BELL'S INEQUALITIES AND QUANTUM FIELD THEORY*

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Abstract. The present state of mathematically rigorous results about Bell's inequalities in relativistic quantum field theory is reviewed. In addition, the nature of the statistical independence of algebras of observables associated to spacelike separated spacetime regions is discussed.

I. INTRODUCTION.

Motivated by the desire to bring into the realm of testable hypotheses at least some of the important matters concerning the interpretation of quantum mechanics that were evoked in the controversy surrounding the Einstein-Podolsky-Rosen paradox [11,26], Bell discovered the first version [9,10] of a series of related inequalities that are now generally called Bell's inequalities and that have received a great deal of attention (for reviews see [16,7]). These inequalities provide an upper bound on the strength of correlations between systems that are no longer interacting but have interacted in their past.

The class of correlation experiments involved in these inequalities can be described briefly. A source provides an ensemble of identically prepared systems, one after another, and, as part of the preparation, splits each system into two subsystems, directing these to separate arms of the experiment. At one arm the arriving subsystem is subjected to a measuring device chosen from a class $\mathcal{A}$ of suitable devices, and at the other arm the incident subsystem interacts with a measuring device from a class $\mathcal{B}$. In the simplest situations, $\mathcal{A}$ and $\mathcal{B}$ each consists of two devices. For each device $A \in \mathcal{A}$ and each $B \in \mathcal{B}$ with possible outcome sets $\hat{A}$ and $\hat{B}$, the relative frequencies $p(\alpha, \beta)$ of the measurement of $\alpha \in \hat{A}$ on one arm and $\beta \in \hat{B}$ on the other arm are determined. An operational condition of independence of the two arms of the experiment is required:

$$\sum_{\beta \in \hat{B}} p(\alpha, \beta) \equiv p(\alpha)$$

must be independent of the choice of $B \in \mathcal{B}$, and

$$\sum_{\alpha \in \hat{A}} p(\alpha, \beta) \equiv p(\beta)$$

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must be independent of the choice of $A \in \mathcal{A}$. If there are $|\mathcal{A}|$ devices in $\mathcal{A}$ and $|\mathcal{B}|$ devices in $\mathcal{B}$, the one must carry out $|\mathcal{A}| \cdot |\mathcal{B}|$ correlation experiments to obtain the necessary data.

Bell's inequality, in the form of Clauser and Horne [15], is:

$$p(\alpha_1, \beta_1) + p(\alpha_1, \beta_2) + p(\alpha_2, \beta_1) - p(\alpha_2, \beta_2) \leq p(\alpha_1) + p(\beta_1), \tag{1.1}$$

for all $\alpha_i \in \hat{A}_i$, $\beta_j \in \hat{B}_j$, $A_i \in \mathcal{A}$, $B_j \in \mathcal{B}$. Bell's theorem (and the many generalizations that followed) is a metatheoretical theorem that states that all theories of a certain class that describe such a correlation experiment must provide predictions satisfying (1.1). Hence, if in a real experiment correlation probabilities are measured that violate (1.1), then one must conclude that there are real physical processes that cannot be described by any theory in the said class. If a theory predicts a violation of (1.1), then Bell's theorem implies that not all predictions of this theory can be reproduced by any theory in the said class.

What is the class of theories that must produce only correlation probabilities satisfying (1.1)? The details of the answer to this question depend on the particular set of hypotheses used to prove Bell's inequality, and Bell's theorem appears in many forms in the literature. Because it is not the object of this paper to review this multitude of theorems, let it suffice simply to say that most versions assume, explicitly or tacitly, that the theories in this class are "classical" and "local". Roughly speaking, this means that all the correlation probabilities are given by a single classical measure (the significance of this assumption was particularly emphasized in [52]; see also [53]) and the theory treats the two subsystems as independent of each other (independence is unfortunately a theory-dependent concept, hence it cannot be further specified here without entering into metatheoretic terrain). We refer to the review [16] for a detailed discussion of some of these "Bell's theorems" and to [44] for a very general approach to Bell's inequalities that enters more into the metatheoretic considerations that are necessary.

In this paper we are concerned with what standard theories, particularly quantum field theory, predict about the violation of (1.1). We first recall, in order to fix notation, how such theories model the experimental situation described above. There is a C*-algebra $\mathcal{C}$ (with identity 1) of observables for the system and a pair $(\mathcal{A}, \mathcal{B})$ of mutually commuting subalgebras of $\mathcal{C}$ (each containing 1) for the algebra of observables of the independent subsystems. The possible outcomes α, β are modelled by basic observables $\tilde{A} \in \mathcal{A}$, $\tilde{B} \in \mathcal{B}$ satisfying $0 \leq \tilde{A} = \tilde{A}^* \leq 1$, $0 \leq \tilde{B} = \tilde{B}^* \leq 1$ (projections are examples of such basic observables). To each device A, resp. B, there corresponds a collection $\{\tilde{A}_i\}$, resp. $\{\tilde{B}_j\}$, of such basic observables such that $\Sigma\tilde{A}_i = 1$, resp. $\Sigma\tilde{B}_j = 1$. Corresponding to the preparation of the ensemble of systems there is a state ϕ, a positive, normalized, linear functional on $\mathcal{C}$. Then the correlation probabilities $p(\alpha, \beta)$ are given by $\phi(\tilde{A}\tilde{B})$. Built into this model are the relations

$$\sum_j \phi(\tilde{A}_i\tilde{B}_j) = \phi(\tilde{A}_i) \qquad , \qquad \sum_i \phi(\tilde{A}_i\tilde{B}_j) = \phi(\tilde{B}_j)\,,$$

for all devices $\{\tilde{A}_i\}$, $\{\tilde{B}_j\}$. Hence the operational condition of independence of the two subsystems is an integral part of the model.

Making the obvious substitutions into (1.1), one obtains Bell's inequality for the standard theories. For the sake of convenience, we rewrite this as:

$$-1 \leq \phi(A_1B_1) + \phi(A_1B_2) + \phi(A_2B_1) - \phi(A_2B_2) \leq 1, \tag{1.2}$$

with $-1 \leq A_i = A_i^* \leq 1$, $-1 \leq B_j = B_j^* \leq 1$, $A_i \in \mathcal{A}$, $B_j \in \mathcal{B}$. Note that $2\tilde{A} - 1 = A$ is a selfadjoint contraction if and only if $\tilde{A}$ is a basic observable. Bell's inequality for the standard theories is thus the requirement that (1.2) is satisfied for all pairs $\{A_1, A_2\} \subset \mathcal{A}$, $\{B_1, B_2\} \subset \mathcal{B}$ of selfadjoint contractions. We therefore make the following definition.

<u>Definition 1.1</u>: The maximal Bell correlation of the pair $(\mathcal{A}, \mathcal{B})$ of commuting subalgebras of the C*-algebra $\mathcal{C}$ in the state $\phi \in \mathcal{C}^*$ is

$$\beta(\phi, \mathcal{A}, \mathcal{B}) \equiv \sup \frac{1}{2}\phi(A_1(B_1 + B_2) + A_2(B_1 - B_2)),$$

where the supremum is taken over all selfadjoint contractions $A_i \in \mathcal{A}$, $B_j \in \mathcal{B}$. (Note that $\beta(\phi, \mathcal{A}, \mathcal{B})$ is a convex functional in ϕ and if $\mathcal{A}_1 \subset \mathcal{A}$, $\mathcal{B}_1 \subset \mathcal{B}$, then $\beta(\phi, \mathcal{A}_1, \mathcal{B}_1) \leq \beta(\phi, \mathcal{A}, \mathcal{B})$.)

Bell's inequality (1.2) can thus be expressed as

$$\beta(\phi, \mathcal{A}, \mathcal{B}) \leq 1. \tag{1.3}$$

We recall the following result.

<u>Proposition 1.2</u> [14,43,45,30]: For any C*-algebra $\mathcal{C}$, commuting subalgebras $\mathcal{A}$ and $\mathcal{B}$ and state ϕ on $\mathcal{C}$,

$$\beta(\phi, \mathcal{A}, \mathcal{B}) \leq \sqrt{2}. \tag{1.4}$$

Hence, in standard theories the maximal Bell correlation is never greater than $\sqrt{2}$. A measured violation of (1.4) could serve to exclude all theories providing C*-algebras as models, just as a violation of (1.3) excludes "classical, local" theories. (For further information on this point, see [14,32].) However, here we are working within the standard theories.

<u>Definition 1.3</u>: The pair $(\mathcal{A}, \mathcal{B})$ of commuting subalgebras of a C*-algebra $\mathcal{C}$ maximally violates Bell's inequality in the state ϕ if $\beta(\phi, \mathcal{A}, \mathcal{B}) = \sqrt{2}$.

Of course, $\beta(\phi, \mathcal{A}, \mathcal{B}) > 1$ already entails violation of Bell's inequality. The following theorem collects a number of general situations where Bell's inequality must be satisfied in standard theories.

Theorem 1.4 [45]: Let $(\mathcal{A}, \mathcal{B})$ be a pair of commuting subalgebras of a C*-algebra $\mathcal{C}$.

a) If $\phi|\mathcal{A} \vee \mathcal{B}$, the restriction of ϕ to the C*-algebra generated by $\mathcal{A}$ and $\mathcal{B}$, is a convex sum of product states over $(\mathcal{A}, \mathcal{B})$, then $\beta(\phi, \mathcal{A}, \mathcal{B}) = 1$.

b) If $\mathcal{A}$ or $\mathcal{B}$ is abelian, then $\beta(\phi, \mathcal{A}, \mathcal{B}) = 1$, for all states ϕ.

c) If $\phi|\mathcal{A}$ or $\phi|\mathcal{B}$ is a pure state, then $\beta(\phi, \mathcal{A}, \mathcal{B}) = 1$.

Part (a) asserts that if the preparation of the subsystems is such that the state is a (sum of) product state over $(\mathcal{A}, \mathcal{B})$, i.e. $\phi(AB) = \phi(A)\phi(B)$ for all $A \in \mathcal{A}$, $B \in \mathcal{B}$, then the correlations between the observables of the subsystems in this state are weak enough to satisfy Bell's inequality. From (b) we learn that if at least one of the two subsystems is classical, i.e. all observables commute, then Bell's inequality is satisfied in every state. And in (c) the purity of the restriction to one of the subsystems entails weak Bell correlations.

It has been known for some time [9,10] that quantum mechanics, to state the matter in our language, predicts the existence of $(\mathcal{A}, \mathcal{B})$ and ϕ such that $\beta(\phi, \mathcal{A}, \mathcal{B}) = \sqrt{2}$. We state and prove this fact.

Theorem 1.5: Let $\mathcal{A}$ and $\mathcal{B}$ be mutually commuting copies of the two by two complex matrices $M_2(\mathbf{C})$ acting on the Hilbert space $\mathcal{H}$. Then there exists a normal state ϕ on $\mathcal{B}(\mathcal{H})$, the algebra of all bounded, linear operators on $\mathcal{H}$, such that $\beta(\phi, \mathcal{A}, \mathcal{B}) = \sqrt{2}$.

Proof: In $\mathcal{A}$, resp. $\mathcal{B}$, there is a copy $\{\sigma_x, \sigma_y, \sigma_z\}$, resp. $\{\sigma'_x, \sigma'_y, \sigma'_z\}$ of the Pauli spin matrices. Let $\Phi_\pm$, resp. $\Phi'_\pm$, satisfy $\sigma_z\Phi_\pm = \pm\Phi_\pm$, resp. $\sigma'_z\Phi'_\pm = \Phi'_\pm$, and let $\mathcal{K}$ be the four-dimensional Hilbert subspace of $\mathcal{H}$ generated by $\{\Phi_\pm, \Phi'_\pm\}$. Furthermore, let $\mathcal{D}$ be the C*-algebra generated by $\{\sigma_x, \sigma_y, \sigma_z\} \cup \{\sigma'_x, \sigma'_y, \sigma'_z\}$. Then $\mathcal{K}$ is unitarily equivalent to $\mathbf{C}^2 \otimes \mathbf{C}^2$ and $\mathcal{D}|\mathcal{K}$ is unitarily equivalent to $\mathcal{B}(\mathbf{C}^2) \otimes \mathcal{B}(\mathbf{C}^2) = M_2(\mathbf{C}) \otimes M_2(\mathbf{C})$. Let $\chi_+ = \binom{1}{0} \in \mathbf{C}^2, \chi_- = \binom{0}{1} \in \mathbf{C}^2$, and similarly for $\chi'_\pm$. Define $\Psi \equiv 2^{-1/2}(\chi_+ \otimes \chi'_- - \chi_- \otimes \chi'_+) \in \mathbf{C}^2 \otimes \mathbf{C}^2$. If $\hat{x}$, $\hat{z}$ are the obvious unit vectors in $\mathbf{R}^3$ and $\hat{\alpha} \equiv \cos\alpha\,\hat{z} + \sin\alpha\,\hat{x}$, then $\vec{\sigma} \cdot \hat{\alpha} = \cos\alpha\sigma_z + \sin\alpha\sigma_x$. A straightforward calculation yields

$$< \Psi, (\vec{\sigma} \cdot \hat{\alpha}) \otimes (\vec{\sigma} \cdot \hat{\beta})\Psi > = -\hat{\alpha} \cdot \hat{\beta}.$$

Using the mentioned unitary equivalence, this can be understood to obtain for some $\Psi \in \mathcal{K} \subset \mathcal{H}$. Choosing $\hat{\alpha} = \hat{z}$, $\hat{\alpha}' = \hat{x}$, $\hat{\beta} = \cos\frac{\pi}{4}\hat{z} + \sin\frac{\pi}{4}\hat{x}$, and $\hat{\beta}' = \cos\frac{3\pi}{4}\hat{z} + \sin\frac{3\pi}{4}\hat{x}$, and also $A_1 = \vec{\sigma} \cdot \hat{\alpha}'$, $A_2 = \vec{\sigma} \cdot \hat{\alpha}$, $B_1 = \vec{\sigma}' \cdot \hat{\beta}$, $B_2 = oh\sigma' \cdot \hat{\beta}'$, then one sees that if ϕ denotes the vector state on $\mathcal{B}(\mathcal{H})$ generated by Ψ, $\phi(A_1(B_1 + B_2) + A_2(B_1 - B_2)) = \sqrt{2}$.□

Hence, if two commuting algebras $\mathcal{A}$, $\mathcal{B}$ contain copies of $M_2(\mathbf{C})$ on a Hilbert space $\mathcal{H}$, they maximally violate Bell's inequalities in some normal state on $\mathcal{B}(\mathcal{H})$. Landau showed the following result.

Theorem 1.6 [30]: Let $(\mathcal{A}, \mathcal{B})$ be a pair of commuting von Neumann algebras on a Hilbert space $\mathcal{H}$ such that if $A \in \mathcal{A}$, $B \in \mathcal{B}$ and $AB = 0$, then either $A = 0$ or $B = 0$. Then if neither $\mathcal{A}$ nor $\mathcal{B}$ is abelian, there exists a normal state ϕ on $\mathcal{B}(\mathcal{H})$ such that $\beta(\phi, \mathcal{A}, \mathcal{B}) = \sqrt{2}$.

Sketch of proof: For any projection P, $2P - 1$ is a selfadjoint contraction. Let $P_i \in \mathcal{A}$, $Q_j \in \mathcal{B}$ be projections and $A_i = 2P_i - 1$, $B_j = 2Q_j - 1$. Then

$$\|A_1(B_1 + B_2) + A_2(B_1 - B_2)\| = 2\sqrt{1 + 4\,\|\,[P_1, P_2][Q_1, Q_2]\,\|}.$$

One can find a normal state ϕ on $\mathcal{B}(\mathcal{H})$ such that

$$\frac{1}{2}\,|\phi(A_1(B_1 + B_2) + A_2(B_1 - B_2))\,| = \sqrt{1 + 4\,\|\,[P_1, P_2][Q_1, Q_2]\,\|}.$$

The condition that $A \in \mathcal{A}, B \in \mathcal{B}$ and $AB = 0$ imply either $A = 0$ or $B = 0$ entails that $\|\,[P_1, P_2][Q_1, Q_2]\,\| = \|\,[P_1, P_2]\,\|\;\|\,[Q_1, Q_2]\,\|$ [38]. Since in any nonabelian von Neumann algebra $\mathcal{M}$ two projections $P_1, P_2 \in \mathcal{M}$ can be found such that $\|\,[P_1, P_2]\,\| = \frac{1}{2}$, the theorem's claim follows. □

Hence, one has only two possible situations. Either $\mathcal{A}$ or $\mathcal{B}$ is abelian, so from Theorem 1.4 (b) Bell's inequality is satisfied in all states, or both are nonabelian and (up to the additional hypothesis in Theorem 1.6) there exists a normal state in which Bell's inequality is maximally violated. The next result establishes the interesting fact that only copies of the Pauli spin matrices provide maximal violation.

Proposition 1.7 [45]: Let $(\mathcal{A}, \mathcal{B})$ be a pair of commuting subalgebras of a C*-algebra $\mathcal{C}$ and let $A_i \in \mathcal{A}$, $B_j \in \mathcal{B}$ be selfadjoint contractions such that for a state ϕ on $\mathcal{C}$ with $\phi|\mathcal{A}$ and $\phi|\mathcal{B}$ faithful,

$$\frac{1}{2}\phi(A_1(B_1 + B_2) + A_2(B_1 - B_2)) = \sqrt{2}.$$

Then $A_i^2 = 1$ and $A_1A_2 + A_2A_1 = 0$ (similarly for B_j), so A_1, A_2 and $A_3 \equiv -\frac{i}{2}[A_1, A_2]$ form a realization of the Pauli spin matrices in $\mathcal{A}$ (similarly for B_j in $\mathcal{B}$). (Moreover, A_1, A_2, A_3, resp. B_1, B_2, B_3, are contained in the centralizer of $\mathcal{A}$ in ϕ, resp. centralizer of $\mathcal{B}$ in ϕ).

Sketch of proof: Under the stated assumptions, we may identify $\mathcal{A}$ and $\mathcal{B}$ with a pair of commuting von Neumann algebras on a Hilbert space $\mathcal{H}$ with ϕ realized as a vector state by $\Phi \in \mathcal{H}$, where Φ is cyclic and separating for $\mathcal{A}$ and $\mathcal{B}$. Let $A_i \in \mathcal{A}$, $B_j \in \mathcal{B}$ be selfadjoint contractions such that

$$\frac{1}{2} < \Phi, (A_1(B_1 + B_2) + A_2(B_1 - B_2))\Phi >= \sqrt{2},$$

and let $\tilde{A} = \frac{1}{2}(A_1 + iA_2)$, $\tilde{B} = \frac{1}{2\sqrt{2}}(B_1 + B_2 + i(B_1 + B_2))$. Then $\tilde{A}\Phi = \tilde{B}\Phi$, $\tilde{A}^*\Phi = \tilde{B}^*\Phi$, and

$$\frac{1}{2} < \Phi, (A_1^2 + A_2^2)\Phi > = < \Phi, (\tilde{A}^*\tilde{A} + \tilde{A}\tilde{A}^*)\Phi >= 1.$$

Hence $A_i^2\Phi = \Phi$ and $(\tilde{A}^*\tilde{A} + \tilde{A}\tilde{A}^*)\Phi = \Phi$. Therefore, for any $A \in \mathcal{A}$, $\phi(AA_i^2) = < \Phi, AA_i^2\Phi > = \phi(A)$, $\phi(A(A_1+iA_2)) = 2 < \Phi, A\tilde{A}\Phi > = 2 < \Phi, A\tilde{B}\Phi > = 2 < \tilde{B}^*\Phi, A\Phi > = 2 < \tilde{A}^*\Phi, A\Phi > = \phi((A_1 + iA_2)A)$, and $(A_1A_2 + A_2A_1)\Phi = -2i(\tilde{A}^2 - \tilde{A}^{*2})\Phi = -2i(\tilde{B}^2 - \tilde{B}^{*2})\Phi = (B_1^2 - B_2^2)\Phi = 0$. □

Therefore, if one is designing an experiment to test violation of Bell's inequalities, one should only choose observables (like particle spins, polarizations, etc.) that can be modelled in standard theories by Pauli spin matrices. This is, in fact, what was done in the experiments carried out to date [6, 7, 8, 16], and to an extremely high accuracy, the prediction of a Bell correlation equal to $\sqrt{2}$ was verified. A natural question now is: what does quantum field theory predict about Bell's inequalities?

This question has been examined in a mathematically rigorous manner in very few publications, the first such paper appearing as late as 1985 [43]. Presently the papers that directly address the topic of Bell's inequalities and quantum field theory are [43, 44, 45, 46, 47, 30, 31, 48, 33]. The main results of these papers are reviewed in the next section. We provide a brief overview of these papers.

Paper [43] was an announcement of some of the main results from [44, 45, 46], where it was first proven that any free quantum field theory predicts that Bell's inequalities are maximally violated in the vacuum. In other words, already the vacuum fluctuations in any noninteracting quantum field model entail correlations (for spacelike separated and thus commuting observables) that maximally violate Bell's inequalities. This result indicated that maximal violation of Bell's inequalities had nothing to do with interaction or with special preparation of the system. In [30] it was emphasized that already the nonabelian character of the local algebras of observables sufficed to conclude maximal violation in some (unspecified) state, and the nonclassical nature of the vacuum state was re-established in [31] (however, <u>not</u> by showing that Bell's inequalities were maximally violated in the vacuum - see Theorem 2.1). In the paper [47] (the generality of which was significantly extended in [48]) it was shown that, in fact, the axioms of quantum field theory actually entailed that Bell's inequalities were maximally violated in <u>every</u> (normal) state in essentially every quantum field model. This is a result that is not true in nonrelativistic quantum mechanics. Finally, in paper [33] Landau exhibited, using the construction of [46], (exponentials of) quadratic expressions in free quantum field operators which violate Bell's inequality (not maximally) in thermal states for all sufficiently low temperatures and which have a physical interpretation as 'local' charges associated with symmetry transformations. In Section III we briefly discuss Bell's inequalities and quantum field theory in the more general context of statistical independence.

II. QUANTUM FIELD THEORY.

Ordinary quantum field theory on Minkowski space, formalized in the Wightman axioms [41], provides models of the type considered in the previous section. It is known that up to minor technical assumptions (see e.g. [25]) quantum field theories provide nets of C*-algebras assigning to each open region O of Minkowski space a C*-algebra $\mathcal{A}(O)$ such that the net $\{\mathcal{A}(O)\}$ satisfies certain standard axioms [3, 29] (isotony, locality, Poincaré covariance, and the existence of a Poincaré-covariant representation with positive energy satisfying the relativistic spectrum condition) that were naturally motivated by the interpretation of each $\mathcal{A}(O)$ as the algebra generated by all the observables that can be measured in the spacetime region O. In a certain technical sense [25], the quantum field operators smeared with test functions having support in O generate the algebra $\mathcal{A}(O)$. Since in this section all results refer to normal states, we may consider $\{\mathcal{A}(O)\}$ to be a net of von Neumann algebras in a Hilbert space $\mathcal{H}$ satisfying the mentioned axioms. In this section the algebra $\mathcal{C}$ is the C*-algebra generated by all the algebras in the net $\{\mathcal{A}(O)\}$.

By the locality axiom, if $O_1 \times O_2$, i.e. if all points in O_1 are spacelike separated from all points in O_2, then $\mathcal{A}(O_1) \subseteq \mathcal{A}(O_2)'$, the commutant of $\mathcal{A}(O_2)$ in $\mathcal{B}(\mathcal{H})$. Hence $(\mathcal{A}(O_1), \mathcal{A}(O_2))$ is a pair of commuting C*-algebras as in the previous section. Since at this level of generality we can only consider spacelike separated regions, we have in mind only correlation experiments where the measurements on the two arms are performed far enough apart and in a short enough time that they are spacelike separated (as in [8]).

Since local algebras $\mathcal{A}(O)$ in quantum field theories are very nonabelian, it is clear from Theorems 1.5 and 1.6 that there are going to be many states in which Bell's inequalities are maximally violated. In fact, typical local algebras contain an <u>infinite</u> product of copies of $M_2(\mathbf{C})$ [5,2,47,48], so that by Theorem 1.5 whenever $O_1 \times O_2$ there are infinitely many normal states ϕ such that $\beta(\phi, \mathcal{A}(O_1), \mathcal{A}(O_2)) = \sqrt{2}$. Landau demonstrated the following proposition using Theorem 1.6. The region O_1 is said to be strictly spacelike separated from O_2 if there exists a neighborhood $\mathcal{N}$ of the origin in $\mathbf{R}^4$ such that $O_1 + \mathcal{N} \times O_2$. (This relation is symmetric.)

<u>Theorem 2.1</u> [30,31]: Let $\{\mathcal{A}(O)\}$ be a net of local algebras in a physical representation with

a unique vacuum vector Ω.

a) For any strictly spacelike separated regions O_1, O_2 and any $A_i \in \mathcal{A}(O_1)$, $B_j \in \mathcal{A}(O_2)$ selfadjoint contractions such that $[A_1, A_2] \neq 0$ and $[B_1, B_2] \neq 0$, there exists a normal state ϕ such that

$$\frac{1}{2}\phi(A_1(B_1 + B_2) + A_2(B_1 - B_2)) > 1.$$

b) If Ω is cyclic for all $\mathcal{A}(O)$, $O \neq \emptyset$, and O_1, O_2, O_3 are any three mutually strictly separated spacetime regions, there exists a dense set $\mathcal{S}$ of normal states on $\mathcal{B}(\mathcal{H})$ (containing all states with bounded energy with respect to the vacuum) such that for any selfadjoint contractions $A_i \in \mathcal{A}(O_1)$, $B_j \in \mathcal{A}(O_2)$ satisfying $[A_1, A_2] \neq 0$ and $[B_1, B_2] \neq 0$ there is a projection $P \in \mathcal{A}(O_3)$ and a translation $x \in \mathbf{R}^4$, depending on $\phi \in \mathcal{S}$, so that the translates $A_1(x)$, $A_2(x)$, $B_1(x)$, $B_2(x)$ and $P(x)$ do not have a joint classical distribution in the state ϕ.

Remarks: (1) If the regions O_i above are not very pathological, for example if both are bounded and $O_i'' = O_i$, then they need only be spacelike separated from each other.

(2) The conclusion in (b) is weaker than that in (a), but it also illustrates an aspect of the nonclassical behavior of quantum field theory. Note that because the vacuum state is translation invariant, the assertion in part (b) simplifies somewhat for the vacuum. Since this paper is about Bell's inequalities, we shall sketch only the proof of part (a).

Proof of Theorem 2.1 (a): Under the stated assumptions, it is known [39] that $A \in \mathcal{A}(O_1)$, $B \in \mathcal{A}(O_2)$ and $AB = 0$ imply $A = 0$ or $B = 0$. Hence part (a) is a direct corollary of the proof to Theorem 1.6. Note that if $O_i'' = O_i$ and O_1 is spacelike separated from O_2 (and not necessarily strictly spacelike separated) then it follows from Theorem 3.5 in [20] that $AB = 0$ if and only if $A = 0$ or $B = 0$, so that Theorem 1.6 may be applied once again to yield the claim in Remark (1).□

Although a few natural questions remain open here, it is now clear that quantum field theory predicts the violation of Bell's inequalities in many states for any pair of algebras associated to spacelike separated spacetime regions, no matter how far apart the regions are. If, however, the spacelike separated regions are tangent, then we shall see below that the corresponding algebras of observables maximally violate Bell's inequalities in all normal states. Tangent spacetime regions are spacelike separated regions whose closures intersect, and we shall consider two classes of such regions in this section.

Let $W_R = \{x \in \mathbf{R}^4 | x_1 > |x_0|\}$ denote the "right wedge". Then $\mathcal{W}$, the collection of all "wedge" regions, is the set of all Poincaré transforms of W_R. If O is a spacetime region, O' denotes the interior of its causal complement (the set of all points spacelike separated from O). Then for any $W \in \mathcal{W}$, one has $W' \in \mathcal{W}$. Moreover, the pair (O, O') is always tangent for ordinary regions. The set $\mathcal{K}$ of double cones is described as follows. Let $x,\ y \in \mathbb{R}^4$ be timelike separated with x in y's future light cone. Then a double cone is obtained as the interior of the intersection of x's past light cone with y's future light cone. Taking all such x, y one generates all double cones. Note that double cones are bounded regions, while wedges are not. For both classes of regions, $O = O''$.

Since we are examining situations in which Bell's inequalities are maximally violated in all normal states, we make the following definition.

Definition 2.2: A pair $(\mathcal{A}, \mathcal{B})$ of commuting subalgebras of a W*-algebra $\mathcal{C}$ is called maximally correlated if for any normal state ϕ on $\mathcal{A} \vee \mathcal{B}$, one has $\beta(\phi, \mathcal{A}, \mathcal{B}) = \sqrt{2}$.

Theorem 2.3 [47]: a) In any vacuum sector, in any superselection sector of a global gauge group, in any massive particle representation, $(\mathcal{A}(W),\ \mathcal{A}(W)')$ is maximally correlated, for all $W \in \mathcal{W}$. Hence, if $\mathcal{A}(W)$ is weakly associated to a Wightman field in the sense of [25], $(\mathcal{A}(W),\ \mathcal{A}(W'))$ is maximally correlated for all $W \in \mathcal{W}$.

b) In any free field theory, in any local Fock field theory (e.g. $P(\phi)_2$ [28], Yukawa_2 [40], etc.) and in any dilatation-invariant theory, $(\mathcal{A}(O_1),\ \mathcal{A}(O_2))$ is maximally correlated for any pair $(O_1,\ O_2)$ of tangent double cones.

The three cases that enter into the hypothesis in part (a) above – vacuum sectors (a physical representation with at least one cyclic vacuum vector), superselection sectors [18], and massive particle representations [13] – include all physically interesting situations except for the charged sectors of a gauge theory with local gauge group and a massless particle (like quantum electrodynamics). This latter type of physical setting is not included in this theorem because it is still not known how to describe such a sector rigorously in terms of algebras of observables, not because the theorem is false in such a sector. In part (b) more restrictive conditions are assumed for technical reasons arising from limitations in the method of proof, not because the conclusion is believed to be false more generally. In fact, at the end of this section we shall describe our conjecture on the generality of the result in part (b). But first we shall sketch some aspects of the proof of this theorem in order to give the reader a sense of the ideas behind such results. However, we are obliged to refer the reader to the original papers for complete details. We begin with a discussion of some abstract structure properties.

Definition 2.4: Let $\mathcal{A}$ be a C*-algebra with unit 1. Then $N \in \mathcal{A}$ is called a I_2-generator if $N^2 = 0$ and $NN^* + N^*N = 1$.

Let $V_{\mathcal{A}}$ denote the set of I_2-generators in $\mathcal{A}$. Clearly, if N is contained in $V_{\mathcal{A}}$, then N^*N and NN^* are nonzero complementary projections, i.e. their sum is 1 and their product is 0, and the C*-algebra generated by N is isomorphic to $M_2(\mathbb{C})$ and contains the unit 1 of $\mathcal{A}$. Conversely, if $\mathcal{A}$ contains a copy of $M_2(\mathbb{C})$ containing 1, then $V_{\mathcal{A}} \neq \emptyset$. Note that if $A_i \in \mathcal{A}$ satisfies $A_i^* = A_i$, $A_i^2 = 1$ and $A_1A_2 + A_2A_1 = 0$ (which is the case if A_1, A_2 are maximal violators of Bell's inequalities in some faithful state on $\mathcal{A}$ (Prop. 1.7)), then $N \equiv \frac{1}{2}(A_1 + iA_2)$ is an element of $V_{\mathcal{A}}$. We introduce some standard definitions.

Definition 2.5: A von Neumann algebra $\mathcal{A}$ is said to have the property L_λ (resp. L'_λ) with $\lambda \in [0, 1/2]$ if for every $\epsilon > 0$ and any normal state $\phi \in \mathcal{A}^*$ (resp. finite family $\{\phi_i\}_{i=1}^{n}$ of normal states on $\mathcal{A}$), there exists an $N \in V_{\mathcal{A}}$ such that for any $A \in \mathcal{A}$,

$$|\lambda\phi(AN) - (1-\lambda)\phi(NA)| \leq \epsilon\|A\| \tag{2.1}$$

(resp. for any $A \in \mathcal{A}$ and $i = 1, \dots, n$

$$|\lambda\phi_i(AN) - (1-\lambda)\phi_i(NA)| \leq \epsilon\|A\|.$$

Definition 2.6: The asymptotic ratio set $r_\infty(\mathcal{A})$ of a von Neumann algebra $\mathcal{A}$ is the set of all $\alpha \in [0,1]$ such that $\mathcal{A}$ is W^*-isomorphic to $\mathcal{A} \otimes \mathcal{R}_\alpha$, where $\{\mathcal{R}_\alpha\}_{\alpha\in[0,1]}$ is the family of hyperfinite factors constructed by Powers [35].

It is known that property L'_λ is strictly stronger than property L_λ [4], that property L'_λ implies property $L'_{1/2}$ [4,5], and that property L'_λ for $\mathcal{A}$ is equivalent to $\lambda/1-\lambda \in r_\infty(\mathcal{A})$ [4]. Using Prop. 1.7 one easily sees that if $A_1, A_2 \in \mathcal{A}$ are maximal violators of Bell's inequalities in the normal state ϕ on $\mathcal{A} \vee \mathcal{B}$, where $\mathcal{B} \subset \mathcal{A}'$, then $N \equiv \frac{1}{2}(A_1 + iA_2) \in V_{\mathcal{A}}$ satisfies (2.1) with $\epsilon = 0$ and $\lambda = 1/2$.

These properties are intimately related to the occurrence of $\beta(\phi, \mathcal{A}, \mathcal{A}') = \sqrt{2}$.

Theorem 2.7 [47,48]: For a von Neumann algebra $\mathcal{A}$ with a cyclic and separating vector in a separable Hilbert space $\mathcal{H}$, the following conditions are equivalent.

(a) $\mathcal{A} \approx \mathcal{A} \otimes \mathcal{R}_1$, i.e. $\mathcal{A}$ has property $L'_{1/2}$.

(b) The pair $(\mathcal{A}, \mathcal{A}')$ is maximally correlated.

(c) There exist sequences of selfadjoint contractions $\{A_{1,\alpha}\}_{\alpha\in N}$, $\{A_{2,\alpha}\}_{\alpha\in N} \subset \mathcal{A}$, $\{B_{1,\alpha}\}_{\alpha\in N}$, $\{B_{2,\alpha}\}_{\alpha\in N} \subset \mathcal{A}'$ such that $T_\alpha \equiv \frac{1}{2}(A_{1,\alpha}(B_{1,\alpha} + B_{2,\alpha}) + A_{2,\alpha}(B_{1,\alpha} - B_{2,\,ga}))$ converges to $\sqrt{2}\cdot 1$ in the σ-weak operator topology on $\mathcal{B}(H)$ as $\alpha \to \infty$.

Also the following conditions are equivalent.

(d) $\mathcal{A}$ has the property $L_{1/2}$.

(e) For any vector state $\omega(A) = \langle \Omega,\, A\Omega \rangle$, $\Omega \in \mathcal{H}$, one has $\beta(\omega,\, \mathcal{A},\, \mathcal{A}') = \sqrt{2}$.

Remark: Condition (c) means that there exists a sequence of admissible observables that in the limit maximally violate Bell's inequalities in all normal states at once. Note also that $\mathcal{R}_1$ is an infinite product of copies of $M_2(\mathbf{C})$ [5].

Contained in Theorem 2.7 is a characterization of von Neumann algebras $\mathcal{A}$ such that $(\mathcal{A}, \mathcal{A}')$ is maximally correlated. If $\mathcal{A}$ and $\mathcal{B} \subset \mathcal{A}'$ are von Neumann algebras, then $(\mathcal{A}, \mathcal{B})$ maximally correlated implies that $(\mathcal{A}, \mathcal{A}')$ and $(\mathcal{B}, \mathcal{B}')$ are both maximally correlated. The converse is false [48], so we mention a characterization of maximally correlated pairs of von Neumann algebras $(\mathcal{A}, \mathcal{B})$.

Theorem 2.8 [48]: Let $(\mathcal{A}, \mathcal{B})$ be a pair of commuting von Neumann algebras acting on a separable Hilbert space $\mathcal{H}$. Then the pair $(\mathcal{A}, \mathcal{B})$ is maximally correlated if and only if there exists a type I factor $\mathcal{M} \subset \mathcal{A} \vee \mathcal{B}$ such that $\mathcal{A} \cap \mathcal{M}$ and $\mathcal{B} \cap \mathcal{M}$ are (spatially) isomorphic to $\mathcal{R}_1$ and are relative commutants of each other in $\mathcal{M}$.

Now that the connection between maximal violation of Bell's inequalities and structure properties of the algebras is somewhat clearer, we can proceed to the situation in quantum field theory.

Theorem 2.9 [48]: Let $\{\mathcal{A}(O)\}$ be a net of observable algebras in an irreducible vacuum representation such that $[\bigcup_{O \in \mathcal{K}} \mathcal{A}(O)]\ \Omega$ is dense in the representation space $\mathcal{H}$, where Ω is the (up to a factor) unique vacuum vector. Then for each $W \in \mathcal{W}$, $\mathcal{A}(W)$ is a type III_1 factor that has property L'_λ for all $\lambda \in [0, 1/2]$.

Proof: Under the above assumptions each wedge algebra $\mathcal{A}(W)$ is nontrivial [24] and must be a type III_1 factor [20]. Let $\{V(t)\}_{t \in R}$ denote the strongly continuous unitary group on $\mathcal{H}$ implementing the velocity transformation subgroup of the Poincaré group that leaves W invariant. Then Ω is the (up to a factor) unique $V(\mathbf{R})$-invariant vector in $\mathcal{H}$ and

$$\operatorname*{w-lim}_{|a| \to \infty} V(a) A V(a)^{-1} = < \Omega, A\Omega > \cdot 1 \tag{2.2}$$

for every $A \in \mathcal{A}$ (Prop. I.1.6 in [22]).

By [17], because $\mathcal{A}(W)$ is a type III_1 factor, for any $\epsilon > 0$ and $\lambda \in [0, 1/2]$ there exists a I_2-generator $N \in \mathcal{A}(W)$ such that for every $A \in \mathcal{A}(W)$,

$$|\lambda < \Omega,\ AN\Omega > -(1-\lambda) < \Omega,\ NA\Omega > | \leq \epsilon \|A\|.$$

Since Ω is invariant under $V(R)$ and since

$$V(a)\ \mathcal{A}(W)\ V(a)^{-1} \equiv \alpha_a(\mathcal{A}(W)) = \mathcal{A}(W)$$

for all $a \in \mathbf{R}$, one also has

$$|\lambda < \Omega,\ A\alpha_a(N)\Omega > -(1-\lambda) < \Omega,\ \alpha_a(N)A\Omega > | \leq \epsilon\|A\|, \tag{2.3}$$

for all $A \in \mathcal{A}(W)$ and $a \in \mathbf{R}$. Let $\{\omega_i\}_{i=1}^{n}$ be a finite family of normal states on $\mathcal{A}(W)$. Again by [17] there exist unitaries $U_i, i = 1, \dots, n$, in $\mathcal{A}(W)$ such that

$$| < \Omega,\ U_i A U_i^* \Omega > -\omega_i(A)| \leq \epsilon\|A\|, \tag{2.4}$$

for all $A \in \mathcal{A}(W)$, $i = 1, \dots, n$. Choosing $b \in \mathbf{R}$ such that

$$\|[\alpha_b(N), U_i]\Omega\| \leq \epsilon \tag{2.5}$$

and

$$\|[\alpha_b(N), U_i^*]\Omega\| \leq \epsilon \tag{2.6}$$

which is possible by (2.2), locality and the cyclicity of Ω for $\mathcal{A}(W)$ (see, e.g. the proof of (A) in [21]), one has

$$\begin{aligned}
&|\lambda\omega_i(A\alpha_b(N)) - (1-\lambda)\omega_i(\alpha_b(N)A)| \\
&\leq \lambda|\omega_i(A\alpha_b(N)| - < \Omega, U_i\, A\, \alpha_b(N)\, U_i^*\Omega > | \\
&+ \lambda| < \Omega, U_i A\alpha_b(N)\, U_i^*\Omega > - < \Omega, U_i A\, U_i^*\, \alpha_b(N)\Omega > | \\
&+ |\lambda < \Omega, U_i A\, U_i^* \alpha_b(N)\Omega > -(1-\lambda) < \Omega, \alpha_b(N) U_i\, A\, U_i^*\Omega > | \\
&+ (1-\lambda)| < \Omega,\ \alpha_b(N)\, U_i\, A\, U_i^*\Omega > - < \Omega,\ U_i\alpha_b(N) A\, U_i^*\Omega > | \\
&+ (1-\lambda)| < \Omega, U_i\alpha_b(N) A\, U_i^*\Omega > -\omega_i(\alpha_b(N)A)| \\
&\leq 5\epsilon\|A\|,
\end{aligned}$$

for all $A \in \mathcal{A}(W)$ and $i = 1, \dots, n$, using (2.3) – (2.6). Since $\alpha_b(N) \in V_{\mathcal{A}(W)}$, the theorem is proved.□

Remarks: (1) The above proof is a modification of an argument sketched by Testard in [50].

(2) By using methods of [24], the assumption that the vacuum vector is unique can be dropped and one can still conclude that for each $W \in \mathcal{W}$ $\mathcal{A}(W)$ is type III and has property L'_λ for all $\lambda \in [0, 1/2]$. (See [48])

Since the property L'_λ, $\lambda \in [0, 1/2]$, is an isomorphic invariant, the above theorem is also true for nets of local algebras in representations such as those occuring in the Doplicher, Haag, Roberts theory of superselection structure [18] and also the massive single particle representations of Buchholz and Fredenhagen [13], which include, in principle, charged sectors of theories like quantum chromodynamics. Hence by evoking Theorem 2.7 we have finished the sketch of the proof of part (a) of Theorem 2.3.

We commence the discussion of part (b) of Theorem 2.3 by proving the following result for dilatation-invariant theories. The dilatation invariance of a theory with unique vacuum is expressed by the existence of a strongly continuous, unitary representation $D(\mathbf{R}_+)$ of the dilatation group on $\mathbf{R}^d$ acting such that

$$\delta_\lambda(\mathcal{A}(O)) \equiv D(\lambda)\,\mathcal{A}(O)\,D(\lambda)^{-1} = \mathcal{A}(\lambda O), \lambda > 0.$$

where $\lambda O \equiv \{\lambda x | x \in O\}$ and $D(\lambda)\Omega = \Omega$ for any $\lambda \in \mathbf{R}_+$ (Ω is the unique vacuum vector of the theory).

Theorem 2.10 [48]: Let $\{\mathcal{A}(O)\}$ be a net of local von Neumann algebras in an irreducible vacuum representation of a dilatation-invariant theory such that the wedge algebras are locally generated [24] and $\mathcal{A}(W)' = \mathcal{A}(W')$ for each $W \in \mathcal{W}$ (both of which are true if the net is locally associated to a Wightman field in the sense of [25]). Then for any tangent double cones O_1, O_2 the pair $(\mathcal{A}(O_1), \mathcal{A}(O_2))$ is maximally correlated and thus all double cone algebras have property $L'_{1/2}$.

Proof: It is known [37] that under the given assumptions, for any $A \in \mathcal{C}$, $\delta_\lambda(A)$ converges weakly to $\phi_0(A)\cdot 1$ as $\lambda \downarrow 0$, where $\phi_0 = \phi_0 \circ \delta_\lambda$ is the vacuum state on $\mathcal{C}$. Thus, for any locally normal state $\phi \in \mathcal{B}(\mathcal{H})^*$, $\phi \circ \delta_\lambda \to \phi_0$ pointwise on $\mathcal{C}$ as $\lambda \downarrow 0$. Without loss of generality, it maybe assumed that the point of tangency for O_1 and O_2 is the origin and that $O_1 \subset W_R$, $O_2 \subset W'_R$.

$\mathcal{A}(W_R)$ is a type III_1 factor [20, 21]. Since type III_1 factors have property $L_{1/2}$ [17, 47], it follows from Theorem 2.7 that $\beta(\phi, \mathcal{A}(W_R), \mathcal{A}(W'_R)) = \sqrt{2}$ for every vector state ϕ on $\mathcal{B}(\mathcal{H})$. In particular, $\beta(\phi_0, \mathcal{A}(W_R), \mathcal{A}(W'_R)) = \sqrt{2}$. Let $\epsilon > 0$ be arbitrary and pick selfadjoint contractions $A_i \in \mathcal{A}(W_R)$, $B_j \in \mathcal{A}(W'_R)$, $i,j = 1,2$, such that with $T_\epsilon \equiv \frac{1}{2}(A_1(B_1+B_2)+A_2(B_1-B_2))$ one has $\phi_0(T_\epsilon) \geq \sqrt{2}-\epsilon$. Let also $\delta > 0$ be arbitrary and pick two sufficiently large tangent double cones $\hat{O}_1$, $\hat{O}_2$ (with $O_1 \subset \hat{O}_1 \subset W_R$ and $O_2 \subset \hat{O}_2 \subset W'_R$) such that there exist selfadjoint contractions $\hat{A}_i \in \mathcal{A}(\hat{O}_1)$, $\hat{B}_j \in \mathcal{A}(\hat{O}_2)$ satisfying $|\phi_0(T_\epsilon - \hat{T}_{\epsilon,\delta})| < \delta$, where $\hat{T}_{\epsilon,\delta} \equiv \frac{1}{2}(\hat{A}_1(\hat{B}_1+\hat{B}_2)+\hat{A}_2(\hat{B}_1-\hat{B}_2))$ (this is possible by Kaplansky's density theorem and the assumption that the wedge algebras are locally generated).

Then for any locally normal state $\phi \in \mathcal{B}(\mathcal{H})^*$,

$$\phi \circ \delta_\lambda(\hat{T}_{\epsilon,\delta}) \underset{\lambda\to 0}{\to} \phi_0(\hat{T}_{\epsilon,\delta}) \geq \sqrt{2}-\epsilon-\delta.$$

But for every $\lambda \in \mathbf{R}_+$, one has $\delta_\lambda(\hat{A}_i) \in \mathcal{A}(\lambda\hat{O}_1)$ and $\delta_\lambda(\hat{B}_j) \in \mathcal{A}(\lambda\hat{O}_2)$, and there exists a $\lambda_0 > 0$ such that $\lambda\hat{O}_1 \subset O_1$ and $\lambda\hat{O}_2 \subset O_2$ for all $\lambda < \lambda_0$. Hence the assertion of the theorem follows at once.□

Because the scaling limit of the models mentioned in part (b) of Theorem 2.3 is the massless, free field, which is dilatation-invariant, one can extend the result of Theorem 2.10 to include such models as well. Let $f(x) \to f_\lambda(x) \equiv f(\lambda^{-1}x)$ be the induced action of the dilatation group on the test function space $\mathcal{S}(\mathbf{R}^d)$. It is well known that there exists a scaling function $N(\lambda)$ (monotone, nonnegative for $\lambda > 0$) such that for all $f_1, f_2 \in \mathcal{S}(\mathbf{R}^d)$,

$$\lim_{\lambda \to 0} N(\lambda)^2 \, W_m{}^{(2)}(f_{1,\lambda}, f_{2,\lambda}) = W_0{}^{(2)}(f_1, f_2),$$

where $W_m{}^{(2)}(.,.)$ is the two-point Wightman function of the free field with mass m.

Sufficient conditions in terms of test functions have been given in [46] that insure that Bell's inequalities are maximally violated in the vacuum state by any free field algebras containing the spectral projections of field operators smeared with test functions satisfying said conditions. It is shown in [48] that with the above scaling one can insure that for any pair of tangent double cones (O_1, O_2) one can find test functions with appropriate support satisfying the said conditions. The proof of maximal correlation is then completed by the following theorem.

Theorem 2.11 [48]: Let $(\mathcal{A}, \mathcal{B})$ be a pair of commuting von Neumann algebras acting on a separable Hilbert space. Then the following are equivalent.

(a) $(\mathcal{A}, \mathcal{B})$ is maximally correlated.

(b) There exists a faithful state $\omega \in (\mathcal{A} \vee \mathcal{B})_*$ such that $\beta(\omega, \mathcal{A}, \mathcal{B}) = \sqrt{2}$.

Proof: The implication (a) $\to$ (b) is trivial. To verify the other implication, first note that for a faithful state $\omega \in (\mathcal{A} \vee \mathcal{B})_*$ and an arbitrary normal state $\phi \in (\mathcal{A} \vee \mathcal{B})_*$, ϕ can be arbitrarily well approximated in norm by elements of the set of all states $\psi \in (\mathcal{A} \vee \mathcal{B})_*$ such that there is some $\lambda > 0$ with $\psi \leq \lambda\omega$ (see the proof of Theorem 2.1 in [48]). Let $\{A_1{}^{(n)}, A_2{}^{(n)}, B_1{}^{(n)}, B_2{}^{(n)}\}_{n \in N}$ be a sequence of selfadjoint contractions with $A_i{}^{(n)} \in \mathcal{A}$, $B_j{}^{(n)} \in \mathcal{B}$, $i, j = 1, 2, n \in N$, satisfying

$$\frac{1}{2}\omega(A_1{}^{(n)}(B_1{}^{(n)} + B_2{}^{(n)}) + A_2{}^{(n)}(B_1{}^{(n)} - B_2{}^{(n)})) \to \sqrt{2}$$

as $n \to \infty$, and let $\psi \in (\mathcal{A} \vee \mathcal{B})_*$ be a state with $\psi \leq \lambda\omega$ for some $\lambda > 0$. Then with

$$T_n \equiv \sqrt{2} - \frac{1}{2}(A_1{}^{(n)}(B_1{}^{(n)} + B_2{}^{(n)}) + A_2{}^{(n)}(B_1{}^{(n)} - B_2{}^{(n)})) \geq 0$$

(by Prop. 1.2) one has $\psi(T_n) \leq \lambda\omega(T_n) \to 0$ as $n \to \infty$. Since such states $\psi \in (\mathcal{A} \vee \mathcal{B})_*$ are norm dense in the normal states on $(\mathcal{A} \vee \mathcal{B})$, the desired implication follows. □

Thus, to verify that $(\mathcal{A}, \mathcal{B})$ is maximally correlated, it suffices to check that $\beta(\omega, \mathcal{A}, \mathcal{B}) = \sqrt{2}$ for one, conveniently chosen faithful state $\omega \in (\mathcal{A} \vee \mathcal{B})_*$. In particular, since in quantum field theory the vacuum state ϕ_0 is typically faithful on the local algebras of observables, it follows that if already the vacuum fluctuations are such that Bell's inequalities are maximally violated, then all other preparations of the system will lead to violation of Bell's inequalities, as well.

It is worth emphasizing that the above scaling arguments show that the spacetime supports of the observables that give a Bell correlation converging to $\sqrt{2}$ converge to the point of tangency of the pair (O_1, O_2).

We expect that the following conjecture is true. Let $\{\mathcal{A}(O)\}$ be a net of local observable von Neumann algebras in a vacuum representation to which is locally associated a quantum field in the sense of [25] and for which assumption (A) below holds. Then $(\mathcal{A}(O_1), \mathcal{A}(O_2))$ is maximally correlated for any tangent double cones $O_1, O_2 \in \mathcal{K}$.

(A) There exists a scaling function $N(\lambda)$ (monotone, nonnegative for $\lambda > 0$) such that for all test functions f_i

$$\lim_{\lambda \to 0} N(\lambda)\phi_0(\varphi(f_\lambda)) = {W_0}^{(1)}(f) \ (= 0),$$

$$\lim_{\lambda \to 0} N(\lambda)^2 \phi_0(\varphi(f_{1,\lambda})\varphi(f_{2,\lambda})) = {W_0}^{(2)}(f_1, f_2),$$

and

$$\lim_{\lambda \to 0} N(\lambda)^4 \phi_0(\varphi(f_{1,\lambda})\varphi(f_{2,\lambda})\varphi(f_{3,\lambda})\varphi(f_{4,\lambda})) = {W_0}^{(4)}(f_1, f_2, f_3, f_4),$$

where $\{{W_0}^{(j)}\}_{j=1,2,4}$ are the Wightman functions corresponding to the vacuum state of the free, massless field.

Condition (A) is known to be true in most of the quantum field models that have been constructed. It is a weak, rigorous way of saying that the theory has a well-defined Gell-Mann-Low limit.

If we may briefly summarize: The results above show that in relativistic quantum field theory, how ever the field has been prepared and no matter what the particular dynamics of the field may be, there are observables associated to spacelike tangent regions that maximally violate Bell's inequalities. The maximal violation of Bell's inequalities in every normal state is a consequence of the most basic axioms of quantum field theory. The same axioms imply that though it is true that pairs of algebras of observables associated with regions that are spacelike separated by an arbitrary nonzero distance will have many states satisfying Bell's inequalities, nonetheless even in that case there are (infinitely) many states on the same pairs in which Bell's inequalities are maximally violated.

III. MAXIMAL CORRELATION, SPLIT PROPERTY AND STATISTICAL INDEPENDENCE

In this section we shall briefly contrast maximal correlation with the split property and place them both in the context of statistical independence. For a more complete discussion, see [54]. Having stated maximal correlation as a property of a pair of commuting algebras, we shall do the same for the split property (but see [12, 19]).

Definition 3.1: Let $(\mathcal{A}, \mathcal{B})$ be a pair of commuting von Neumann algebras on a Hilbert space $\mathcal{H}$. The pair $(\mathcal{A}, \mathcal{B})$ is split if there exists a type I factor $\mathcal{M}$ such that $\mathcal{A} \subset \mathcal{M} \subset \mathcal{B}'$.

As we shall recall below more formally, $(\mathcal{A}, \mathcal{B})$ is split if and only if there are many normal product states across the algebra $\mathcal{A} \vee \mathcal{B}$. Hence by Theorem 1.4(a), the split property and maximal correlation of $(\mathcal{A}, \mathcal{B})$ are mutually exclusive, indeed they are each other's opposite, in a sense that we want to indicate in this section. However, we first weave the thread of independence of algebras of observables into the discussion.

In quantum mechanics if the algebras of observables $\mathcal{A}, \mathcal{B}$ of two systems mutually commute, they are viewed as independent, insofar as all measurements on one system are compatible with all measurements on the other system. However, there are stronger conditions of independence that are also of interest.

Definition 3.2: A pair $(\mathcal{A}, \mathcal{B})$ of commuting subalgebras of a C*-algebra $\mathcal{C}$ is said to be C*-independent if for each state $\phi_1 \in \mathcal{A}^*$ and each state $\phi_2 \in \mathcal{B}^*$ there exists a state $\phi \in \mathcal{C}^*$ such that $\phi|\mathcal{A} = \phi_1$ and $\phi|\mathcal{B} = \phi_2$.

Hence two systems with associated algebras of observables $\mathcal{A}, \mathcal{B}$ that are C^*- independent can each be prepared in any state independently of the state of the other system. Roos showed [38] that in fact a pair $(\mathcal{A}, \mathcal{B})$ of commuting C*-algebras is C*-independent if and only if any pair of states $\phi_1 \in \mathcal{A}^*$, $\phi_2 \in \mathcal{B}^*$ has a common extension ϕ that is a **product** state across $\mathcal{A} \vee \mathcal{B}$. This is entailed if, for example, $\mathcal{A} \vee \mathcal{B}$ is naturally isomorphic to the tensor product of $\mathcal{A}$ with $\mathcal{B}$. One has the following result in quantum field theory.

Theorem 3.3 [39, 20, 38, 49]: In an irreducible vacuum representation, for any strictly spacelike separated regions $\mathcal{O}_1$, $\mathcal{O}_2$, $(\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2))$ is C*-independent. For any tangent double cones $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{K}$, resp. any wedge $W \in \mathcal{W}$, the pair $(\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2))$, resp. $(\mathcal{A}(W), \mathcal{A}(W'))$, is C*-independent.

So C*-independence is typical in quantum field theory. There is a yet stronger condition of independence.

<u>Definition 3.4</u>: Let $\mathcal{A}$ and $\mathcal{B}$ be commuting subalgebras of a W^*-algebra $\mathcal{C}$. The pair $(\mathcal{A}, \mathcal{B})$ is said to be W^*-independent (in the product sense) if for every normal state $\phi_1 \in \mathcal{A}_*$ and every normal state $\phi_2 \in \mathcal{B}_*$ there exists a product normal state $\phi \in \mathcal{C}^*$ such that $\phi|(AB) = \phi_1(A)\phi_2(B)$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$

It is known [39, 49] that if $(\mathcal{A}, \mathcal{B})$ is W^*-independent, then $(\mathcal{A}, \mathcal{B})$ is C*-independent. We collect the following characterizations of W^*-independence.

<u>Theorem 3.5</u> [12, 1, 2, 51, 49]: In an irreducible vacuum representation for which the vacuum vector is cyclic for all local algebras, the following are equivalent for any two spacelike separated double cones or wedges O_1, O_2.

(1) Local preparability of all normal states: for every normal state ϕ_0 there is a normal positive map $T : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ such that $T(A) = \phi_0(A)T(1)$ for all $A \in \mathcal{A}(O_1)$ and $T(B) = T(1)B$ for $B \in \mathcal{A}(O_2)$.

(2) $(\mathcal{A}(O_1), \mathcal{A}(O_2))$ is W^*-independent (in the product sense).

(3) $(\mathcal{A}(O_1), \mathcal{A}(O_2))$ is split.

(4) $\mathcal{A}(O_1) \vee \mathcal{A}(O_2) \approx \mathcal{A}(O_1) \otimes \mathcal{A}(O_2)$.

By Theorem 1.4(a), the pair $(\mathcal{A}(O_1), \mathcal{A}(O_2))$ of algebras associated to the tangent spacetime regions considered in Theorem 2.3 are <u>not</u> W^*-independent, even though, by Theorem 3.3, they are C*-independent. On the other hand, strictly spacelike separated pairs of double cones are known in many cases [12, 42, 2] to be split, hence W^*-independent. $(\mathcal{A}(O_1), \mathcal{A}(O_2))$ maximally correlated implies $(\mathcal{A}(O_1), \mathcal{A}(O_2))$ is very badly nonsplit. Being nonsplit is a property that is strictly weaker than being maximally correlated [48]. It is known that algebras of observables associated to tangent, spacelike separated spacetime regions are not split, in general. We mention only the following result.

<u>Theorem 3.6</u> [48]: Let O_1 and O_2 be (Poincaré transforms of) tangent spacelike separated spacetime regions for which there exists a $\lambda_0 > 0$ such that $\lambda O_i \subset O_i$ for all $0 < \lambda < \lambda_0$, $i = 1, 2$. And let $\{\mathcal{A}(O))\}$ be a net of local von Neumann algebras in an irreducible vacuum representation, to which is locally associated a quantum field in the sense of [25]. With assumption (A) at the end of the previous section, then is no (locally) normal state ϕ on $\mathcal{C}$ such that $\phi(AB) = \phi(A)\phi(B)$ for all $A \in \mathcal{A}(O_1)$, $B \in \mathcal{A}(O_2)$.

So we see that very generally, tangent spacetime regions have associated to them nonsplit algebras; however, it is known [49] that for a class of such regions O_1, O_2, one has $\beta(\phi, \mathcal{A}(O_1), \mathcal{A}(O_2) < \sqrt{2} - \epsilon$ for all normal states ϕ, where ϵ depends on geometric properties of the regions at the point of tangency. Hence they are nonsplit but not maximally correlated.

We summarize: If ϕ is a product state over $(\mathcal{A}(O_1), \mathcal{A}(O_2))$ then $\beta(\phi, \mathcal{A}(O_1), \mathcal{A}(O_2)) = 1$. Hence, if $\beta(\phi, \mathcal{A}(O_1), \mathcal{A}(O_2)) > 1$ then ϕ is not a product state over $(\mathcal{A}(O_1), \mathcal{A}(O_2))$. The larger the number $\beta(\phi, \mathcal{A}(O_1), \mathcal{A}(O_2))$ the stronger the correlations between $\mathcal{A}(O_1)$ and $\mathcal{A}(O_2)$ in the state ϕ and the less "product-like" ϕ is across $(\mathcal{A}(O_1), \mathcal{A}(O_2))$. If $\beta(\phi, \mathcal{A}(O_1), \mathcal{A}(O_2)) = \sqrt{2}$, the correlation between $\mathcal{A}(O_1)$ and $\mathcal{A}(O_2)$ is the maximum possible. In the case of some tangent spacetime regions one has $\beta(\phi, \mathcal{A}(O_1), \mathcal{A}(O_2)) = \sqrt{2}$ for all normal states, whereas in the same situation $\beta(\phi, \mathcal{A}(O_1), \mathcal{A}(O_2)) = 1$ for many nonnormal states. The ultraviolet effect that is responsible for the maximal violation of Bell's inequalities in all normal states over tangent spacetime regions entails that tangent spacetime regions are quantitatively (as measured by $\beta(\phi, \mathcal{A}(O_1), \mathcal{A}(O_2))$) and maximally far from W^*-independence while remaining C*-independent.

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