# Bell's Theorem for Temporal Order

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Time has a fundamentally different character in quantum mechanics and in general relativity. In quantum theory events unfold in a fixed time order while in general relativity temporal order is influenced by the distribution of matter. When the distribution of matter requires a quantum description, temporal order is expected to become non-classical – a scenario beyond the scope of current theories. Here we provide a direct description of such a scenario. We consider a massive body in a spatial superposition and show how it leads to "entanglement" of temporal orders between time-like events in the resulting space-time. This entanglement enables accomplishing a task, violation of a Bell inequality, that is impossible under classical temporal order. Violation of the inequality means that temporal order becomes non-classical – it cannot be described by locally defined classical variables. Our approach provides a quantitative method for investigating quantum aspects of space-time and gravity.

## I. INTRODUCTION

Quantum mechanics forces us to question the view that physical quantities (such as spin, positions or energy) have predefined values: Bell's theorem shows that if observable quantities were determined by some locally-defined classical variables, it would be impossible to accomplish certain tasks – such as the violation of Bell's inequalities – whereas such tasks are possible according to quantum mechanics [1, 2] and have been realised in experiments [3–6]. However, the *causal relations* between events remain fixed in quantum theory: whether an event $A$ is in the past, in the future, or space-like separated from another event $B$ is predefined by the location of such events in space-time [7, 8]. In contrast, in general relativity, space-time itself is dynamical: the presence of massive objects affects local clocks and thus causal relations between events defined with respect to them. Nonetheless, the dynamical causal structure of general relativity is still classically predefined: the causal relation between any pair of events is uniquely determined by the distribution of matter-energy degrees

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of freedom in their past light-cone. In other words, causal relations are always determined by local classical variables. The picture is expected to change if we consider *quantum states* of gravitating degrees of freedom: if a massive system is prepared in a superposition of two distinct states, each yielding an observably different causal structure for future events, would it be possible to observe causal relations which display genuine quantum features? This work provides a direct example that this is the case: We show how temporal order between time-like events can become superposed or even entangled. To quantify the non-classicality of these causal relations we formulate a Bell-type theorem for temporal order: We define a task that cannot be accomplished if the time order between the events was predetermined by local variables, while the task becomes possible if the events are in a space-time region affected by the gravitational field of a massive object in a specific quantum state.

## II.   COORDINATE SYSTEMS AND GRAVITATIONAL TIME DILATION

In general relativity an event is an *operationally* defined point in space-time [9]: An event can only be meaningfully specified in relation to a physical system, e.g. it can be defined in terms of the time and location of a physical clock. The presence of massive bodies in general alters the relative rates at which different clocks tick. For example, in a weak field limit, a clock in a gravitational potential $\Phi$ will run slower by a factor $\sqrt{1 + 2\frac{\Phi}{c^2}}$ than an identical clock far away from the mass, where gravitational potential effectively vanishes. In classical physics, this leads to the well-tested time dilation [10, 11] and redshift effects [12]. When the clocks are described as quantum systems, new effects arise from the combination of quantum and general relativistic theories. For a clock in superposition of different distances to the mass, its proper time becomes entangled to the clock's position [13–15]. This entanglement implies a universal decoherence mechanism for generic macroscopic systems under time dilation [16, 17]. The regime of low-energy quantum systems in curved space-time can be described within a framework of general-relativistic composite quantum particles [18]. Here we additionally exploit the fact that only the distance between a clock and a mass has physical significance and due to linearity of quantum theory this must hold also for a superposition of different distances[1].

Consider two agents, $a$ and $b$, with two initially synchronised clocks, each following a fixed world-line (defined by a third, far-way agent). A massive body is brought in the vicinity of the two agents to induce time dilation between them. The position of the mass is decided by a third agent, who chooses between two configurations: $K_{A \prec B}$ and $K_{B \prec A}$. For the choice $K_{A \prec B}$, the massive body is positioned such that the event $A$, defined by the clock of agent $a$ showing proper time $t_a = \tau^*$, will be in the past light cone of the event $B$, which is defined in an analogous way: by the clock of agent $b$ showing proper time $t_b = \tau^*$. If $K_{B \prec A}$ is chosen, the mass is prepared in a different configuration, such that event $B$ ends up in the past of event $A$.

A possible way to realise configuration $K_{A \prec B}$ is to place an approximately point-like body of mass $M$ closer to $b$ than to $a$, see Fig. 1. The light-cone structure of the re-

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[1] There is no difference in the relative ticking rates of two clocks whether we think that the clocks are being positioned at different distances – possibly in a superposition – from the mass, or that the mass is positioned at different distances from the clocks.
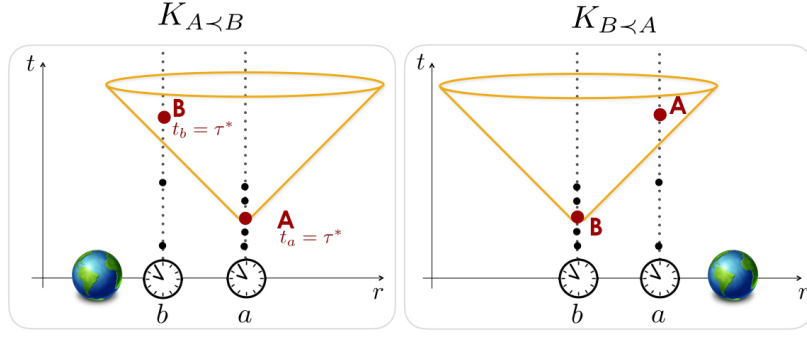
FIG. 1: Causal relations between space-time space can be "engineered" by preparing different mass configurations. Initially synchronised clocks $a$ and $b$ are positioned at fixed distances from a far-away agent whose time coordinate is $t$. Event $A$ ($B$) is defined by the clock at $a$ ($b$) showing proper time $\tau^*$. In configuration $K_{A \prec B}$ (left) a mass is placed closer to $b$ than to $a$. Due to gravitational time dilation, event $A$ can end up in a causal past of event $B$: for a sufficiently large $\tau^*$ the time difference between the clocks becomes greater than it takes light to travel between them. Light emitted at event $A$ reaches clock $b$ before the event $B$ occurs. Configuration $K_{B \prec A}$ (right) is fully analogous to $K_{A \prec B}$: the mass is placed closer to clock $a$ and the event $B$ can end up in the causal past of the event $A$.

sulting space-time is fully determined by the metric tensor $g_{\mu\nu}$, for which we adopt the sign convention $(-, +, +, +)$. Using Schwarzschild coordinates, in the first-order post-Newtonian expansion, the relevant components of the metric are $g_{00}(r) = -\left(1 + 2\frac{\Phi(r)}{c^2}\right)$ and $g_{rr}(r) = \left(1 + 2\frac{\Phi(r)}{c^2}\right)^{-1}$, where $r$ is the radial coordinate (centred at the massive body's position) and $\Phi(r) = -\frac{GM}{r}$ is the gravitational potential [19]. Assuming that $a$ and $b$ are at fixed coordinate distances from the mass, $r_a$ and $r_b = r_a - h$ respectively, we want to find the parameters for which event $A$ ends up in the past light-cone of $B$ for $K_{A \prec B}$ (and vice versa for $K_{B \prec A}$). An infinitesimal proper time element along a world line at a fixed distance $r$ from the mass is given by $d\tau(r) = \sqrt{-g_{00}(r)}dt$, where $t$ is the coordinate time, and a photon travelling in the radial direction from $r_a$ reaches $r_b$ after an elapsed coordinate time $T_c = \frac{1}{c} \int_{r_b}^{r_a} dr' \sqrt{-\frac{g_{rr}(r')}{g_{00}(r')}}$. Thus, if the photon is emitted at the local time $t_a = \tau^*$, it reaches $r_b$ when $b$'s local time is $\bar{t}_b = \sqrt{-g_{00}(r_b)}(\frac{\tau^*}{\sqrt{-g_{00}(r_a)}} + T_c)$, assuming that the local clocks are synchronised so that $t_a = 0$ and $t_b = 0$ coincide with the coordinate time $t = 0$. For

$$\tau^* > T_c \frac{\sqrt{-g_{00}(r_b)}}{1 - \sqrt{\frac{g_{00}(r_b)}{g_{00}(r_a)}}} \tag{1}$$

we have $\bar{t}_b \leq \tau^*$, which means that there is enough time for a not-faster-than-light signal emitted at event $A$ (defined by $t_a = \tau^*$) to travel the distance $h$ and reach agent $b$ at event $B$ (defined by $t_b = \tau^*$). This means that event $A$ is in the causal past of event $B$ as required. For example, for $h \ll r_a$ the condition (1) is satisfied for $\tau^* > \frac{2r_a^2 c}{GM}$. Configuration $K_{B \prec A}$ can be arranged analogously, by placing the mass closer to $a$ than to $b$. Then, the condition $\tau^* > \frac{2r_b^2 c}{GM}$, for $h \ll r_b$, ensures that $B$ is in the causal past of $A$.

The example above simply illustrates that in general relativity causal structure is dynamical and depends on the stress-energy tensor of the matter degrees of freedom. Preparing different matter distributions on a space-like hypersurface can result in different causal relations between events in the causal future of the preparation event.

## III.   QUANTUM CONTROL OF TEMPORAL ORDER

When $A$ is in the past light-cone of $B$, a physical system can in principle be transferred from $A$ to $B$. Consider a quantum system $S$ initially prepared in state $|\psi\rangle^S$ which undergoes a unitary $U_A$ at event $A$ (at the space-time location where the clock of agent $a$ marks proper time $\tau^*$) and a unitary $U_B$ at event $B$. (We ignore a possible additional time evolution between the two events for simplicity.) Such ordered events $A, B$ can therefore result in the following state of $S$:

$$|\widetilde{\psi}_1\rangle^S = U_B U_A |\psi\rangle^S. \tag{2}$$

If, however, $B$ is before $A$ and $S$ is prepared in the same initial state, the final state of $S$ is

$$|\widetilde{\psi}_2\rangle^S = U_A U_B |\psi\rangle^S. \tag{3}$$

A situation can therefore be arranged such that state (2) is produced for configuration $K_{A \prec B}$ and (3) is produced for $K_{B \prec A}$. Different mass configurations can result in different temporal orders of local operations both in quantum and in classical theory. However, if quantum theory applies to massive objects, the two mass configurations can be assigned quantum states $|K_{A \prec B}\rangle, |K_{B \prec A}\rangle$. These two states will be orthogonal, if the corresponding mass configurations are macroscopically distinguishable. Furthermore, in quantum theory a superposition of two physical states is also a valid state of the system. Thus, at least in principle, a superposition state $|K_+\rangle := \frac{1}{\sqrt{2}} (|K_{A \prec B}\rangle + |K_{B \prec A}\rangle)$ also represents a physical mass configuration. By a straightforward application of the superposition principle, the final state of the mass $M$ and of the system $S$ then reads

$$|\psi_{\text{sup}}\rangle^{MS} = \frac{1}{\sqrt{2}} \left( |K_{A \prec B}\rangle^M U_B U_A |\psi\rangle^S + |K_{B \prec A}\rangle^M U_A U_B |\psi\rangle^S \right). \tag{4}$$

The state above is the result of a process wherein the order of operations on a '*target*' system ($S$) is determined by the quantum state of a '*control*' system ($M$). Such a process is known as a *quantum switch* [20] and has been studied as a possible quantum-information resource [21–27]. The state $|\psi_{\text{sup}}\rangle^{MS}$ is a superposition of two amplitudes corresponding to different predefined, classical orders between events $A$ and $B$. Note that, if the control system is discarded, the reduced state of $S$ is

$$\frac{1}{2} \left( |\widetilde{\psi}_1\rangle\langle\widetilde{\psi}_1|^S + |\widetilde{\psi}_2\rangle\langle\widetilde{\psi}_2|^S \right), \tag{5}$$

which is indistinguishable from a *probabilistic mixture* of $|\widetilde{\psi}_1\rangle$ and $|\widetilde{\psi}_2\rangle$. The state in Eq. (5) can be interpreted as arising from events $A$ and $B$ with a classical, albeit unknown, temporal order. Therefore, any protocol aimed at testing operationally quantum features of temporal order necessarily requires a measurement of the control system.

## IV. BELL'S THEOREM FOR TEMPORAL ORDER

The above example shows that superpositions of massive objects can result in a coherent quantum control of temporal order between events. This conclusion relies on the assumption that quantum formalism is valid. The question thus arises if it is possible to probe the nature of temporal order irrespective of the validity of quantum theory? To achieve this a *theory independent* argument would be needed – which does not rely on the quantum framework (e.g. since quantum mechanics may need to be modified in a complete quantum-gravity theory). Below we provide such an argument, which could exclude the very possibility of explaining data from a hypothetical experiment in terms of a definite temporal order, with no assumption about the validity of quantum mechanics.

We introduce a task that allows refuting definite temporal order in conjunction with a few additional, physically plausible assumptions. Our formulation is analogous to Bell's theorem for local hidden variables, and we thus refer to the theorem below as Bell's theorem for temporal order of events. The core of the argument is simple: Given a bipartite system prepared in a separable state, it is not possible to violate any bipartite Bell inequality by performing *local operations* (transformations and measurements) on the two parts, as long as the local operations are applied in a *definite order*.

The theorem we formulate is theory independent, but not fully device-independent, as it refers to the notions of a "physical state" and a "physical transformation", in addition to the measured probability distributions. See Appendix A for a discussion of the present work in the context of the theory-dependent framework of causally non-separable quantum processes [28–30] and the fully theory- and device-independent approach of causal inequalities [28, 31].

### A. Framework

We consider a sufficiently broad framework to describe physical systems that can undergo transformations and measurements, similar to generalised probabilistic theories [32–34]. In this framework, a *state* $\omega$ is a complete specification of the probabilities $P(o|i,\omega)$ for observing outcome $o$ given that a measurement with setting $i$ is performed on the system. We are interested in situations where a system can be split up in subsystems, say $S_1$ and $S_2$, with space-like separated agents performing independent operations on $S_1$ and $S_2$. We say $\omega$ is a *product state*, and write $\omega = \omega_1 \otimes \omega_2$, if probabilities for local measurements factorise as $P(o_1, o_2|i_1, i_2, \omega) = P(o_1|i_1, \omega_1)P(o_2|i_2, \omega_2)$. If state $\omega_1^f$ is prepared for system $S_1$ and state $\omega_2^f$ is prepared for system $S_2$, according to a probability distribution $P(f)$ for some variable $f$, we write $\omega = \int df\, P(f)\omega_1^f \otimes \omega_2^f$ and say the state is *separable*. Probabilities are then given by the corresponding mixture: $P(o_1, o_2|i_1, i_2, \omega) = \int df\, P(o_1|i_1, \omega_1^f)P(o_2|i_2, \omega_2^f)P(f)$. Note that for such a decomposition Bell inequalities cannot be violated [1, 35].

A physical transformation of the system is represented by a function $\omega \mapsto T(\omega)$. To make our arguments precise we will need a notion of *local transformations*, namely, realised at the time and location defined by a local clock. If $S_1$ is the subsystem on which a local transformation $T_1$ acts, and $S_2$ labels the degrees of freedom space-like separated from $T_1$,

then, by definition, $T_1$ transforms product states as $\omega_1 \otimes \omega_2 \mapsto T_1(\omega_1) \otimes \omega_2$ and separable states by convex extension[2]. We further need to define how different local transformations combine. This depends on their relative spatio-temporal locations: If transformations $T_1, T_2$ are space-like separated they combine as $(T_1 \otimes T_2)(\omega_1 \otimes \omega_2) = T_1(\omega_1) \otimes T_2(\omega_2)$, which follows from the definition above; if $T_1$ is in the future of $T_2$, we define their combination as $T_1 \circ T_2(\omega) = T_1(T_2(\omega))$. (For simplicity, we omit possible additional transformations taking place between the specified events, as they are of no consequence for our argument).

## B. Bell's theorem for temporal order

The scenario for which the theorem is formulated involves a bipartite system with subsystems $S_1$ and $S_2$ and a system $M$ that can influence the temporal order of events. For $j = 1, 2$, each system $S_j$ undergoes two transformations, $T_{A_j}$ and $T_{B_j}$, at space-time events $A_j$, $B_j$, respectively. Each system is then measured at an event $C_j$ according to some measurement setting $i_j$, producing a measurement outcome $o_j$. Additionally, $M$ is measured at an event $D$, space-like separated from both $C_1$ and $C_2$, producing an outcome $z$, see Fig. 2. We now
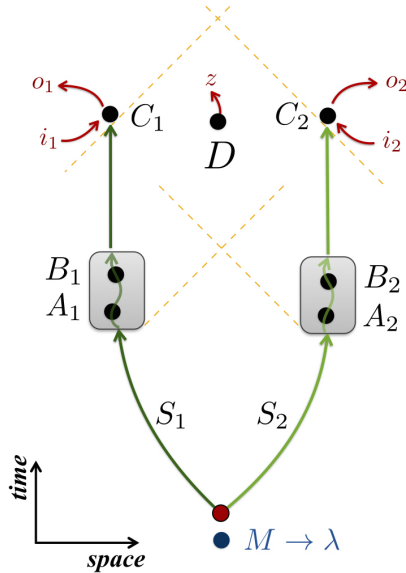


FIG. 2: Bell's theorem for temporal order. A bipartite system, made of subsystems $S_1$ and $S_2$, is sent to two groups of agents. Operations on $S_1$ ($S_2$) are performed at events $A_1, B_1$ ($A_2, B_2$). At event $C_1$ ($C_2$), a measurement with setting $i_1$ ($i_2$) and outcome $o_1$ ($o_2$) is performed. Events $A_1, B_1$ are space-like separated from $A_2, B_2$ and $C_1$ is space-like to $C_2$; light cones are marked by dashed yellow lines. The order of events $A_j, B_j$, $j = 1, 2$, is described by a variable $\lambda$ defined by a system $M$. The system $M$ is measured at event $D$, producing an output bit $z$. If the initial state of the systems $S_1, S_2, M$ is separable, and $\lambda$ is a classical variable (possibly dynamical and probabilistic), the resulting bipartite statistics of the outcomes $o_1, o_2$ cannot violate any Bell inequality, even if conditioned on $z$.

define the notion of classical order between events:

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[2] How local operations act on general, non-separable states can depend on the particular physical theory; however, action on separable states will suffice for our purposes.

**Definition 1.** *A set of events is* classically ordered *if, for each pair of events $A$ and $B$, there exist a space-like surface and a classical variable $\lambda$ defined on it that determines the causal relation between $A$ and $B$: for each given $\lambda$, either $A \preceq B$ ($A$ in the past causal cone of $B$), $B \preceq A$ ($A$ in the past causal cone of $B$), or $A||B$ ($A$ and $B$ space-like separated).*

Notice that *classically ordered* events do not necessarily form a partially ordered set: classical order can be *dynamical* (the order between two events can depend on some operation performed in the past, i.e. some agent can "prepare" $\lambda$) and *stochastic* (the variable $\lambda$ might be distributed according to some probability, and not specified deterministically) [30, 36].

The assumption of *classical order* is sufficient to derive *causal inequalities* [28, 31]: tasks that, without any further assumptions, cannot be performed on a classical causal structure. However, it is not possible to violate causal inequalities using quantum control of order [29, 30], this is why we need additional assumptions in the present context. It is an open question whether a gravitational realisation/implementation of a scenario that does allow a violation of causal inequalities is possible.

**Bell's theorem for temporal order.** *No states, set of transformations and measurements which obey assumptions 1—5 below can result in a violation of the Bell inequalities.*

1) **Local state:** *The initial state $\omega$ of $S_1, S_2$ and $M$ is separable (as defined in Sec. IV A).*

2) **Local operations:** *All transformations performed on the systems are local (as defined in Sec. IV A).*

3) **Classical order:** *The events at which operations (transformations and measurements) are performed are classically ordered.*

4) **Space-like separation:** *Events $(A_1, B_1)$ are space-like separated from events $(A_2, B_2)$; $C_1$, $C_2$, and $D$ are pair-wise space-like separated.*

5) **Free-choice:** *The measurement choices in the Bell measurement are independent of the rest of the experiment (This is a standard assumption necessary in Bell-like theorems).*

More formally, let us denote by $\mathbb{T} = (T_{A_1}, T_{B_1}, T_{A_2}, T_{B_2})$ the set of all local transformations irrespective of their order. The thesis of the theorem can be rephrased as: the conditional probability

$$P\left(o_1, o_2 | i_1, i_2, z, \mathbb{T}, \omega\right) \tag{6}$$

produced under assumptions 1–5 does not violate Bell's inequalities for any value of $z$.

*Proof.* Assumption (1) says that there is a random variable $f$ determining the local states $\omega_1^f$, $\omega_2^f$ of systems $S_1$, $S_2$, respectively. Assumption (3) says there is a random variable $\lambda$ that determines the order of events. In general, the two variables can be correlated by some joint probability distribution $P(\lambda, f)$. By assumption (4), events labelled $A_1, B_1$ are space-like separated from events $A_2, B_2$ and the order between events within each set $(A_j, B_j)$, $j = 1, 2$ can be defined by a permutation $\sigma_j$. Most generally, there is a probability $P(\sigma_j|\lambda)$ that the permutation $\sigma_j$ is realised for a given $\lambda$. By assumption (2), for each given order the system undergoes a transformation $T^{\sigma_1} \otimes T^{\sigma_2}$, where $T^{\sigma_1}$ is the transformation obtained by

composing $T_{A_1}$ and $T_{B_1}$ in the order corresponding to the permutation $\sigma_1$ and similarly for $T^{\sigma_2}$. (For example, if $\sigma_1$ corresponds to the order $A_1 \prec B_1$, then $T^{\sigma_1} = T_{B_1} \circ T_{A_1}$.) Furthermore, at event $D$ an outcome $z$ is obtained with a probability $P(z|\lambda, f, \sigma_1, \sigma_2)$. Finally, using assumption (1), we write the probabilities for all outcomes as

$$
\begin{aligned}
&P\left(o_1, o_2, z | i_1, i_2, \mathbb{T}, \omega\right) = \\
&\sum_{\sigma_1 \sigma_2} \int d\lambda \, df \, P(o_1|i_1, T^{\sigma_1}(\omega_1^f)) P(o_2|i_2, T^{\sigma_2}(\omega_2^f)) P(\sigma_1|\lambda) P(\sigma_2|\lambda) P(z|\lambda, f, \sigma_1, \sigma_2) P(\lambda, f),
\end{aligned}
\tag{7}
$$

A simple Bayesian inversion $P(\sigma_1|\lambda) P(\sigma_2|\lambda) P(z|\lambda, f, \sigma_1, \sigma_2) P(\lambda, f) = P(\lambda, f, \sigma_1, \sigma_2|z) P(z)$, where we used $P(\sigma_j|\lambda) = P(\sigma_j|\lambda, f)$, gives the desired probabilities

$$
\begin{aligned}
P\left(o_1, o_2 | i_1, i_2, z, \mathbb{T}, \omega\right) = \\
\sum_{\sigma_1 \sigma_2} \int d\lambda \, df \, P(o_1|i_1, T^{\sigma_1}(\omega_1^f)) P(o_2|i_2, T^{\sigma_2}(\omega_2^f)) P(\lambda, f, \sigma_1, \sigma_2|z) = \\
\int d\tilde{f} P(o_1|i_1, T^{\sigma_1}) P(o_2|i_2, T^{\sigma_2}) P(\tilde{f}|z), \quad (8)
\end{aligned}
$$

where $\tilde{f}$ is a short-hand for the variables $\lambda, f, \sigma_1, \sigma_2$ The above probability distribution satisfies the hypothesis of Bell's theorem and thus cannot violate any Bell inequality. $\qquad \square$

### C. Violation of Bell inequalities for temporal order

Here we show how gravitational quantum control of temporal order, Sec. III, can result in events whose temporal order is "entangled". We devise a quantum protocol where local operations are performed on an initially separable system, such that assumptions 1, 2, 4 and 5 are respected, but Bell's inequalities are nonetheless violated.

A bipartite quantum system, initially in a product state $|\psi_1\rangle^{S_1}|\psi_2\rangle^{S_2}$, is sent to two different regions of space such that $a_1$, $b_1$, and $c_1$ only interact with $S_1$, while $a_2$, $b_2$, and $c_2$ only interact with $S_2$. Agents $a_1$, $a_2$ perform respectively the unitaries $U_{A_1}$, $U_{A_2}$ at the events $A_1$, $A_2$, while agents $b_1$, $b_2$, perform the unitaries $U_{B_1}$, $U_{B_2}$ at the events $B_1$, $B_2$. Finally, $c_1$ and $c_2$ measure $S_1$ and $S_2$ at events $C_1$ and $C_2$, respectively, see Fig. 3. Assume that a massive system can be prepared in two configurations, $K_{A \prec B}$ and $K_{B \prec A}$, such that $A_1 \prec B_1 \prec C_1$ ($A_1$ in the past light-cone of $B_1$, etc) and $A_2 \prec B_2 \prec C_2$ for $K_{A \prec B}$, while $B_1 \prec A_1 \prec C_1$ and $B_2 \prec A_2 \prec C_2$ for $K_{B \prec A}$; and such that the events are space-like separated as per assumption 4, which can always be achieved by having the groups sufficiently separated. If the mass is prepared in superposition $\frac{1}{\sqrt{2}} (|K_{A \prec B}\rangle + |K_{B \prec A}\rangle)$, the joint state of the mass and the systems after the application of the unitaries is

$$
\frac{1}{\sqrt{2}} \left( |K_{A \prec B}\rangle^M U_{B_1} U_{A_1} |\psi_1\rangle^{S_1} U_{B_2} U_{A_2} |\psi_2\rangle^{S_2} + |K_{B \prec A}\rangle^M U_{A_1} U_{B_1} |\psi_1\rangle^{S_1} U_{A_2} U_{B_2} |\psi_2\rangle^{S_2} \right).
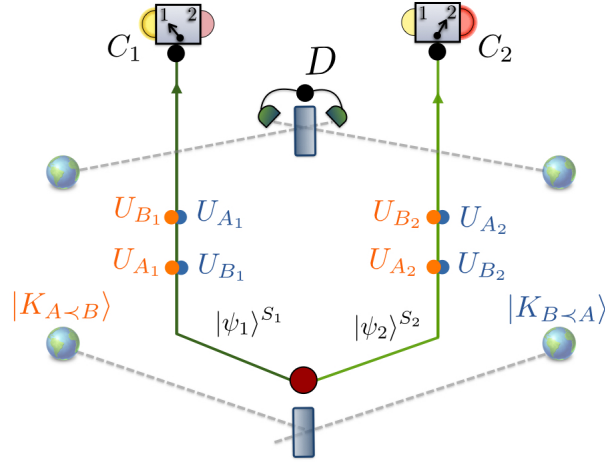\tag{9}
$$

FIG. 3: Schematics of a protocol for a violation of Bell's inequalities for temporal order. Systems $S_1, S_2$ are prepared in a product state $|\psi_1\rangle^{S_1}|\psi_2\rangle^{S_2}$ and sent to space-like separated regions. One pair of agents performs unitary operations $U_{A_1}$, $U_{B_1}$ on $S_1$ at the correspondingly marked space-time events; another pair acts on $S_2$ with unitary operations $U_{A_2}$ and $U_{B_2}$. Each operation is applied only once, at an event defined by the specific proper time of the local clock of the agent. A massive body is prepared in a superposition of two configurations $|K_{A \prec B}\rangle$ and $|K_{B \prec A}\rangle$ which define different causal structures for future events. For the amplitude $|K_{A \prec B}\rangle$, the operations $U_{A_i}$ $i = 1, 2$ applied on $S_i$ are in the causal past of the operations $U_{B_i}$ (orange dots); and vice versa for $|K_{B \prec A}\rangle$ (blue dots). The operations can be chosen such that $U_{A_i}U_{B_i}|\psi_i\rangle^{S_i}$ is orthogonal to $U_{B_i}U_{A_i}|\psi_i\rangle^{S_i}$ (for both $i = 1$ and $i = 2$), resulting in a maximally entangled final state. Bell measurements are performed at events $C_1$ and $C_2$ on $S_1$ and $S_2$, respectively. At event $D$ the mass is measured in a superposition basis. Conditioned on the outcome this measurement, the results of the measurements at $C_1, C_2$ can maximally violate Bell's inequalities, which would not be possible if the order of events was classical (even if probabilistic).

Agent $d$ at the event $D$ measures the mass in the superposition basis $|\pm\rangle = \frac{1}{\sqrt{2}}(|K_{A \prec B}\rangle \pm |K_{B \prec A}\rangle)$. Conditioned on the outcome, the joint state of $S_1$ and $S_2$ reads

$$\frac{1}{\sqrt{2}}\left(U_{B_1}U_{A_1}|\psi_1\rangle^{S_1}U_{B_2}U_{A_2}|\psi_2\rangle^{S_2} \pm U_{A_1}U_{B_1}|\psi_1\rangle^{S_1}U_{A_2}U_{B_2}|\psi_2\rangle^{S_2}\right). \qquad (10)$$

If the states $U_{B_1}U_{A_1}|\psi_1\rangle^{S_1}$, $U_{B_2}U_{A_2}|\psi_2\rangle^{S_2}$ are orthogonal to $U_{A_1}U_{B_1}|\psi_1\rangle^{S_1}$, $U_{A_2}U_{B_2}|\psi_2\rangle^{S_2}$, respectively, then the state (10) is maximally entangled. Local measurements can thus be performed on subsystems $S_1, S_2$ whose outcomes will violate Bell inequalities, conditioned on the measurement outcome at $D$ (see Appendix B for an example).

## D. Physical realisation

To achieve large time dilation between a pair of clocks one can use a very heavy object or, alternatively, any mass $M$ dense enough to put one of the clocks close to its Schwarzschild radius $R_S := \frac{2GM}{c^2}$. The ticking rate of a clock at $R_S + \epsilon$ with $\epsilon \ll R_S$ differs from the ticking rate of an identical clock at $R_S + l$, $l > \epsilon$, by a factor approximately $\sqrt{\frac{R_S}{\epsilon}(1 + 2\frac{\phi(R_S + l)}{c^2})}$,

which becomes arbitrarily large for a small $\epsilon$. One thus needs a dense but not necessarily a heavy object. Laboratory realisation of the protocol outlined above will nevertheless pose a formidable challenge, but it is in principle possible. Below we give an example.

Consider the mass configurations $K_{A\prec B}, K_{B\prec A}$ realised using an effectively point-like body with a fixed mass. The distance between agents $b_i$, $i = 1, 2$ and the mass is the same for both $K_{A\prec B}$ and $K_{B\prec A}$, while agents $a_i$ are closer to the mass in configuration $K_{B\prec A}$ than in $K_{A\prec B}$, as illustrated in Fig. 4 a). The subsystems $S_i$ can be realised as two identically
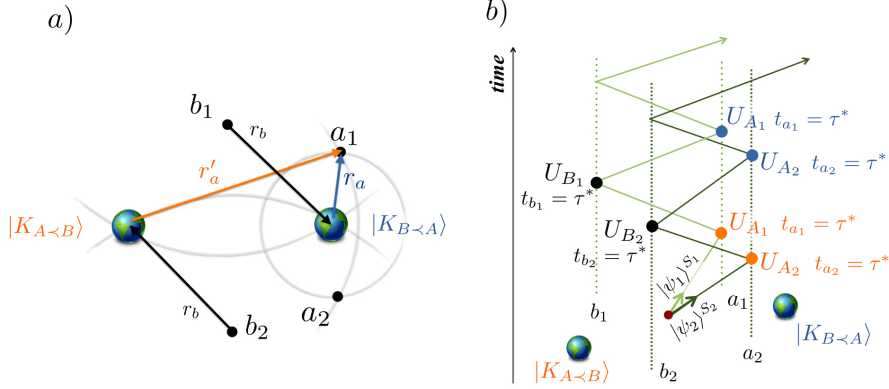


FIG. 4: A protocol for a violation of Bell's inequalities for temporal order. a) Mass configurations $K_{A\prec B}, K_{B\prec A}$ and location of the agents $a_i, b_i$, $i = 1, 2$. $b_i$ are at a distance $r_b$ from both configurations, while $a_i$ are at a distance $r_a$ from $K_{B\prec A}$ and $r'_a > r_a$ from $K_{A\prec B}$. b) Space-time diagram of the protocol. Systems $S_1, S_2$ are implemented in the polarisation of two photons, initially in a product state $|\psi_1\rangle^{S_1}|\psi_2\rangle^{S_2}$. Green lines are the photons' world lines; green dotted lines are world lines of the agents. Orange (blue) dots mark events when agents $a_i$ apply unitaries $U_{A_i}$ for the configuration $K_{A\prec B}$ ($K_{B\prec A}$); black dots mark events when $b_i$ apply $U_{B_i}$. The photons bounce twice between the agents, but each operation is applied on the photon only once – when the local clocks of the agents show proper time $\tau^*$. Due to time dilation induced by the mass, $U_{A_i}$ are applied before $U_{B_i}$ for configuration $K_{A\prec B}$ (and the events $A_i$ coincide with the photons reaching $a_i$ for the first time) – and are applied after $U_{B_i}$ for configuration $K_{B\prec A}$ (and the events $A_i$ coincide with the photons reaching $a_i$ for the second time).

prepared, uncorrelated photons and the local operations can be performed on their polarisation degrees of freedom (DOF). The photon source is equally distant from $K_{A\prec B}$ and $K_{B\prec A}$, (equidistant to the extent that the local clock of the source remains sufficiently uncorrelated with the mass). All clocks involved in the protocol are initially synchronised with the clock of the source.

At a pre-defined time $T_s$ according to the clock at the source, the source emits the photon pair – the emission time is thus uncorrelated with the mass configuration. Photon $S_i$ is directed towards agent $a_i$, then to $b_i$, again to $a_i$, back to $b_i$, and exits towards agent $c_i$ (or $c_i$ simply replaces $a_i$), see Fig. 4 b). The agents interact with the relevant DOF of the photon only once, at the time $\tau^*$ as measured by their local clocks. The unitary transformations are assumed to be independent of the mass configuration (or other aspects of the experiment). As discussed in sec. II, the emission time $T_s$ of the photons can be chosen such that event $A_i$

(at which $U_{A_i}$ is applied) is before the event $B_i$ (at which $U_{B_i}$ is applied) and so that event $A_i$ coincides with the photon reaching $a_i$ for the first time for configuration $K_{A \prec B}$, and when the photon reaches $a_i$ for the second time for $K_{B \prec A}$. The photon reaches $a_i$ twice, before or after $\tau^*$ – depending on the mass configuration, at which no operation is performed: The photon is reflected with no transformation on the polarisation. The event when the operation $U_{B_i}$ is applied always coincides with the photon reaching $b_i$ for the first time, since agents $b_i$ are at the same distances to the mass for both configurations.

In general, the travel time of the photon can depend on the mass configuration due to the Shapiro delay [37, 38]. In order to mitigate this effect, after the emission time $T_s$ – sufficient to induce the required time dilation between the clocks – the mass can be (coherently) moved such that it is at the same distance from each agent (for both $K_{A \prec B}$ and $K_{B \prec A}$), or such that it is sufficiently far away from both. Moreover, in order to de-correlate the time-dilated clocks from the systems $S_i$, the amplitudes of the mass can be swapped and the mass can be measured by the agent $d$ after a time interval equal to $T_s$ – when the clocks of $a_i$ and $b_i$ become synchronised again. Appendix C provides details of a protocol that provides both: suppression of the Shapiro effect and decorrelation of the clocks. In Appendix D we also discuss a quantum field realisation of the protocol.

Our examples show that it is in principle possible to prepare events with entangled temporal order in a protocol that satisfies assumptions 1, 2, 4 and 5. It is therefore in principle possible to prove existence of this entanglement by the violation of the Bell inequality for temporal order.

## V. RESOURCE FOR THE VIOLATIONS

Time dilation induced by the mass results in correlations between the order in which local operations are applied on the subsystems. A way to see this is to use a time coordinate that is independent of the location of the mass (e.g. the local time of a far-away agent, coordinate $t$ in Fig. 1): the superposition state of the massive body induces correlations between the proper times of different clocks, corresponding to the same time coordinate.

These correlations are transferred to the subsystems when agents apply their operations at a fixed proper time of their local clock. Moreover, if the state of the mass is described in terms of number states in spatial modes, the violation can be explained as due to "entanglement swapping" from the mass to the two subsystems – since in this approach a spatial superposition is described as an entangled state of the corresponding modes. Independently of the description of the mass and the used coordinates, the violation of the inequalities is due to the correlations between the space-time metric and the mass. The key differences between the gravitational and other methods for a quantum control of temporal order are discussed in Appendix E.

## VI. DISCUSSION

The non-classical causal structures discussed in this work arise in a semi-classical, albeit non-perturbative, regime where no explicit quantisation of the gravitational field is needed

(which is complementary to the regime of most quantum gravity frameworks [39]). Our approach shows that classical general relativity and standard quantum mechanics are sufficient to analyse scenarios involving superpositions of macroscopically different classical backgrounds. Not only is there no tension between the two frameworks, but there is also *no ambiguity* in the prediction of physical effects that arise: For each probability amplitude the time-dilation effects introduced by the mass can be treated classically. The considered processes involve a simple superposition of such amplitudes and the final probability amplitude is given by the usual Feynman sum.

A realisation of the Bell-test for time order is very challenging, but we note that experiments to prepare superposition states of massive objects and test their gravitational interactions are already under development [40–44]. However, assuming that the violation is *fundamentally* impossible would have far reaching consequences: it would imply that time *could* be described with a classical parameter even in space-times originating from a quantum state of a massive object – with no need to invoke any other mechanism that would decohere such quantum states, such as [45–49] (see also Appendix F). On the other hand, since these mechanisms postulate a specific decoherence time of spatial superpositions of massive bodies, one could think that they preclude the preparation of non-classical causal structures. This is not the case: the time required to complete our protocol can be shorter than the decoherence time postulated by these models (see the last two paragraphs in Appendix C). Thus, contrary to some motivations [47, 49], these models do not enforce classical space-time with a fixed causal structure.

## VII. CONCLUSION

We have shown how non-classical causal structures can be engineered by exploiting time dilation from a massive body in a quantum state. This non-classicality can be quantified independently of whether the quantum formalism is trusted – by the violation of the Bell-like inequality for temporal order. Our analysis exposed a close connection between the abstract framework of process matrices, the quantum switch, and joint effects of quantum mechanics and general relativity. The work thus opens a new route for exploring quantum aspects of space-time and gravity. Our results show that classical notions of time and temporal order are untenable in the light of the basic principles of quantum theory and general relativity.

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## Appendix A: Causally non-separable quantum processes

Non-classical causal relations can be studied within a recent framework for quantum mechanics with no pre-defined causal structure introduced in ref. [28]. The starting point of the framework is the notion of *local events* that take place in *local regions*, with spatial and

temporal boundaries of the region defined by local clocks. An event is identified with an operation performed in the local region (for example a unitary transformation, or a projection on a given state obtained as the result of a measurement). A physical scenario, comprising the space-time geometry in which the local regions are embedded, the initial state, and the dynamics connecting the regions, is compactly represented by a *process*—a specification of the probabilities for any possible event/local operation to take place in each region.

At a formal level, a local region $X$ is defined by an input Hilbert space $\mathcal{H}^{X_I}$ and an output Hilbert space $\mathcal{H}^{X_O}$, identified with the quantum degrees of freedom on space-like surfaces on the past and future of $X$, respectively. Quantum operations are represented as operators $M^{X_I X_O} \in \mathcal{L}(\mathcal{H}^{X_I}) \otimes \mathcal{L}(\mathcal{H}^{X_O})$, where $\mathcal{L}(\mathcal{H})$ is the space of linear operators on the Hilbert space $\mathcal{H}$. Probabilities for events in regions $A$, $B$,... are then given by a generalisation of the Born rule:

$$P(M^{A_I A_O}, M^{B_I B_O}, \dots) = \mathrm{Tr}\left[\left(M^{A_I A_O} \otimes M^{B_I B_O} \otimes \dots\right) \cdot W^{A_I A_O B_I B_O \cdots}\right], \qquad (A1)$$

where $W^{A_I A_O B_I B_O \cdots} \in \mathcal{L}(\mathcal{H}^{A_I}) \otimes \mathcal{L}(\mathcal{H}^{A_O}) \otimes \mathcal{L}(\mathcal{H}^{B_I}) \otimes \mathcal{L}(\mathcal{H}^{B_O})$ is the *process matrix*.

In this formalism, causal relations between local regions are encoded in the process matrix. For example, the process matrix

$$W^{A_I A_O B_I B_O} = \rho^{A_I} \otimes [[\mathbb{1}]]^{A_O B_I} \otimes \mathbb{1}^{B_O}, \text{ where} \qquad (A2)$$

$$[[\mathbb{1}]]^{A_O B_I} := |\mathbb{1}\rangle\!\rangle\langle\!\langle\mathbb{1}|^{A_O B_I} \text{ and} \qquad (A3)$$

$$|\mathbb{1}\rangle\!\rangle^{A_O B_I} := \sum_j |j\rangle^{A_O} |j\rangle^{B_I}, \qquad (A4)$$

represents a situation where an agent at $A$ receives a state $\rho$, while the output of $A$'s operation is sent to $B$ through the identity channel. Such a process is only compatible with the order of events $A \preceq B$; more general processes compatible with an order of events given by a permutation $\sigma$ are denoted $W^\sigma$. If the order is determined by a classical variable $\lambda$, defined in some region in the past of all events, the process matrix has the form

$$W = \int d\lambda \, W^{\sigma_\lambda} P(\lambda) \qquad (A5)$$

for some probability distribution $P(\lambda)$.

The question of whether a certain quantum scenario can be embedded in a classical space-time, with a classical order of events, thus reduces to the question whether the corresponding process matrix can be decomposed in a mixture of the form (A5), which we call *causally separable*[3]. The *quantum switch*, described in the section "Quantum control of temporal order" is represented by the process matrix

$$W = |\omega\rangle\langle\omega| \qquad (A6)$$

$$|\omega\rangle = \frac{1}{\sqrt{2}}\left(|K_{A \prec B}\rangle^{M_I}|ABC\rangle + |K_{B \prec A}\rangle^{M_I}|BAC\rangle\right), \qquad (A7)$$

$$|ABC\rangle = |\psi\rangle^{A_I}|\mathbb{1}\rangle\!\rangle^{A_O B_I}|\mathbb{1}\rangle\!\rangle^{B_O C_I}, \quad |BAC\rangle = |\psi\rangle^{B_I}|\mathbb{1}\rangle\!\rangle^{B_O A_I}|\mathbb{1}\rangle\!\rangle^{A_O C_I}, \qquad (A8)$$

---

[3] A more general definition [28] where the order of future events can depend on past events, is not necessary for our analysis.

where $M$ labels the control system and $C$ is the region where the system is measured after the operations in regions $A$, $B$ are performed. As shown in ref. [29], it is possible to find an experimental procedure, namely a set of operations and measurements for $A$, $B$, $C$, $M$, that allows proving the causal non-separability of the switch. However, such a *causal witness* is both device and theory dependent, namely it relies on the quantum description of the operations performed. *Causal inequalities*[28, 31] on the other hand, provide a device and theory independent test for causal order; however, no quantum-control of causal order can violate causal inequalities, as proven in refs. [29, 30] and it is an open question whether any physically realisable process can.

The process matrix corresponding to the scenario with "entangled orders", introduced in the main text is

$$W = |\varpi\rangle\langle\varpi| \tag{A9}$$

$$|\varpi\rangle = \frac{1}{\sqrt{2}} \left( |K_{A\prec B}\rangle^{M_I} |A_1 B_1 C_1\rangle |A_2 B_2 C_2\rangle + |K_{B\prec A}\rangle^{M_I} |B_1 A_1 C_1\rangle |B_2 A_2 C_2\rangle \right), \tag{A10}$$

using definitions similar to (A8). Just as for the switch, it is easy to prove that process (A9) is not causally separable[4]. However, a process of this type cannot be used to violate causal inequalities, see e.g. ref. [29]. The procedure described in the main text can nonetheless prove the causal non-separability of process (A9) in a theory-independent, albeit device-dependent, way.

### Appendix B: State and measurements for the CHSH inequality violation

Consider a two-qubit system in an initial state $|\psi_1\rangle^{S_1} \otimes |\psi_2\rangle^{S_2} \equiv |z+\rangle^{S_1} \otimes |z+\rangle^{S_2}$. As local unitaries we can choose

$$U_{A_1} = U_{A_2} \equiv U_A = \frac{\mathbb{1} + i\sigma_x}{\sqrt{2}}, \qquad U_{B_1} = U_{B_2} \equiv U_B = \sigma_z, \tag{B1}$$

where $\sigma_x$ and $\sigma_z$ are the usual Pauli matrices. Notice that $U_A U_B = \frac{\sigma_z + \sigma_y}{\sqrt{2}}$, while $U_B U_A = \frac{\sigma_z - \sigma_y}{\sqrt{2}}$. The final state (10) is in this case

$$\frac{1}{\sqrt{2}} \left( |x+\rangle^{S_1} |x+\rangle^{S_2} \pm |x-\rangle^{S_1} |x-\rangle^{S_2} \right), \tag{B2}$$

where the sign depends on the outcome $|\pm\rangle$ of the measurement on the massive system. In order to violate Bell inequalities, the agent $c_1$ measures the observable $\mathcal{C}_1^0 = \frac{\sigma_y - \sigma_z}{\sqrt{2}}$ for the setting $i_1 = 0$ and the observable $\mathcal{C}_1^1 = \frac{\sigma_y + \sigma_z}{\sqrt{2}}$ for $i_1 = 1$, while $c_2$ measures $\mathcal{C}_2^0 = \sigma_y$ for $i_2 = 0$ and $\mathcal{C}_2^1 = \sigma_z$ for $i_2 = 1$. With these measurement choices, the expectation value of the CHSH quantity is

$$\langle CHSH \rangle_\pm = \langle \mathcal{C}_1^0 \otimes \mathcal{C}_2^0 + \mathcal{C}_1^0 \otimes \mathcal{C}_2^1 + \mathcal{C}_1^1 \otimes \mathcal{C}_2^0 - \mathcal{C}_1^1 \otimes \mathcal{C}_2^1 \rangle_\pm = \mp 2\sqrt{2}, \tag{B3}$$

---

[4] As it is a rank-one projector, it cannot be decomposed as a non-trivial mixture of orders. Yet it does not describe a process with a definite order, because the signalling relations between parties do not define a partial order.

for the two outcomes $z = \pm 1$ of the measurement at $D$. Thus, conditioned on the outcome $z$, the measurements at $C_1$ and $C_2$ violate the CHSH inequality $|\langle CHSH \rangle| \leq 2$. Notice that the measurement settings at $C_1$ and $C_2$ are independent of $z$ and thus the three measurements can be performed at space-like separation. The violation of the inequality is recovered when all the data are compared.

### Appendix C: Disentangling clocks from the mass

The protocol allowing for the violation of Bell's inequalities for temporal order exploits correlations between the clocks of the agents $a_1, b_1$ and the agents $a_2, b_2$, created due to time dilation induced by the mass. It should be noted that the protocol allows maximal violation of the Bell inequality if the joint state of the systems $S_1$ and $S_2$ is pure (and maximally entangled) when the Bell measurements are realised. Thus, for a maximal violation, the clocks need to decorrelate from the mass after the application of the unitaries. Below we sketch a scenario that can achieve this.

The space-time arrangement of the mass and the agents in this example is presented in Figure 5. It can be realised in one spatial dimension: agents acting on the system $S_1$ are located at distance $h$ from each other, and the mass is placed at distance $r$ (configuration $K_{B \prec A}$) or $r + L$ (configuration $K_{A \prec B}$) from agent $a_1$. Agents acting on system $S_2$ are placed symmetrically on the opposite side of the mass, such that the mass is at a distance $r + L$ from $a_2$ in configuration $K_{B \prec A}$ and $r$ in configuration $K_{A \prec B}$. Here, events $A_j$ are defined by the local time $\tau_a$ that differs from the local time $\tau_b$ defining $B_j$, $j = 1, 2$. In such a case, even though the mass is always closer to $a_j$ than to $b_j$, the two mass configurations can lead to different event orders – as they induce different relative time dilations. (Equivalently, one can introduce an initial offset in the synchronisation of the clocks.) Note that the time orders between the two groups are here "anti-correlated": $A_1 \prec B_1$ and $B_2 \prec A_2$ for $K_{A \prec B}$, and vice versa for $K_{B \prec A}$. Since otherwise the scenario is the same for $S_1$ and $S_2$, we focus on the events and operations performed on $S_1$. The key observation is that swapping the mass distribution, as depicted in Figure 5, will eventually disentangle the clocks from the mass, and since the clocks must be suitably time-dilated when the operations are performed, the operations must not take place in the future light cone of the swapped mass state.

The proper time $\tau_a$ that has to elapse for the clock of $a_1$ such that the order of events is $A_1 \prec B_1$ for $|K_{A \prec B}\rangle$ and $B_1 \prec A_1$ for $|K_{B \prec A}\rangle$ for the present case reads

$$
\tau_a = \sqrt{-g_{00}(r)} \frac{T_c(r, h) + T_c(r + L, h)\sqrt{\frac{g_{00}(r+L+h)}{g_{00}(r+h)}}}{1 - \sqrt{\frac{g_{00}(r)g_{00}(r+L+h)}{g_{00}(r+h)g_{00}(r+L)}}},
$$

where $T_c(r, L/2)$ is the coordinate travel time of light between radial distances $r$ and $r + L/2$ from the mass. The coordinate time corresponding to $\tau_a$ is $T_a = \tau_a / \sqrt{-g_{00}(r)}$. The proper
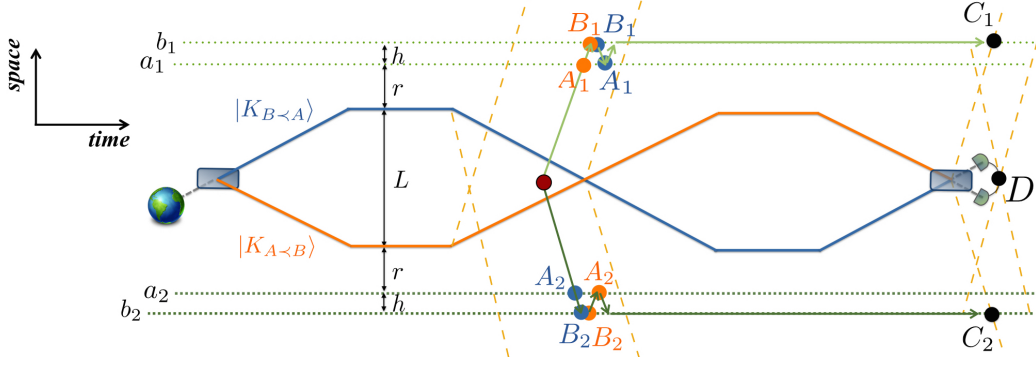
FIG. 5: **Space-time diagram of a protocol for disentangling the clocks from the mass.** In configuration $K_{A \prec B}$ the mass is at a distance $r + L$ from $a_1$, and at $r + L + h$ from $b_1$. In $K_{B \prec A}$ – it is at $r$ from $a_1$ and at $r + h$ from $b_1$. The opposite holds for $a_2, b_2$. The initial mass superposition is swapped (after sufficient time to prepare the clocks in the correlated state) so that they finally show the same time. At the local time $\tau_a$ of $a_1$ (at event $A_1$) the agent applies $U_{A_1}$ on $S_1$. At the local time $\tau_b$ of $b_1$ the agent applies $U_{B_1}$ on $S_1$. For the mass configuration $K_{A \prec B}$ $A_1$ is before $B_1$ (orange-coloured events), while for $K_{B \prec A}$ event $B_1$ is before $A_1$ (blue-colored events). The opposite order holds for events $A_2, B_2$ occurring on the opposite side of the mass, where agents $a_2, b_2$ act on $S_2$. Unitary operations should be applied in the future light-cone of the event where the clocks get correlated and outside the future light cone of the event when the mass amplitudes are swapped, Bell measurements (at $C_1, C_2$) should be made when the clocks become disentangled (at future light-like events to when the mass amplitudes are brought together), and the measurement at event D should be space-like to $C_1, C_2$; dashed yellow lines represent the relevant light-cones.

time of event $B_1$ is then defined as[5]:

$$\tau_b = \sqrt{-g_{00}(r + L + h)}\left(\frac{\tau_a}{\sqrt{-g_{00}(r + L)}} + T_c(r + L, h)\right).$$

The coordinate time required for the application of the operations can be estimated as twice the travel time of light between the agents, $T_o = 2T_c(r + L/2, h)$.

The world lines of the mass can be arranged such that: a) the mass is moving slow so that the two amplitudes of the mass are swapped in a time interval longer than $T_o$; b) during the application of the operations the distance of each agent to the mass is approximately the same for both mass configurations (as in Figure 5). The first guarantees that there is enough time to apply the operations after the clocks get correlated, the second – that the Shapiro delay (see Sec. IV D) can be neglected.

The coordinate-time duration of the entire protocol can be estimated as $T_p = 2T_a + 4L/2c$, where $L/2c$ is the minimal time required to put the mass in superposition of amplitudes separated by the distance $L/2$. Taking as an example $r = 10^{10}R_S$, $L = 5r$, $h = r$ and $M = 1$mg where $R_S \approx 10^{-30}$m, the protocol from Figure 5 takes $T_p \approx 7 \times 10^{-18}$s.

Within some approaches a spatial superposition state of a mass, such as used in our protocol, is postulated to decohere [45–49]. Different motivations were given for this deco-

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[5] It can directly be checked that when the mass is placed in configuration $K_{A \prec B}$ – at a distance $r + L$ from $a_1$– the event $A_1$ defined by local clock of $a_1$ showing proper time $\tau_a$ is in the past light cone of event $B_1$, which is defined by the local clock of $b_1$ showing proper time $\tau_b$. When the mass is placed in configuration $K_{B \prec A}$, event $B_1$ ends up in the past of the event $A_1$.

herence, e.g. that space-time should be compatible with a classical description (global field of time-like vectors), or that some process must enforce irreversibility of the quantum time evolution to explain observation of definite measurement outcomes (solve the "measurement problem"). However, even if endorsed, these models do not preclude realisation of our protocol: the decoherence time scale from these models is the Diosi-Penrose time [46, 47] $T_{DP} = \frac{2\delta^3 \hbar}{G(mL)^2}$, where $\delta$ is a free parameter. For every value of $\delta$ one can find the mass and the relevant distances $(M, r, L, h)$ so that the duration of our entire protocol is shorter than $T_{DP}$. For example, following the recent ref. [50] and taking $\delta = 10^{-7}$m, for our specific example above one gets $T_{DP} \approx 0.5$s. Taking instead originally proposed value $\delta = 10^{-15}$m [46], the desired regime is achieved e.g. for $M = 0.1\mu$g, $r = 10^7 R_S$, $L = 5 \times 10^5 r$, $h = 10^5 r$; with $T_p \sim 10^{-23}$s and $T_{DP} \sim 10^{-13}$s. Since the above decoherence models allow for the preparation of events with entangled temporal order, they do not enforce the classicality of the causal structure of space-time.

We conclude that it is in principle possible to achieve the required entanglement of orders, swap the mass distribution so as to finally disentangle the clocks form the mass, and satisfy the locality conditions on the events.

### Appendix D: Field theory realisation of the protocol

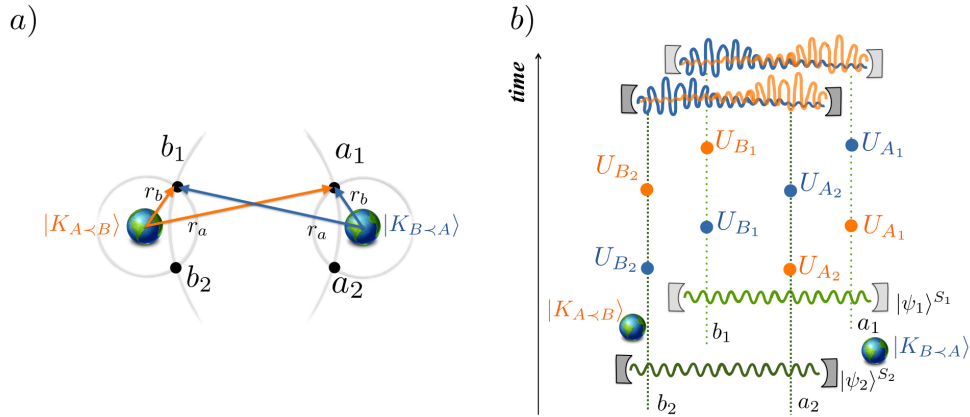Here we discuss another possibility for the realisation of the protocol. The mass distri-



FIG. 6: **Protocol for the violation of Bell's inequalities for temporal order with quantum fields.** a) Mass configurations: In configuration $K_{A \prec B}$ the mass is at the same distance $r_b$ from both $b_i$ and at a distance $r_a > r_b$ from both $a_i$. For the configuration $K_{B \prec A}$ the mass is at $r_a$ from both $b_i$, and at $r_b$ from both $a_i$. b) Space-time diagram. Systems $S_i$ are implemented in two regions of an electromagnetic field, prepared in a vacuum state. Agents apply coarse-grained operations $U_{A_i}, U_{B_i}$ (in the subspace spanned by the vacuum and a single-photon state) locally on the field at correspondingly marked events. For configuration $K_{A \prec B}$ ($K_{B \prec A}$) only orange (blue) events occur, and the final state of the field is represented in orange (blue).

bution is such that for $K_{A \prec B}$ the mass is closer to $b_1$ than to $a_1$ and for $K_{B \prec A}$ the relative distances are reversed. The same holds for agents $a_2, b_2$ who are placed symmetrically to

agents $a_1, b_1$ with respect to the mass; see Figure 6 a).

The local operations can be performed in the Fock space of a photon field, more precisely in the two-level subspace spanned by the vacuum and a single-photon state of a chosen field mode. The field is prepared in some mode $\alpha$ at event $A_1$, and in mode $\beta$ at event $B_1$. The modes are chosen such, that the two final states of the field at event $C_1$ – obtained depending on the order between events $A_1, B_1$ – are distinguishable. The situation for the agents $a_2, b_2$ is the same and they prepare the modes $\alpha, \beta$, at the events $A_2, B_2$, respectively, see Figure 6 b).

One needs to note, that the vacuum state of a relativistic quantum field is entangled with respect to the local subsystems [51–53]. However, this entanglement is effectively inaccessible under coarse-grained operations [54]. Thus, if the operations performed by the agents are sufficiently coarse-grained, the initial state consisting of the local regions of a vacuum of a quantum field is effectively separable and does not violate the assumptions of the protocol. This implementation differs from the example given in the main text in that it does not need a source that would produce a state at a specific time, and distribute it to the agents.

## Appendix E: Gravitation vs other methods for a quantum control of temporal order

The superposition or entanglement of temporal orders was discussed here in the context of "relativistic quantum engineering": a far away agent prepares a quantum state of a massive system which due to relativistic gravity effects yields a desired quantum causal structure for future events. Control over temporal order of applying operations on a system can also be achieved when agents control the positions of clocks that define when the operations are applied. For example, by placing clock $A$ closer and $B$ further away from a fixed mass in superposition with $B$ closer and $A$ further, proper times of the clocks become entangled as in the gravitational switch. Quantum control of the time order can also be achieved without any use of gravitational interaction, e.g. in the extended model of a quantum circuit: Quantum gates can be applied on a system in different orders in a superposition [20–23] The latter has already been practically implemented using an interferometer to route a photon through two gates (acting on its polarisation) in different orders [24, 27]. The key difference to the gravitational scheme presented in this work is that in both the above alternatives the events would be embedded in a classical space-time: In the example of an entangled clock pair, only these specific clocks could be used to label events for which temporal order is non-classical, while any other nearby clock would define classically order events. In the example of an extended quantum circuit, only the photon that went through a beam splitter will undergo different transformations in a non-classical order. In contrast, in the scenario considered in this work *any* local system in the spacetime region "affected" by the superposition state of the mass will have classically undefined proper time. Thus, any two pairs of clocks in this region will define events with an entangled order.

The above can be highlighted by considering the scenario leading to the violation of the Bell-like inequality for temporal order in a coordinate system defined by the massive body, which is here in superposition. (We note, however, that there is no complete theory of such "quantum coordinate transformations"). Space-time coordinates of the events would then be

defined with respect to the position of the mass and a local clock at its location – rather than with respect to the positions and proper times of the clocks of the agents. By definition, in these coordinates the location of the mass is fixed. As a result, all local operations performed in the local regions of the agents would appear to be embedded in a fixed space-time metric but performed at *different* space-time *events* in superposition – such that the orders of events in different space-time regions are always entangled.

Moreover, in the gravitational scenario introduced in this work, all operations are performed at fixed local times, independently of the event order. No local time measurement can reveal whether a given agent is acting first or second. Furthermore, the mass distributions can be prepared such that the same force is exerted at various events but the potential is different, e.g. by using spherical shells of matter of different radii [55] (instead of point-like mass distributions). In that case *no local measurement* can reveal which of the two mass configurations was prepared, and thus what the event order was. This is different in both the above scenarios, where local non-demolition measurements of the positions of the clocks or the photon's time of arrival to a gate in a circuit, would reveal the event order, and alter the outcomes of the protocol.

The above shows that there is a fundamental difference between the gravitational control of temporal order discussed here and other methods. Although the final state of a system undergoing some transformations in a non-classical order is independent of how the order was controlled, only when the mass controls temporal relations is the effect universal – applying to *all* events in some space-time region. Thus, only in the gravitational case one would conclude that non-classical temporal order indicates non-classicality of space-time.

### Appendix F: What if it is *fundamentally not possible* to violate the Bell inequality for time order?

Since a test of Bell's inequalities for temporal order has never been performed and would be very challenging, one can also ask what if it is not possible even in principle to violate the corresponding Bell's inequalities, or if it is fundamentally not possible to satisfy assumptions of the theorem other than 3?

If that were the case, a classical description of temporal order could always be given, e.g. in terms of the classical variable $\lambda$ (introduced in Definition 1) – even in space-times originating from a quantum state of a massive body. Moreover, the classical variable describing temporal order of events could be used to define a classical time parameter according to which the systems evolve, even in scenarios involving macroscopic masses in quantum superposition states. Interestingly, this would imply that models forbidding spatial superpositions of large masses on the ground that it is not possible to define time evolution in the resulting space-time, such as refs [45–49] are redundant: Time would be compatible with a classical description (in terms of a hidden variable) even in the presence of quantum states of massive bodies.