of Clebsch–Gordan coefficients [10]. This method therefore admits the treatment of a many body relativistic system with spin [14]. It is interesting to note that the little group rotations defined by (2.23) are in a spacelike surface defined by n^μ. The vector n^μ may be thought of as the normal to the spacelike surfaces defined by Schwinger [16] in the discussion of his variational principle for quantum field theory, thus providing a natural framework for the development of a covariant spinor formalism without reference to the momentum representation.

There are two fundamental representations of $SL(2,C)$ which are inequivalent [15]. Multiplication by the operator $\sigma \cdot p$ of a two dimensional spinor representing one of these results in an object transforming like the second representation. Such an operator could be expected to occur in a dynamical theory, and therefore the state of lowest dimension in spinor indices of a physical system should contain both representations [5]. As we shall emphasize, however, in our treatment of the more than one particle system, for the rotation subgroup, both of the fundamental representations yield the same $SU(2)$ matrices up to a unitary transformation, and therefore the Clebsch–Gordan decomposition of the product state into irreducible representations may be carried out independently of which fundamental $SL(2,C)$ representation is associated with each of the particles [14].

We now discuss the construction of Dirac spinors.

The defining relation for the fundamental $SL(2,C)$ matrices is

$$\Lambda^\dagger \sigma^\mu n_\mu \Lambda = \sigma^\mu (\Lambda^{-1} n)_\mu, \tag{2.24}$$

where $\sigma^\mu = (\sigma^0, \sigma)$; σ^0 is the unit 2×2 matrix, and σ are the Pauli matrices. Since the determinant of $\sigma^\mu n_\mu$ is the Lorentz invariant ${n^0}^2 - \mathbf{n}^2$, and the determinant of Λ is unity in $SL(2,C)$, the transformation represented on the left hand side of (2.24) must induce a Lorentz transformation on n^μ. The inequivalent second fundamental representation may be constructed by using this defining relation with σ^μ replaced by $\underline{\sigma}^\mu \equiv (\sigma^0, -\sigma)$. For every Lorentz transformation Λ acting on n^μ, this defines an $SL(2,C)$ matrix $\underline{\Lambda}$ (we use the same symbol for the Lorentz transformation on a four-vector as for the corresponding $SL(2,C)$ matrix acting on the 2-spinors).

Since both fundamental representations of $SL(2,C)$ should occur in the general quantum wave function representing the state of the system, the norm in each n-sector of the Hilbert space must be defined as [13]

$$N = \int d^4x(|\hat{\psi}_n(x)|^2 + |\hat{\phi}_n(x)|^2), \tag{2.25}$$

where $\hat{\psi}_n$ transforms with the first $SL(2,C)$ and $\hat{\phi}_n$ with the second. From the construction of the little group (2.21), it follows that $L(n)\psi_n$ transforms with Λ, and $\underline{L}(n)\phi_n$ transforms with $\underline{\Lambda}$; making this replacement in (2.23), and using the fact, obtained from the defining relation (3.22), that $L(n)^{\dagger^{-1}}L(n)^{-1} = \mp\sigma^\mu n_\mu$ and $\underline{L}(n)^{\dagger^{-1}}\underline{L}(n)^{-1} = \mp\underline{\sigma}^\mu n_\mu$, one finds that

$$N = \mp\int d^4x\bar{\psi}_n(x)\gamma\cdot n\psi_n(x), \tag{2.26}$$

where $\gamma\cdot n \equiv \gamma^\mu n_\mu$ (for which $(\gamma\cdot n)^2 = -1$), and the matrices γ^μ are the Dirac matrices as defined in the books of Bjorken and Drell [17]. Here, the four-spinor $\psi_n(x)$ is defined by

$$\psi_n(x) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} L(n)\hat{\psi}_n(x) \\ \underline{L}(n)\hat{\phi}_n(x) \end{pmatrix}, \tag{2.27}$$

and the sign $\mp$ corresponds to n^μ in the positive or negative light cone. The wave function defined in (2.26) transforms as

$$\psi'_n(x) = S(\Lambda)\psi_{\Lambda^{-1}n}(\Lambda^{-1}x) \tag{2.28}$$

and $S(\Lambda)$ is a (nonunitary) transformation generated infinitesimally, as in the standard Dirac theory (see, for example [17]), by $\Sigma^{\mu\nu} \equiv \frac{i}{4}[\gamma^\mu, \gamma^\nu]$.

The Dirac operator $\gamma\cdot p$ is not Hermitian in the (invariant) scalar product associated with the norm (2.16). It is of interest to consider the Hermitian and anti-Hermitian parts

$$\begin{aligned} K_L &= \frac{1}{2}(\gamma\cdot p + \gamma\cdot n\gamma\cdot p\gamma\cdot n) = -(p\cdot n)(\gamma\cdot n) \\ K_T &= \frac{1}{2}\gamma^5(\gamma\cdot p - \gamma\cdot n\gamma\cdot p\gamma\cdot n) = -2i\gamma^5(p\cdot K)(\gamma\cdot n), \end{aligned} \tag{2.29}$$

where $K^\mu = \Sigma^{\mu\nu}n_\nu$, and we have introduced the factor $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, which anticommutes with each γ^μ and has square -1 so that K_T is Hermitian and commutes with the Hermitian K_L. Since

$$K_L^2 = (p\cdot n)^2 \tag{2.30}$$

and

$$K_T^2 = p^2 + (p\cdot n)^2, \tag{2.31}$$

we may consider

$$K_T^2 - K_L^2 = p^2 \tag{2.32}$$

to pose an eigenvalue problem analogous to the second order mass eigenvalue condition for the free Dirac equation (the Klein Gordon condition). For the Stueckelberg equation of evolution corresponding to the free particle, we may therefore take [13]

$$K_0 = \frac{1}{2M}(K_T^2 - K_L^2) = \frac{1}{2M}p^2. \tag{2.33}$$

In the presence of electromagnetic interaction, gauge invariance under a spacetime dependent gauge transformation, the expressions for K_T and K_L given in (2.29), in gauge covariant form, then imply, in place of (2.33),

$$K = \frac{1}{2M}(p - eA)^2 + \frac{e}{2M}\Sigma_n^{\mu\nu}F_{\mu\nu}(x), \tag{2.34}$$

where

$$\Sigma_n^{\mu\nu} = \Sigma^{\mu\nu} + K^\mu n^\nu - K^\nu n^\mu \equiv \frac{i}{4}[\gamma_n^\mu, \gamma_n^\nu], \tag{2.35}$$

where the γ_n^μ are defined in (2.39). The expression (2.34) is quite similar to that of the second order Dirac operator; it is, however, Hermitian in the scalar product defined by (2.26); it has no direct *electric* coupling to the electromagnetic field in the special frame for which $n^\mu = (1,0,0,0)$ in the minimal coupling model we have given here (note that in his calculation of the anomalous magnetic moment, Schwinger [18] puts the electric field to zero; a non-zero electric field would lead to a non-Hermitian term in the standard Dirac propagator, the inverse of the Klein–Gordon square of the interacting Dirac equation). The matrices $\Sigma_n^{\mu\nu}$ are, in fact, a relativistically covariant form of the Pauli matrices.

To see this, we note that the quantities K^μ and $\Sigma_n^{\mu\nu}$ satisfy the commutation relations

$$\begin{aligned} [K^\mu, K^\nu] &= -i\Sigma_n^{\mu\nu} \\ [\Sigma_n^{\mu\nu}, K^\lambda] &= -i[(g^{\mu\lambda} + n^\nu n^\lambda)K^\mu - (g^{\mu\lambda} + n^\mu n^\lambda)K^\nu, \\ [\Sigma_n^{\mu\nu}, \Sigma_n^{\lambda\sigma}] &= -i[(g^{\nu\lambda} + n^\nu n^\lambda)\Sigma_n^{\mu\sigma} + (g^{\sigma\mu} + n^\sigma n^\mu)\Sigma_n^{\lambda\nu} \\ &\quad - (g^{\mu\lambda} + n^\mu n^\lambda)\Sigma_n^{\nu\sigma} + (g^{\sigma\nu} + n^\sigma n^\nu)\Sigma_n^{\lambda\nu}]. \end{aligned} \tag{2.36}$$

Since $K^\mu n_\mu = n_\mu\Sigma_n^{\mu\nu} = 0$, there are only three independent K^μ and three $\Sigma_n^{\mu\nu}$. The matrices $\Sigma_n^{\mu\nu}$ are a covariant form of the Pauli matrices, and the last of (2.36) is the Lie algebra of $SU(2)$ in the spacelike surface orthogonal to n^μ. The three independent K^μ correspond to the non-compact part of the algebra which, along with the $\Sigma_n^{\mu\nu}$ provide a representation of the Lie algebra of the full Lorentz group. The covariance of this representation follows from

$$S^{-1}(\Lambda)\Sigma_{\Lambda n}^{\mu\nu}S(\Lambda)\Lambda_\mu^\lambda\Lambda_\nu^\sigma = \Sigma_n^{\lambda\sigma}. \tag{2.37}$$

In the special frame for which $n^\mu = (1,0,0,0))$, $\Sigma_n^{i,j}$ become the Pauli matrices $\frac{1}{2}\sigma^k$ with (i,j,k) cyclic, and $\Sigma_n^{0j} = 0$. In this frame there is no direct electric interaction with the spin in the minimal coupling model (2.34). We remark that there is, however, a natural spin coupling which becomes pure electric in the special frame, given by

$$i[K_T, K_L] = -ie\gamma^5(K^\mu n^\nu - K^\nu n^\mu)F_{\mu\nu}. \tag{2.38}$$

It is easy to see that the value of this commutator reduces to $\mp e\sigma\cdot\mathbf{E}$ in the special frame for which $n^0 = -1$; this operator is Hermitian and would correspond to an electric dipole interaction with the spin.

Note that the matrices

$$\gamma_n^\mu = \gamma_\lambda\pi^{\lambda\mu}, \tag{2.39}$$

where the projection

$$\pi^{\lambda\mu} = g^{\lambda\mu} + n^\lambda n^\mu, \tag{2.40}$$

appearing in (2.36), play an important role in the description of the dynamics in the induced representation. In (2.34), the existence of projections on each index in the spin coupling term implies that $F^{\mu\nu}$ can be replaced by $F_n{}^{\mu\nu}$ in this term, a tensor projected into the foliation subspace.

We further remark that in relativistic scattering theory, the S-matrix is Lorentz covariant [17]. The asymptotic states can be decomposed according to the conserved projection operators

$$P_\pm = \frac{1}{2}(1 \mp \gamma \cdot n)$$
$$P_{E\pm} = \frac{1}{2}(1 \mp \frac{p \cdot n}{|p \cdot n|}) \tag{2.41}$$
and
$$P_{n\pm} = \frac{1}{2}(1 \pm \frac{2i\gamma^5 K \cdot p}{[p^2 + (p \cdot n)^2]^{1/2}}).$$

The operator

$$\frac{2i\gamma^5 K \cdot p}{[p^2 + (p \cdot n)^2]^{1/2}} \to \sigma \cdot \mathbf{p}/|\mathbf{p}| \tag{2.42}$$

when $n^\mu \to (1, 0, 0, 0)$. *i.e.*, $P_{n\pm}$ corresponds to a helicity projection. Therefore the matrix elements of the S-matrix at any point on the orbit of the induced representation is equivalent (by replacing S by $U(L(n))SU^{-1}(L(n))$) to the corresponding helicity representation associated with the frame in which n^μ is n^0.

The anomalous magnetic moment of the electron can be computed in this framework (Bennett [19]) without appealing to the full quantum field theory of electrodynamics.

3. The many body problem with spin, and spin-statistics

As in the nonrelativistic quantum theory, one represents the state of an N-body system in terms of a basis given by the tensor product of N one-particle states, each an element of a one-particle Hilbert space. The general state of such an N-body system is given by a linear superposition over this basis [4]. Second quantization then corresponds to the construction of a Fock space, for which the set of all N body states, for all N, are imbedded in a large Hilbert space, for which operators that change the number N are defined [4]. In order to construct the tensor product space corresponding to the many-body system, we must consider, as for the nonrelativistic theory, only the product of wave functions which are elements of the same Hilbert space. In the nonrelativistic theory, this corresponds to functions at equal time; in the relativistic theory, the functions are taken to be at equal τ. Thus, in the relativistic theory, there are correlations at unequal t, within the support of the Stueckelberg wave functions. Moreover, for particles with spin we argue that in the induced representation, these functions must be taken at *identical values of* n^μ, *i.e.*, taken at the same point on the orbits of the induced representation of each particle [20].

This statement lies in the observation that the spin-statistics relation appears to be a universal fact of nature. An elementary proof, for example, for a system of two spin 1/2 particles, is that a π rotation of the system introduces a phase factor of $e^{i\frac{\pi}{2}}$ for each particle, thus introducing a minus sign for the two body state. However, the π rotation is equivalent to an interchange of the two identical particles. This argument rests on the fact that each particle is in *the same representation* of $SU(2)$, which can only be achieved in the induced representation with the particles at the same point on their respective orbits. The same argument applies for bosons, which must be symmetric under interchange (in this case the phase of each factor in a pair is $e^{i\pi}$). We therefore see that identical particles must carry the same value of n^μ [20], and the construction of the N-body system must follow this rule. It therefore follows that the two body relativistic system can carry a spin computed by use of the usual Clebsch–Gordan coefficients, and entanglement would follow even at unequal time (within the support of the equal τ wave functions), as in the proposed experiment of Palacios et al. [1]. This argument can be followed for arbitrary N, and therefore the Fock space of quantum field theory carries the properties usually associated with fermion (or boson) fields, with the entire Fock space foliated over the orbit of the inducing vector n^μ.

Let us now construct a two body Hilbert space in the framework of the relativistic quantum theory. The states of this two body space are given by linear combinations over the product wave functions, where the wave functions (for the spin (1/2) case; the formulation is the same for bosons) are of the type described in (2.27), *i.e.* (for equal n and τ),

$$\psi_{ij}(x_1, x_2) = \psi_i(x_1) \otimes \psi_j(x_2), \tag{3.1}$$

where $\psi_i(x_1)$ and $\psi_j(x_2)$ are elements of the one-particle Hilbert space $\mathcal{H}$. Let us introduce the notation, often used in differential geometry, that

$$\psi_{ij}(x_1, x_2) = \psi_i \otimes \psi_j(x_1, x_2), \tag{3.2}$$

identifying the arguments according to a standard ordering. Then, without specifying the spacetime coordinates, we can write

$$\psi_{ij} = \psi_i \otimes \psi_j, \tag{3.3}$$

formally, an element of the tensor product space $\mathcal{H}_1 \otimes \mathcal{H}_2$. The scalar product is carried out by pairing the elements in the two factors according to their order, since it corresponds to integrals over x_1, x_2, *i.e.*,

$$(\psi_{ij}, \psi_{k,\ell}) = (\psi_i, \psi_k)(\psi_j, \psi_\ell). \tag{3.4}$$

For two identical particle states satisfying Bose–Einstein of Fermi–Dirac statistics, we must write, according to our argument given above,

$$\psi_{ijn} = \frac{1}{\sqrt{2}}[\psi_{in} \otimes \psi_{jn} \pm \psi_{jn} \otimes \psi_{in}], \tag{3.5}$$

where $n \equiv n^\mu$ is the timelike four vector labelling the orbit of the induced representation. This expression has the required symmetry or antisymmetry only if both functions are on the same points of their respective orbits in the induced representation. Furthermore, they transform under the *same* $SU(2)$ representation of the rotation subgroup of the Lorentz group, and thus for spin 1/2 particles, under a π spatial rotation (defined by the space orthogonal to the timelike vector n^μ) they both develop a phase factor $e^{i\frac{\pi}{2}}$. The product results in an over all negative sign. As in the usual quantum theory, this rotation corresponds to an interchange of the two particles, but here with respect to a "spatial" rotation around the timelike vector n^μ. The spacetime coordinates in the functions are rotated in this (foliated) subspace of spacetime, and correspond to an actual exchange of the positions of the particles in space time, as in the formulation of the standard spin-statistics theorem. It therefore follows that the interchange of the particles occurs in the foliated space defined by n^μ, and, furthermore:

> *The antisymmetry of identical spin 1/2 (fermionic) particles, at equal τ, remains at unequal times (within the support of the wave functions). This is true for the symmetry of identical spin zero (bosonic) particles as well.*

The construction we have given enables us to define the spin of a many body system, even if the particles are relativistic and moving arbitrarily with respect to each other.

> *The spin of an N-body system is well-defined, independent of the state of motion of the particles of the system, by the usual laws of combining representations of $SU(2)$, i.e., with the usual Clebsch–Gordan coefficients, if the states of all the particles in the system*

are in induced representations at the same point of the orbit n^μ and equal τ.

[14]

Furthermore, as we have pointed out, the generators of the rotation groups in the fibre n of the foliation, act in the spacelike subspace orthogonal to n^μ. Therefore, orbital angular momenta can as well be combined using standard Clebsch–Gordan addition for any number of particles, independently of the fact that they are in relative motion

4. The Palacios et al. experiment

The Palacios et al. prediction for the measurement of existence of entanglement of spin 1/2 electrons emitted by double ionization of helium rests on the interference that can be observed for the space–time configuration part of the wave functions, which are symmetric, since the spin part is antisymmetric in the spin zero state. As we have pointed out, the antisymmetry of the spin state at *unequal* times (within the support of the wave function) is valid in the SHP theory, and the corresponding spacetime parts of the wave function will be, in the same way, symmetric. This experiment would then show interference between parts of the wave function carrying different values of the t variable in the same way as in the Lindner et al. experiment. The orders of magnitude of time intervals in the Palacios et al. configuration are, in fact, due to the characteristic properties of helium, very close to those of the Lindner et al. experiment. The time intervals involved are therefore also of the order of femtoseconds. Our discussion, in the framework of the SHP theory, considers the time t as an observable, with a spread (rigorously obeying the uncertainty relation $\Delta t \Delta E \geq \frac{1}{2}\hbar$) in the wavepackets (on the Hilbert space over the measure d^4x), *for both particles at equal* τ.

The two entangled electrons are considered to be emitted, with the same polarization, with energies $E_1 = 35$ eV and $E_2 = 69$ eV (about 10.4 eV and 14.6 eV after atomic physics corrections), separated by time intervals of the order of .75 fs (femtoseconds), with emission pulse widths of the order of 0.5 fs (non-overlapping), here necessarily within the time width of the two-body wave packet. Since this time interval is of the order of the time intervals in the Lindner et al. experiment in the emission of a single electron, the structure of the two-body wave packet should have similar spread in time, the characteristic uncertainty in energy determined by the atomic decay mechanism. As we have remarked in our study [6] of the Lindner et al. experiment, Floquet theory [21] (for which the time t becomes an observable in a nonrelativistic framework) then would not explain the interference.

As formulated by Palacios et al., the antisymmetric spin zero state is antisymmetric in the spin factors and therefore symmetric in the spacetime factors in the two-body state. We write the spacetime factor for the wave function with both functions in the same foliation sheet n^μ (we suppress the normalization factor $1/\sqrt{2}$)

$$\begin{aligned}\Psi &= \varphi_1(x_1)\varphi_2(x_2) + \varphi_1(x_2)\varphi_1(x_1) \\ &\cong \varphi_1(k_1)\varphi_2(k_2)\big[e^{i(\mathbf{k}_1\cdot\mathbf{x}_1+\mathbf{k}_2\cdot\mathbf{x}_2-E_1t_1-E_2t_2)} \\ &+ e^{i(\mathbf{k}_1\cdot\mathbf{x}_2+\mathbf{k}_2\cdot\mathbf{x}_1-E_1t_2-E_2t_1)}\big],\end{aligned} \tag{4.1}$$

where we have interchanged the spacetime locations of the two identical electrons in the symmetrization (equivalent to interchange of the states). The two states, φ_1 and φ_2 differ in that in the first an electron is emitted from He, the second, the second electron is emitted from He^+.

We now define

$$T = \frac{t_1 + t_2}{2} \tag{4.2}$$

and

$$\Delta t = t_2 - t_1, \tag{4.3}$$

so that Eq. (4.1) becomes

$$\begin{aligned}\Psi &= \varphi_1(x_1)\varphi_2(x_2) + \varphi_1(x_2)\varphi_1(x_1) \\ &\cong \varphi_1(k_1)\varphi_2(k_2)e^{-i(E_1+E_2)T}\big[e^{i(\mathbf{k}_1\cdot\mathbf{x}_1+\mathbf{k}_2\cdot\mathbf{x}_2-\frac{i}{2}(E_2-E_1)\Delta t)} \\ &+ e^{i(\mathbf{k}_1\cdot\mathbf{x}_2+\mathbf{k}_2\cdot\mathbf{x}_1-\frac{i}{2}(E_1-E_2)\Delta t)}\big],\end{aligned} \tag{4.4}$$

in agreement with the structure found by Palacios et al. (we assume equal pulse widths as in their work).

Carrying out the integrals of the wave packets $\varphi_1(k_1), \varphi_2(k_2)$ (here, E_1, E_2 are independent of k_1, k_2), there will be an additional phase (as in the Palacios et al. calculation, but the Δt-dependent phase is proportional to $E_2 - E_1$. We remark that these energies, corresponding to the spectra of the relativistic atomic bound state problem [22] contain to first order the terms M_ic^2 plus the Schrödinger eigenvalue, with additional relativistic corrections (here negligible). The M_ic^2 terms cancel for two electrons, and the remaining bound state level values would be in agreement with the Palacios et al. calculation.

5. Conclusions

We have discussed spin and orbital angular momentum representations in a consistent relativistic quantum theory, generalizing Wigner's construction for the representation of relativistic spin from a foliation over momentum to a foliation over an arbitrary timelike vector n^μ [12] normalized to unity. This formulation admits the construction of representations of relativistic spin and angular momentum in a quantum mechanical Hilbert space for which the generators of both spin and angular momentum act in a spacelike surface orthogonal to the timelike vector n^μ. The standard Clebsch–Gordan methods are applicable to the reduction of direct product representations of two body (or more) states [14], in the fibre labelled by n^μ, and in particular, to relativistic entanglement. The construction of such a state, involving linear combinations of direct products of wave functions *at equal* τ, admit correlations are unequal times since the wave functions have support on both space and time (as we have remarked, in practice it is not possible to prepare a two-body state at precisely equal times).

Since the pulse spacings assumed by Palacios et al. were about 0.75 fs, interference would be supported between the two two-body states in superposition with wave function widths of this order of magnitude. The uncertainty relation then implies that $\Delta E \geq 10^{-3}$ eV. Natural line widths in atomic physics appear to be of order 10^{-6} eV, so that the uncertainty in time in the Stueckelberg wave packet could be much larger than what is needed to account for the observation of interference in time in the entangled state.

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Please cite this article in press as: L. Horwitz, R.I. Arshansky, Relativistic entanglement, Phys. Lett. A (2018), https://doi.org/10.1016/j.physleta.2018.04.004