Beables in Algebraic Quantum Mechanics *

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Michael Redhead is one of the foremost advocates of the tenability of scientific realism in the domain of quantum theory. Particularly inspiring is his deep physical knowledge and intuition, combined with the uncanny ability he has to tease out the essence of a conceptual problem in physics from amidst the often bewildering and mathematically daunting literature. It is a pleasure to be able to offer this piece in honour of the (self-described) 'Cantabridgian dinosaur's' retirement, and to express my desire that he not become extinct just yet!

1   Beables versus 'Observables'

A good deal of Michael's work has focussed on articulating the pitfalls of adopting a 'simple realism of possessed values' in quantum mechanics, which is put under pressure by the no-go theorems of Kochen-Specker and Bell. While I suspect Michael's views on the matter are still tentative and exploratory, in his recent book *From Physics to Metaphysics* (1995a, Ch. 3) Michael appears to favour van Fraassen's (1973) idea of securing determinate values for all observables by 'ontologically contextualizing' physical magnitudes. The idea is to let any given degenerate self-adjoint operator on a system's Hilbert space represent more than one magnitude of the system. Each magnitude is distinguished from the others by the functional relations its values have to different complete commuting sets of self-adjoint operators of which the given self-adjoint operator is a member. Thus, to pick

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1

out a physical magnitude it is not enough to know that its statistics are represented by tracing the density operator of the system with some particular self-adjoint operator; for the degenerate operators, one must also pick a context of definition for the magnitude being measured, specified by some complete commuting set.

Formally, this is enough to prevent Kochen-Specker contradictions. But for Michael the real payoff is that it yields a novel holistic interpretation of quantum nonlocality (1995a, pp. 86-7). Take the example of two correlated spin-1 particles in the entangled singlet state

$$-3^{-\frac{1}{2}}[|S_{1x} = 0\rangle|S_{2x} = 0\rangle - |S_{1y} = 0\rangle|S_{2y} = 0\rangle + |S_{1z} = 0\rangle|S_{2z} = 0\rangle] \quad (1)$$

for which Michael and a former student were able to supply the first purely algebraic proof of Bell's theorem (Heywood and Redhead 1983). Since the self-adjoint operator S_{1n}^2 pertaining to any (squared) spin component of particle 1 is represented by the degenerate operator $S_{1n}^2 \otimes I_2$ on 1 + 2's Hilbert space, ontological contextualism blocks the conclusion (which would otherwise be forced by the Heywood-Redhead argument) that the outcome of a measurement of S_{1n}^2 must causally depend upon measurements performed on particle 2. This conclusion is blocked because the very definition of the spin magnitude being measured rests on which complete commuting set of self-adjoint operators of the composite 1 + 2 system the values of $S_{1n}^2 \otimes I_2$ are referred to. If we cannot specify a subsystem's properties independently of properties relating to the whole combined system, then the question of whether properties intrinsic to a subsystem causally depend on measurements undertaken on spacelike-separated systems cannot even be raised. Michael calls this consequence of ontological contextualism 'ontological nonlocality' to contrast it with 'environmental nonlocality' that would involve an explicit spacelike causal dependence of local properties on distant measurements in apparent conflict with relativity theory.

Despite the lure of this route to peaceful coexistence between relativity and quantum nonlocality, it is hard to be totally at ease with an ontology that entertains the existence of large numbers of distinct physical magnitudes which are in principle statistically indistinguishable. And since it's not obvious how failing to classify quantum nonlocality as a causal connection improves the chances of securing a Lorentz invariant realist interpretation of the theory, it is surely worth seeking an alternative, *simpler* realism of possessed values that takes the functional relations between self-adjoint operators just as seriously.

In fact, one doesn't have to look very far. The key lies in rejecting an assumption that is necessary to prove the Kochen-Specker theorem which Michael dubs the 'Reality Principle' in *Incompleteness, Nonlocality and Realism* (1987):

> If there is an operationally defined number associated with the self-adjoint operator $\hat{Q}$ (i.e. distributed probabilistically according to the statistical algorithm of QM for $\hat{Q}$), then there exists an element of reality . . . associated with that number and measured by it (1987, p. 133-4).

Michael considers (and rightly rejects) only one way to deny the Reality Principle. Faced with incompatible ways to measure a degenerate self-adjoint operator $\hat{Q}$—depending on which complete commuting set it is measured along with—one could say that only one way reveals $\hat{Q}$'s true value and the others 'produce numbers which just "hang in the air" and do not measure anything of ontological significance' (1987, p. 136). But this is not the most natural way to deny the Reality Principle. The most natural way is to regard the measurement of certain self-adjoint operators as yielding results without ontological significance *however* they are 'measured'—the paradigm example being the measurement of 'spin' in Bohm's theory.[1] This is not at all to renounce the realist demand for an explanation of measurement results, but only to abandon the particular form of explanation demanded by the Reality Principle, which dictates that each self-adjoint operator needs to be thought of as having its measurement results determined by a pre-existing element of reality *unique to that operator*. (In Bohm's theory, by contrast, all measurement results are grounded in the pre-existing position of the particle together with its initial wavefunction.) Jettisoning this part of the Reality Principle clears the way for an interpretive programme in quantum mechanics which has received concrete expression recently in various 'modal' interpretations of quantum mechanics,[2] and has figured prominently in the writings of John Bell (1987, Chs. 5, 7, 19).

For Bell, a self-adjoint operator is just a mathematical device which, when traced with the system's density operator, generates the empirically correct statistics in an experiment on the system which orthodox quantum mechanics would loosely call a 'measurement' of the 'observable' represented by the operator (1987, p. 52). Out of the 'observables' of the orthodox interpretation Bell seeks to isolate some subset, the '*be*ables' of a system,

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[1] See Pagonis and Clifton (1995) and Dürr et al. (1996) for further discussion.

[2] See Clifton (1996) and references therein.

which can be ascribed determinate values and about which orthodoxy's loose talk is perfectly precise:

> Many people must have thought along the following lines. Could one not just promote *some* of the 'observables' of the present quantum theory to the status of beables? The beables would then be represented by linear operators in the state space. The values which they are allowed to *be* would be the eigenvalues of those operators. For the general state the probability of a beable *being* a particular value would be calculated just as was formerly calculated the probability of *observing* that value (1987, p. 41).

Bell's thinking is the exact opposite of Michael's in *From Physics to Metaphysics*. While Michael entertains the possibility that there are far more beables than self-adjoint operators, Bell is content with there being far less. Elsewhere, Bell explains how one can get away with this:

> Not all 'observables' can be given beable status, for they do not all have simultaneous eigenvalues, i.e. do not all commute. It is important to realize therefore that most of these 'observables' are entirely redundant. What is essential is to be able to define the positions of things, including the positions of instrument pointers or (the modern equivalent) of ink on computer output (1987, p. 175).

> 'Observables' must be *made*, somehow, out of beables. The theory of local beables should contain, and give precise physical meaning to, the algebra of local observables (1987, p. 52).

In this last passage we see a further contrast with Michael's thinking. While his ontological nonlocality countenances locally measurable but nonlocally defined beables, Bell restricts his considerations to beables that are both locally measurable *and* locally defined.[3] And in at least one other place (1987, p. 42), Bell again expresses his interest in modelling a theory of local beables after Haag's (1992) algebras of local observables in relativistic quantum field theory.

I find this aspect of Bell's thinking particularly intriguing. But given the $C^\star$-algebraic formulation of Haag's theory, it would be of interest to have a purely algebraic characterization of sets of (bounded) observables which are viable candidates for representing the *beables* of a quantum-mechanical system—be it a spin-1 particle in the singlet state or a bounded open region of spacetime. This is the main aim of the present paper.

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[3]See also pgs. 42 and 53 of Bell's (1987).

I begin in Section 2. by considering what subsets of the self-adjoint part of the $C^\star$-algebra $\mathcal{U}$ of a quantum system should be candidates for beable status. No doubt this is partly a matter of taste. But a natural requirement to impose is that sets of beables be closed under the taking of any continuous self-adjoint function of their members. As we shall see, such sets have their own characteristic algebraic structure, and I call them 'Segalgebras' because they conform to the general postulates for algebraic systems of observables laid down and studied by Segal (1947). It turns out that a subset of self-adjoint elements in $\mathcal{U}$ forms a Segalgebra exactly when it is the self-adjoint part of some $C^\star$-subalgebra of $\mathcal{U}$. This is satisfying insofar as there is no reason to expect that a set of beables should have an algebraic structure any different from the full set of observables of the system out of which it is distinguished. And the fact that Segalgebras are none other than the self-adjoint parts of $C^\star$-algebras allows us to carry over facts about $C^\star$-algebras to their Segalgebras.

In Section 3. I discuss what is going to count as an acceptable way of assigning values to beables in a Segalgebra. Again this is partly a matter of taste, and has been hotly debated ever since von Neumann proved his infamous no-hidden-variables theorem. Nevertheless I shall argue that while von Neumann's conception of values as given by linear functionals on the Segalgebra of 'observables' of a system is utterly inappropriate if their assigned 'values' are only defined dispositionally or counterfactually (so I agree wholeheartedly with Bell 1987, Ch. 1), there is no reason not to require the categorically possessed values of beables to be given by linear functionals, regardless of whether they commute. This will not lead in the direction of an algebraic analogue of von Neumann's no-go theorem, such as is proved by Misra (1967), simply because I shall not be supposing that all 'observables' have beable status!

In line with the approach to value definiteness taken by modal interpretations, I will also not be requiring that the beables of a quantum system be the same from one quantum state of the system to another. In the second passage from Bell (1987) quoted above he implies that the self-adjoint operators corresponding to beables must all commute. It will turn out that this only follows (at least in nonrelativistic quantum mechanics) if one requires the beables of a system to be the same for all its quantum states. However, even if that requirement is dropped, we shall see that a Segalgebra of beables still has to be 'almost commutative'.

In Section 4. I shall introduce the new notion of a quasicommutative

Segalgebra to make this idea precise. In Section 5. I go on to show that quasicommutative Segalgebras are both necessary and sufficient for representing the measurement statistics prescribed by a quantum state as an average over the actual values of the beables in the algebra[4]—in line with the first passage from Bell (1987) quoted above. I also discuss two concrete examples of noncommuting Segalgebras of beables employed by the orthodox (Dirac-von Neumann) interpretation of nonrelativistic quantum mechanics and modal interpretations thereof. Section 6. then discusses the question of how 'big' (and noncommutative) a Segalgebra of beables can be consistent with satisfying the statistics of some quantum state. There is a simple characterization of the Segalgebras of beables on a finite-dimensional Hilbert space that are maximal in this sense, but the infinite-dimensional case remains open.

Finally, in Section 7. I discuss two ways one can argue for entertaining only commutative Segalgebras of beables. The first way (as I've already mentioned) is to require that the beables of a system be the same for all its quantum states, or at least for a 'full set' thereof (we'll see that the former demand is too strong on physical grounds). The second way arises in the context of algebraic relativistic quantum field theory, where it turns out that Segalgebras of local beables must be *fully* commutative if they are to satisfy the measurement statistics dictated by a state of the field with bounded energy.

2 Segalgebras of Beables

A $C^\star$-algebra is a normed algebra $\mathcal{U}$ over the complex numbers which is complete in the metric topology induced by the norm $|\cdot|$ and equipped with an involution $\star$ that, together with the norm, satisfies the $C^\star$-norm property,[5]

$$|A^\star A| = |A|^2, \text{ for all } A \in \mathcal{U}. \tag{2}$$

We will hardly ever need to suppose that our $C^\star$-algebras are concrete algebras of bounded operators acting on some Hilbert space, but for convenience

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[4]This result generalizes results obtained by Bub and Clifton (1996) and Zimba and Clifton (forthcoming) in two directions. First and most importantly, I shall not need to restrict myself to sets of beables with discrete spectra. And, second, I shall not need to assume anything about the projection operators with beable status (such as: that they form an ortholattice); indeed, a $C^\star$-algebra need have *no* nontrivial projections.

[5]Note that this property, together with the triangle inequality and product inequality $|AB| \leq |A||B|$, entails that $|A| = |A^\star|$ for any $A \in \mathcal{U}$.

I'll still refer to the elements of $\mathcal{U}$ as operators. Of course, for a quantum system represented by some $\mathcal{U}$, the operators in $\mathcal{U}$'s self-adjoint part—consisting of all $A \in \mathcal{U}$ such that $A = A^\star$—represent the bounded observables of the system. For simplicity, I set aside the possibility of superselection rules and take the term 'observable' to be synonomous with 'self-adjoint operator'. Our task, then, is to lay down some natural guidelines for granting observables in $\mathcal{U}$ beable status.

It seems reasonable to require that a quantum system be such that its beables combine algebraicly to yield other beables, the idea being that if a set of observables have definite values, any self-adjoint function of them ought to have a definite value as well. Thus a set of beables should at least form a real vector space. And, starting with any single beable, one should be able to form polynomials over the reals in that beable which are also beables of the system. If we are going to allow these polynomials to have a constant term, we had better also require that the identity operator I be a beable.[6] Finally, it seems reasonable to require that sets of beables not just be closed under polynomial functions of their members, but all continuous functions thereof. Thus if an observable A is a beable, then $\sin A$, e^A and (if the spectrum of A consists only of nonnegative values) $\sqrt{A}$ should all have beable status too. There is only one way to define a nonpolynomial continuous function of a bounded observable A, viz. as the norm limit of a sequence of polynomials in A by analogy with the Weierstrass approximation theorem from ordinary analysis.[7] Thus sets of beables will need to be closed in norm.[8]

Of course we cannot require sets of beables to be closed under products, since the product of two observables is an observable only if they commute. However, we can always introduce a new symmetric product on $\mathcal{U}$ by

$$A \circ B \equiv 1/4[(A + B)^2 - (A - B)^2] = 1/2[A, B]_+, \tag{3}$$

which is manifestly such that if both A and B are self-adjoint $A \circ B$ will be too. Since the symmetric product of two self-adjoint operators is expressible, as above, in terms of real linear combinations and squares, it follows from

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[6]Not all $C^\star$-algebras have an identity, but one can always be 'adjoined' to any $C^\star$-algebra—see Bratteli and Robinson 1987, Prop. 2.1.5.

[7]Geroch (1985, Ch. 52) contains a complete discussion.

[8]This is consistent with our beables remaining self-adjoint, since the limit of a sequence of self-adjoint operators must itself be self-adjoint due to the continuity of the adjoint operation (which follows from $|A^\star - B^\star| = |(A - B)^\star| = |A - B|$).

our requirements on sets of beables that they *are* closed under the symmetric product.

It is easy to see that the symmetric product on $\mathcal{U}$ is homogeneous (i.e. $r(A \circ B) = (rA) \circ B = A \circ (rB)$) and distributive over addition. Moreover, the symmetric product will be associative on any triple of elements $A, B, C \in \mathcal{U}$ if they mutually commute in the $C^\star$ product (for in that case $\circ$ just reduces to the $C^\star$ product, which of course is associative by definition). However, if a triple of elements do not mutually commute, the symmetric product cannot be assumed associative. A simple example is provided by the $C^\star$-algebra $\mathcal{U}(H_2)$ of all Hermitian operators on complex two-dimensional Hilbert space. If we consider the Pauli spin operators σ_x and σ_y, then since they anti-commute we have $\sigma_x \circ \sigma_y = 0$, thus $\sigma_x \circ (\sigma_x \circ \sigma_y) = 0$; yet $(\sigma_x \circ \sigma_x) \circ \sigma_y = \sigma_x^2 \circ \sigma_y = I \circ \sigma_y = \sigma_y$.

Since raising elements in $\mathcal{U}$ to any desired $C^\star$ power can be re-expressed in terms of the symmetric product as

$$A^n = \underbrace{A \circ A \circ \cdots \circ A}_{n \text{ times}}, \tag{4}$$

we can dispense with reference to the $C^\star$ product in our requirements on beable sets. Thus what we have required, so far, is that any set of beables be a real closed linear subspace of observables taken from $\mathcal{U}$ which forms a (not necessarily associative!) algebra with respect to the symmetric product. This is an instance of the sort of algebraic structure studied by Segal (1947), and a simple concrete example is given by

$$\{a\sigma_x + b\sigma_y + cI \in \mathcal{U}(H_2) | a, b, c \in \mathcal{R}\}. \tag{5}$$

There is one last requirement I need to impose on beable sets. We can also introduce on $\mathcal{U}$ an *antisymmetric* product by

$$A \bullet B \equiv i/2[A, B]_-, \tag{6}$$

which also has the property that if both A and B are self-adjoint so is $A \bullet B$. This product is again homogeneous and distributive by not necessarily associative (e.g. $\sigma_x \bullet (\sigma_x \bullet \sigma_y) = -\sigma_y$, while $(\sigma_x \bullet \sigma_x) \bullet \sigma_y = 0$). If we are serious about wanting sets of beables to contain *all* continuous self-adjoint functions of their members, then they ought to be closed under the antisymmetric product too (its continuity is proven using the triangle and product inequalities). With closure under both the symmetric and antisymmetric products,

8

we then get closure under self-adjoint polynomials in two beables, like

$$cAB + c^*BA = 2\Re(c)A \circ B + 2\Im(c)A \bullet B. \tag{7}$$

It must be admitted, though, that closure under $\bullet$ is a strong assumption; for example, the set in Eqn. 5 is now ruled out, since it fails to contain $\sigma_x \bullet \sigma_y = -\sigma_z$. It might be of interest to investigate what portion of my conclusions can be recovered without assuming sets of beables are closed under $\bullet$, but I shall not do so here.

To summarize, our candidate beable sets are to be real closed linear subspaces of observables in $\mathcal{U}$ that contain the identity and are closed under the (generally nonassociative) symmetric and antisymmetric products. Such structures I call *Segalgebras* to distinguish them within the class of Segal's own algebras, which need not admit an antisymmetric product.[9] Virtually everything about Segalgebras follows from the fact that they are simply the self-adjoint parts of $C^\star$-subalgebras of the $C^\star$-algebras from which their elements are drawn.

To see this, recall that a subalgebra of a $C^\star$-algebra $\mathcal{U}$ is a subset of the algebra (possibly not containing the identity) that is closed under the relevant operations, i.e. a complex norm closed subspace of $\mathcal{U}$ closed under the taking of $C^\star$ products and adjoints. For T any set of observables in $\mathcal{U}$, define

$$T + iT = \{A \in \mathcal{U} | A = X + iY, \text{ with } X, Y \in T\}. \tag{8}$$

Then we have:

Theorem 1 *A subset T of the observables in a $C^\star$-algebra $\mathcal{U}$ is a real closed linear subspace of $\mathcal{U}$ closed under the symmetric and antisymmetric products if and only if $T + iT$ is a $C^\star$-subalgebra of $\mathcal{U}$.*

Proof. 'If'. Assuming $T + iT$ is a subalgebra of $\mathcal{U}$, it is automatic that T is a real linear subspace. Moreover, since any Cauchy sequence $\{A_n\} \subseteq T$ must at least converge to an element $A \in T + iT$, and the limit of a sequence of self-adjoint elements must itself be self-adjoint, A must lie in the self-adjoint part of $T + iT$, which is obviously T. Thus T is closed. Now recall that if D is an element in a $C^\star$-algebra, it has unique real and imaginery parts given by

$$\Re(D) = \frac{1}{2}(D + D^\star), \ \Im(D) = \frac{1}{2}(-iD + iD^\star). \tag{9}$$

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[9]Neither does Segal's (1947) symmetric product (which he calls 'formal product') have to be homogeneous or distributive!

To prove T is closed under symmetric and antisymmetric products, suppose $A, B \in T$. Then $A, B \in T + iT$, and since $T + iT$ is a subalgebra of $\mathcal{U}$, $AB \in T + iT$ has unique real and imaginery parts. Using Eqns. 9, those parts are just $A \circ B$ and $-A \bullet B$ and the conclusion follows.

'Only if'. Given that T is a real subspace of $\mathcal{U}$, it is routine to check that $T + iT$ is a complex subspace closed under $\star$. Next, suppose $\{A_n\} \subseteq T + iT$ is a Cauchy sequence, i.e. $|A_n - A_m| \to 0$. From Eqns. 9, the triangle inequality and the fact that $|D| = |D^\star|$ for any $D \in \mathcal{U}$, we see that $|\Re(D)|, |\Im(D)| \leq |D|$. Therefore,

$$|\Re(A_n) - \Re(A_m)| = |\Re(A_n - A_m)| \leq |A_n - A_m| \to 0, \qquad (10)$$

and, similarly, $|\Im(A_n) - \Im(A_m)| \to 0$. So both $\{\Re(A_n)\}$ and $\{\Im(A_n)\}$ must be Cauchy sequences in T. Letting their respective limits be $A_1, A_2 \in \mathcal{T}$, further use of the triangle inequality establishes that $A_1 + iA_2$ is the limit in $\mathcal{U}$ of $\{A_n\}$. Hence $T + iT$ is norm closed. Finally, for closure of $T + iT$ under $C^\star$-products, let $A, B \in T + iT$, so

$$A = X + iY \text{ and } B = X' + iY' \text{ with } X, Y, X', Y' \in T. \qquad (11)$$

A simple calculation yields

$$\Re(AB) = X \circ X' + X \bullet Y' + Y \bullet X' - Y \circ Y', \qquad (12)$$

$$\Im(AB) = -X \bullet X' + X \circ Y' + Y \circ X' + Y \bullet Y'. \qquad (13)$$

Therefore, since T is a real linear subspace closed under both the symmetric and antisymmetric products, $AB \in T + iT$. QED.

Thm. 1 tells us that a subset of $\mathcal{U}$ is a Segalgebra exactly when it is the self-adjoint part of some subalgebra of $\mathcal{U}$ containing the identity. As a first example of how this makes the 'theory' of Segalgebras parasitic upon facts about the $C^\star$-algebras they generate, consider the maps that preserve these structures. Recall that a mapping of $C^\star$-algebras $\psi : \mathcal{U} \to \mathcal{U}'$ is called a $\star$-*homomorphism* if it preserves the identity, linear combinations, products and adjoints. It is a theorem that $\star$-homomorphisms are continuous,[10] so in fact they preserve *all* the relevant structure of a $C^\star$-algebra. Analogously, call a mapping of Segalgebras $\phi : \mathcal{S} \to \mathcal{S}'$ a *homomorphism* if it preserves the

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[10]Bratteli and Robinson (1987), Prop. 2.3.1.

identity, linear combinations, and symmetric and antisymmetric products. There is an obvious bijective correspondence between *-homomorphisms and homomorphisms. If $\psi : \mathcal{U} \to \mathcal{U}'$ is a *-homomorphism, the restriction of ψ to $\mathcal{U}$'s Segalgebra is a homomorphism into $\mathcal{U}'$'s. Conversely, if $\phi : \mathcal{S} \to \mathcal{S}'$ is a homomorphism, the (unique) linear extension of ϕ to $\mathcal{S} + i\mathcal{S}$ given by $\psi(A) = \phi(\Re(A)) + i\phi(\Im(A))$ is a *-homomorphism into $\mathcal{S}' + i\mathcal{S}'$. (To check that ψ preserves $C^\star$ products, use Eqns. 11–13.) Due to this bijective correspondence, we learn 'for free' that homomorphisms of Segalgebras must also be continuous.[11]

3   Statistical States and Value States

Having decided that our sets of beables will have the algebraic structure of Segalgebras, the next step is to decide how to assign values to beables. For this, we first need to recall the algebraic definition of a quantum state.

An operator A in a $C^\star$-algebra $\mathcal{U}$ is called *positive* if it is self-adjoint and has a non-negative spectrum. It is useful to have on hand two alternative equivalent definitions: A is positive if it is the square of a self-adjoint operator in $\mathcal{U}$, or if there is a $B \in \mathcal{U}$ such that $A = B^\star B$.[12] A *state* on a $C^\star$-algebra $\mathcal{U}$ is a (complex-valued) linear functional on $\mathcal{U}$ that maps positive operators to nonnegative numbers and the identity to 1. It is a theorem that states, so defined, are continuous.[13]

We can define a *state on a Segalgebra* $\mathcal{S}$ in essentially the same way, as a (this time, real-valued) linear functional on $\mathcal{S}$ that maps squares to nonnegative numbers and I to 1. Again, there is the obvious bijective correspondence between states on Segalgebras and on the $C^\star$-algebras they generate, so that Segalgebra states are continuous too. In general, a state on a Segalgebra is

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[11]It is natural to ask whether one could define Segalgebras *independently* of $C^\star$-algebras (not assuming, as I have, that their elements are drawn from a $C^\star$-algebra), and then prove that every Segalgebra (abstractly defined) is isomorphic to the self-adjoint part of a $C^\star$-algebra. (Never mind the fact that were I to have taken such a route, my observations about Segalgebras—such as that their homomorphisms are continuous—would no longer come for free!) I recently learned from Klaas Landsman that the answer is in fact Yes, and that the abstract counterpart of a Segalgebra is called a Jordan-Lie-Banach algebra ($\circ$ is the Jordan product, $\bullet$ the Lie product)—see Landsman (forthcoming), Ch. 1 for a complete discussion and references.

[12]See Bratteli and Robinson (1987), Props. 2.2.10 and 2.2.12 respectively.

[13]Bratteli and Robinson (1987), Prop. 2.3.11.

merely statistical and specifies only the expectation values of the observables the state acts upon. If we want to interpret those expectation values as averages of the actual values of beables, we need a specification of the allowed valuations.

Certainly a valuation on the beables in a Segalgebra should itself be a real-valued function on it. Moreover, valuations ought to assign nonnegative numbers to observables with nonnegative spectra! And valuations should map I to 1, if only because if some 'deviant' set of valuations did not, then since all the states whose statistics we want to recover with our valuations map I to 1, we would have to assign that deviant set measure-zero anyway. But should a valuation of the beables in a Segalgebra $\mathcal{S}$ be a *linear* function on $\mathcal{S}$? That is, should a valuation just be a special kind of state, in the technical sense above, that gives the actual values of beables rather than just their expectations? I believe that the assumption of linearity *is* defensible, yet I completely agree with Bell's (1987, Ch. 1) critique of von Neumann's no-hidden-variables theorem (!). Let me explain.

What Bell objects to in both von Neumann's and Kochen-Specker's no-go theorems is arbitrary assumptions about how the results of measurements undertaken with incompatible experimental arrangements would turn out. For von Neumann, it is the assumption that if an observable C is actually measured, where $C = A + B$ and $[A, B] \neq 0$, then had A instead been measured, or B, their results would have been such as to sum to the value actually obtained for C. For Kochen-Specker, who adopt von Neumann's linearity requirement only when $[A, B] = 0$, it is the assumption that the results of measuring C would be the same independent of whether C is measured along with A and B or in the context of measuring some other pair of compatible observables A' and B' such that $C = A' + B'$. What makes these assumptions arbitrary is that the results of measuring observables $A, B, C, \ldots$ might not reveal separate pre-existing values for them (contra Michael's Reality Principle), but rather realize dispositions of the system to produce those results in the context of the specific experimental arrangements they are obtained in. In other words, the 'observables' at issue need not all be *be*ables in the hidden-variables interpretations the no-go theorems seek to rule out.

Some commentators place undue emphasis on the fact that, in his criticism of von Neumann's linearity assumption, Bell points out that it is mathematically possible for the measurement results dictated by hidden-variables in individual cases to violate linearity for noncommuting observables even while linearity of their expectation values is preserved after averaging over those

variables. But it is wrong to portray Bell's critique as turning on a mathematical possibility, and it misses the reason why, for Bell, the theorems of von Neumann and Kochen-Specker stand or fall together. For having established the mathematical point beyond doubt using a simple toy hidden-variables model, Bell goes on to remark: 'At first sight the required additivity of expectation values seems very reasonable, and it is rather the non-additivity of allowed values (eigenvalues) which requires explanation' (1991, p. 4). Bell then backs up this remark by giving a positive *physical* explanation for the non-additivity, in terms of measurement results displaying dispositions of the system in different experimental contexts rather than pre-existing values for the 'observables' measured.

Now while arbitrary assumptions about the results of measuring 'observables' are certainly to be avoided, it seems to me that there is no good physical reason (short of reintroducing some form of ontological contextuality) to reject linearity as a requirement on the categorically possessed values of *beables*. Of course, linearity for the values of noncommuting beables with discrete spectra is not going to be easy to satisfy, since the sum of any two eigenvalues for A and B needn't even *be* an eigenvalue for $C = A + B$.[14] But rather than assign an eigenvalue to beable C in such cases that floats freely of the values assigned to beables A and B (yet on averaging linearity of expectation values is miraculously restored), it would be better not to have promoted 'observables' A and B to beable status in the first place![15] *That* is what I take the lesson of Bell's critique of the no-hidden-variables theorems to be.

There is one last requirement to impose on our beable valuations. The value assigned to the square of any beable should equal the square of the value assigned to that beable. It appears that Bohm's (1952) theory contradicts this by predicting that a particle confined to a box in a stationary state with energy $(1/2m)(nh/L)^2$ will possess zero momentum, so that its energy could not possibly be proportional to the square of the value of its

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[14]The standard example is $A = \sigma_x, B = \sigma_y$.

[15]One could avoid this conclusion by only withholding beable status from C—thereby rejecting my assumption that beable sets form real vector spaces. But it would be hard to find principled reasons for that rejection. For example, the kinetic and potential energies of a particle in a potential well don't commute, but if both were assumed to have pre-existing possessed values it would be difficult to comprehend the particle's total energy not having a possessed value too.

momentum.[16] However, Bohm's theory also predicts that if the momentum of the particle is measured, it will be found to have nonzero values $\pm nh/L$ with equal probability. It follows that momentum in the Bohm theory is *not* a beable in Bell's sense, since the probability of *finding* a certain value for momentum is not the same as the probability that the particle *has* that momentum.

So we are requiring our beable value states to be, in the well-known jargon, dispersion-free states. A state ω on a Segalgebra $\mathcal{S}$ is called *dispersion-free on an observable* $A \in \mathcal{S}$ if $\omega(A^2) = (\omega(A))^2$, and a state is called *dispersion-free on* $\mathcal{S}$ if it is dispersion-free on *all* observables of $\mathcal{S}$. It is time now to develop some of the consequences of our assumption that valuations on Segalgebras of beables are given by dispersion-free states.

Obviously every homomorphism of $\mathcal{S}$ into the Segalgebra of real numbers $\mathcal{R}$ is a dispersion-free state on $\mathcal{S}$. (In $\mathcal{R}$ the symmetric product is the usual product, the antisymmetric product of any two real numbers is 0, and the norm of an element is its absolute value.) Conversely, every dispersion-free state on $\mathcal{S}$ is a homomorphism into $\mathcal{R}$. In fact, somewhat more is true, as a consequence of the following theorem.[17]

Theorem 2 *Let ω be a state on a Segalgebra $\mathcal{S}$ that is dispersion-free on some $A \in \mathcal{S}$. Then $\omega(A)$ lies in the spectrum of A and for any $B \in \mathcal{S}$, $\omega(A \circ B) = \omega(A)\omega(B)$ and $\omega(A \bullet B) = 0$.*

Proof. Extend ω to a state τ on the $C^\star$-algebra $\mathcal{S} + i\mathcal{S}$ by defining $\tau(C) = \omega(\Re(C)) + i\omega(\Im(C))$. The map which sends each pair of elements $C, D \in \mathcal{S} + i\mathcal{S}$ to $\tau(C^\star D)$ defines a positive semi-definite inner product $\langle C|D \rangle$ on $\mathcal{S} + i\mathcal{S}$. Therefore, we can derive (in the usual way) the Schwartz inequality

$$|\tau(C^\star D)| \leq \sqrt{\tau(C^\star C)}\sqrt{\tau(D^\star D)}, \text{ for any } C, D \in \mathcal{S} + i\mathcal{S}. \qquad (14)$$

Now since τ is dispersion-free on A, $\tau([A - \tau(A)I]^2) = 0$. Replacing C by $A - \tau(A)I$ in the Schwartz inequality above (and remembering that A is self-adjoint), it follows that $\tau([A - \tau(A)I]D) = 0$ for every $D \in \mathcal{S} + i\mathcal{S}$. Thus $\tau(A) = \omega(A)$ must lie in the spectrum of A; for were $A - \tau(A)I$ invertible

[16]Bohm (1952, Sec. 5).

[17]This result is a modification of standard arguments for $C^\star$-algebras given in Kadison (1975, pp. 105-6).

in $\mathcal{S} + i\mathcal{S}$, we could set D equal to the inverse of $A - \tau(A)I$ and derive the contradiction $\tau(I) = 0$.[18]

Continuing, since $\tau([A - \tau(A)I]D) = 0$ for every $D \in \mathcal{S} + i\mathcal{S}$, it follows that $\tau(AD) = \tau(A)\tau(D)$ for every such D. Using the same argument, we can replace D by $A - \tau(A)I$ and C by $D^\star$ in Eqn. 14 to get $\tau(DA) = \tau(D)\tau(A)$ for every $D \in \mathcal{S} + i\mathcal{S}$. Therefore, since τ agrees with ω on $\mathcal{S}$, for any $B \in \mathcal{S}$ we have

$$
\begin{align}
\omega(A \circ B) &= \tau(1/2[AB + BA]) \tag{15} \\
&= 1/2[\tau(AB) + \tau(BA)] \tag{16} \\
&= 1/2[\tau(A)\tau(B) + \tau(B)\tau(A)] \tag{17} \\
&= 1/2[\omega(A)\omega(B) + \omega(B)\omega(A)] \tag{18} \\
&= \omega(A)\omega(B) \tag{19}
\end{align}
$$

and

$$
\begin{align}
\omega(A \bullet B) &= \tau(i/2[AB - BA]) \tag{20} \\
&= i/2[\tau(AB) - \tau(BA)] \tag{21} \\
&= i/2[\tau(A)\tau(B) - \tau(B)\tau(A)] = 0. \tag{22}
\end{align}
$$

QED.

Thm. 2 is a mixed blessing. On the one hand, we only required from the outset that value states assign nonnegative values to beables with nonnegative spectra, so it is reassuring to now see that the value assigned to *any* beable must lie *within* its spectrum. On the other hand, the theorem shows that asking for a Segalgebra of noncommuting beables with decently behaved valuations is a tall order. For suppose $[A, B]_- \neq 0$, so that $A \bullet B \neq 0$. Then since the spectrum of $A \bullet B$ must include $\pm|A \bullet B|$,[19] $A \bullet B$ must have at least one nonzero spectral value; yet Thm. 2 tells us that antisymmetric products of beables are always mapped to 0 by dispersion-free states!

It is exactly this sort of observation that leads Misra (1967, pp. 856-7) to conclude that hidden-variables in algebraic quantum mechanics are impossible. Working in the more general context of Segal's algebras, Misra

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[18]Since invertibility in $\mathcal{S} + i\mathcal{S}$ is equivalent to invertibility in any $C^\star$-algebra with $\mathcal{S} + i\mathcal{S}$ as a subalgebra (Bratteli and Robinson 1987, Prop. 2.2.7), this argument does not assume that $\mathcal{S}$ is the entire self-adjoint part of the $C^\star$-algebra describing some system.

[19]Cf. Geroch (1985, Thm. 60).

introduces the idea of a derivation on the algebra which is a linear mapping **D** from the algebra onto itself that satisfies the Leibniz rule with respect to the symmetric product:

$$\mathbf{D}(A \circ B) = A \circ \mathbf{D}(B) + \mathbf{D}(A) \circ B. \tag{23}$$

Misra does not assume the Segal algebras he works with form the self-adjoint parts of $C^\star$-algebras—that they are *Segalgebras* in my sense—though he *does* need to assume that every observable in the algebra is the difference of two positive observables in the algebra.[20] With that assumption, Misra (1967, Thm. 4) shows dispersion-free states map derivations to zero: for any A in the algebra, any derivation **D**, and any dispersion-free state $\langle \cdot \rangle$, $\langle \mathbf{D}(A) \rangle = 0$. Since the antisymmetric product of any observable in a Segalgebra $\mathcal{S}$ with some given observable defines a derivation on $\mathcal{S}$, Misra's result entails Thm. 2's result that dispersion-free states on Segalgebras map antisymmetric products to zero. But, as we shall see explicitly in the final section of this paper, Misra's conclusion that the result 'excludes hidden variables in the general algebraic setting of quantum mechanics' (p. 857) is based upon a failure to distinguish 'observables' from beables. Indeed, we shall shortly see that noncommuting Segalgebras of beables have *not* been excluded.

4   Quasicommutative Segalgebras

Clearly a Segalgebra $\mathcal{S}$ is *commutative* if its antisymmetric product is trivial, i.e. $\mathcal{S} \bullet \mathcal{S} = \{0\}$. In this section I introduce the idea of a quasicommutative Segalgebra which we will see in the next section captures the precise extent to which beables can fail to commute. To adequately motivate and characterize 'quasicommutativity' from a formal point of view, I'll first need an alternative characterization of commutativity and then I'll need to discuss quotient Segalgebras.

There is a famous representation theorem for commutative $C^\star$-algebras from which an analogous result for commutative Segalgebras can be extracted. Recall that an archetypal example of a $C^\star$-algebra is the algebra $\mathcal{C}(X)$ of all complex-valued continuous functions on a compact Hausdorff space X (e.g. the interval $[0, 1]$). In $\mathcal{C}(X)$ linear combinations and products

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[20]This is true for $C^\star$-algebras, and therefore Segalgebras—see Prop. 2.2.11 of Bratteli and Robinson (1987).

of functions are defined in the obvious (pointwise) way, the adjoint of a function is its complex conjugate, and the norm of a function is the maximum absolute value it takes over X.[21] The point about commutative $C^\star$-algebras is that they all arise in this way. *Every* commutative $C^\star$-algebra (with identity) is *-isomorphic to $\mathcal{C}(X)$ for some compact Hausdorff space X.[22] Since a *-isomorphism of $C^\star$-algebras induces an isomorphism between their Segalgebras, it follows that every commutative Segalgebra $\mathcal{S}$ is isomorphic to the Segalgebra of all *real*-valued continuous functions $\mathcal{C}(X)$ on some compact Hausdorff space X.[23]

Now call a set of states Ω on a Segalgebra $\mathcal{S}$ *full* if Ω 'separates the points of $\mathcal{S}$' in the sense that for any two distinct elements of $\mathcal{S}$ there is a state in Ω mapping them to different expectation values—or, equivalently (by the linearity of states), if for any nonzero $A \in \mathcal{S}$ there is a state $\omega \in \Omega$ such that $\omega(A) \neq 0$. Then the alternative characterization of commutativity I need is the following.[24]

Theorem 3 *A Segalgebra is commutative if and only if it has a full set of dispersion-free states.*

Proof. 'Only if'. If $\mathcal{S}$ is commutative it is isomorphic to the set of real-valued functions $\mathcal{C}(X)$ on some compact Hausdorff space X. Suppose $A \neq 0$. Then $A(x)$, the isomorphic image of A in $\mathcal{C}(X)$, cannot be the zero function (since the isomorphism can only map the zero operator to that function). So there is at least one point $x_0 \in X$ such that $A(x_0) \neq 0$. It is easy to see that the map defined by $\langle B \rangle = B(x_0)$ for all $B \in \mathcal{S}$ is a dispersion-free state on $\mathcal{S}$ satisfying $\langle A \rangle \neq 0$.

'If'. Consider any pair $A, B \in \mathcal{S}$ and their antisymmetric product $A \bullet B$. If $A \bullet B \neq 0$, then by hypothesis there is a dispersion-free state $\langle \cdot \rangle$ on $\mathcal{S}$ such that $\langle A \bullet B \rangle \neq 0$. But this contradicts Thm. 2, therefore $A \bullet B = 0$. *QED.*

In light of this result, the natural way to define quasicommutivity of a Segalgebra is in terms of it admitting a 'nearly full' set of dispersion-free states that only separate the points of $\mathcal{S}$ modulo some ideal in the algebra. It will turn

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[21]$\mathcal{C}(X)$ is closed since the uniform limit of a sequence of (necessarily bounded) continuous functions on a compact X is itself a continuous function on X—see Simmons (1963, pp. 80-85).

[22]Bratteli and Robinson (1987, Prop. 2.1.11A).

[23]See also Segal (1947, Thm. 1).

[24]This result is just a variation on Segal's (1947, Thm. 3).

out that factoring out the ideal yields a quotient Segalgebra that is (fully) commutative—which is what one might expect if the original (unfactored) Segalgebra were already 'close to being commutative'.

I'll introduce the idea of ideals and quotients for Segalgebras by again recalling their $C^\star$-algebra counterparts first. A (two-sided) ideal in $\mathcal{U}$ is a subspace $\mathcal{I}$ of $\mathcal{U}$ which is invariant under multiplication on the left or right by any $A \in \mathcal{U}$, i.e. $\mathcal{U}\mathcal{I} \subseteq \mathcal{I}$ and $\mathcal{I}\mathcal{U} \subseteq \mathcal{I}$. We shall be interested only in closed *-ideals, i.e. ideals closed in norm and under the taking of adjoints. (An example is the collection of all functions in $\mathcal{C}(X)$ which vanish at some point of X.) Clearly such an ideal is a subalgebra of $\mathcal{U}$. The idea is to 'factor out' this subalgebra so that what remains is again a $C^\star$-algebra. So as not to be left with something completely trivial, we will also require that $\mathcal{I}$ be *proper*, i.e. that it be a proper subset of $\mathcal{S}$, which is equivalent to requiring that $\mathcal{I}$ not contain the identity.

Each proper closed *-ideal $\mathcal{I}$ in $\mathcal{U}$ determines an equivalence relation $\cong_\mathcal{I}$ on $\mathcal{U}$ defined by

$$A \cong_\mathcal{I} B \text{ if and only if } A - B \in \mathcal{I}. \tag{24}$$

The equivalence classes of $\cong_\mathcal{I}$ form a $C^\star$-algebra called the *quotient $C^\star$-algebra $\mathcal{U}/\mathcal{I}$ by the ideal $\mathcal{I}$*. To see how, let $\hat{A}$ denote the equivalence class in which A lies, and similarly for $\hat{B}$, $\hat{C}$, etc. Now define the relevant operations in $\mathcal{U}/\mathcal{I}$ by

$$c\hat{A} = \widehat{cA}, \ \hat{A} + \hat{B} = \widehat{A + B}, \ \hat{A}\hat{B} = \widehat{AB}, \ \hat{A}^\star = \widehat{A^\star}. \tag{25}$$

noting that $\hat{I}$ is the identity in $\mathcal{U}/\mathcal{I}$ and $\hat{0}$ the zero element. (The factoring has been implemented by 'collapsing' everything in $\mathcal{I}$ into $\hat{0} \in \mathcal{U}/\mathcal{I}$.) Because $\mathcal{I}$ is a *-ideal, the definitions in Eqn. 25 are well-defined independent of the representatives chosen for the equivalence classes that appear in them. (For example, $\hat{A}\hat{B} = \widehat{AB}$ is well-defined only if $A \cong_\mathcal{I} A'$ and $B \cong_\mathcal{I} B'$ imply $AB \cong_\mathcal{I} A'B'$. But assuming the former, $(A - A')(B + B') \in \mathcal{I}$ and $(A + A')(B - B') \in \mathcal{I}$, therefore

$$\frac{1}{2}[(A - A')(B + B') + (A + A')(B - B')] = AB - A'B' \in \mathcal{I}.) \tag{26}$$

If we define

$$|\hat{A}| = \inf_{B \cong_I A} |B|, \tag{27}$$

18

an elementary argument in the theory of Banach algebras establishes that $\mathcal{U}/\mathcal{I}$ is a complete normed algebra, and a not so elementary argument establishes the $C^{\star}$-norm property.[25]

The analogue of all this for Segalgebras may now be obvious. A proper closed ideal in $\mathcal{S}$ is a closed subspace $\mathcal{I}$ of $\mathcal{S}$ not containing the identity which is invariant under symmetric and antisymmetric multiplication, i.e. $\mathcal{S} \circ \mathcal{I} \subseteq \mathcal{I}$ and $\mathcal{S} \bullet \mathcal{I} \subseteq \mathcal{I}$. (Clearly we do not need to distinguish left from right multiplication.) By an argument virtually identical to the 'Only if' part of Thm. 1's proof, $\mathcal{I}$ extends to a proper closed *-ideal $\mathcal{I} + i\mathcal{I}$ in $\mathcal{S} + i\mathcal{S}$. So we can take the *quotient Segalgebra* $\mathcal{S}/\mathcal{I}$ *by the ideal* $\mathcal{I}$ to be the Segalgebra part of $(\mathcal{S}+i\mathcal{S})/(\mathcal{I}+i\mathcal{I})$. It is easy to see that for $A \in \mathcal{S}$, $\hat{A} \in \mathcal{S}/\mathcal{I}$ and that $A, B \in \mathcal{S}$ lie in the same equivalence class of $\mathcal{S}/\mathcal{I}$, i.e. $\hat{A} = \hat{B}$, if and only if $A \cong_{\mathcal{I}} B$. Furthermore, using Eqns. 25 it is easy to verify that the map which sends $A \in \mathcal{S}$ to $\hat{A} \in \mathcal{S}/\mathcal{I}$ is a homomorphism, and this homomorphism is surjective since $\hat{A} \in \mathcal{S}/\mathcal{I}$ is the image of $\Re(A) \in \mathcal{S}$.

We have now assembled all the necessary machinery to fulfill my promise to introduce a reasonable notion of quasicommutativity.

Theorem 4 *Let $\mathcal{S}$ be a Segalgebra with proper closed ideal $\mathcal{I}$. Then the following are equivalent:*

1. *$\mathcal{S}/\mathcal{I}$ is commutative.*

2. *For any $A \notin \mathcal{I}$ there is a dispersion-free state $\langle \cdot \rangle$ on $\mathcal{S}$ such that $\langle A \rangle \neq 0$.*

3. *$\mathcal{S} \bullet \mathcal{S} \subseteq \mathcal{I}$.*

Proof. 1. $\Rightarrow$ 2. If $A \notin \mathcal{I}$, then $\hat{A} \neq \hat{0}$. Since $\mathcal{S}/\mathcal{I}$ is commutative (by hypothesis), it has a full set of dispersion-free states (Thm. 3). So there is a dispersion-free state $\{\cdot\}$ on $\mathcal{S}/\mathcal{I}$ such that $\{\hat{A}\} \neq 0$. Defining $\langle B \rangle = \{\hat{B}\}$ for all $B \in \mathcal{S}$, the map $\langle \cdot \rangle$—being the composition two homomorphisms, the second into $\mathcal{R}$—is a dispersion-free state on $\mathcal{S}$ satisfying $\langle A \rangle \neq 0$.

2. $\Rightarrow$ 3. Identical to the 'If' part of Thm. 3 with $\mathcal{I}$ now playing the role of $\{0\}$.

3. $\Rightarrow$ 1. Let $\hat{A}, \hat{B} \in \mathcal{S}/\mathcal{I}$. Since $\mathcal{S} \bullet \mathcal{S} \subseteq \mathcal{I}$ (by hypothesis) and $\hat{\mathcal{I}} = \hat{0}$, we have $\hat{A} \bullet \hat{B} = \widehat{A \bullet B} = \hat{0}$, whence $\mathcal{S}/\mathcal{I}$ is commutative. *QED.*

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[25]See Simmons (1963, Thm. 69D) and Bratteli and Robinson (1987, Prop. 2.2.19), respectively.

In the case $\mathcal{I} = \{0\}$, Parts 1. and 3. of Thm. 4 just assert that $\mathcal{S}$ itself is commutative and Part 2. that $\mathcal{S}$ has a full set of dispersion-free states. So Thm. 4 generalizes Thm. 3 by relaxing its requirement of full commutativity. Motivated by this, call a Segalgebra $\mathcal{I}$-*quasicommutative* whenever it satisfies the equivalent conditions of Thm. 4.[26] Notice from 3. of Thm. 4 that if $\mathcal{S}$ is $\mathcal{I}$-quasicommutative and $\mathcal{J}$ is another proper ideal in $\mathcal{S}$ containing $\mathcal{I}$, then $\mathcal{S}$ is $\mathcal{J}$-quasicommutative as well. In particular, if $\mathcal{S}$ is commutative, it is automatically $\mathcal{I}$-quasicommutative with respect to any proper ideal $\mathcal{I} \in \mathcal{S}$. Since the converse fails, quasicommutativity is genuinely weaker than commutativity.

5   Beable Subalgebras

It is high time I spelled out the connection between quasicommutativity and the problem of *beables* in quantum mechanics. For a Segalgebra of beables to satisfy the statistics prescribed by some quantum state, we must be able to interpret the state's expectation values as averages over the actual values of the beables in the algebra. In other words, the quantum state must be a mixture of dispersion-free states on the algebra.

Let x be a variable in a measure space X, μ a positive measure on X such that $\mu(X) = 1$, and ω_x $(x \in X)$ a collection of states on a Segalgebra $\mathcal{S}$. Then the mapping defined by

$$\omega(A) = \int_X \omega_x(A)\mathrm{d}\mu(x), \text{ for any } A \in \mathcal{S}, \tag{28}$$

will also be a state on $\mathcal{S}$. A state ω is called *mixed* if it can be represented, in the above way, as a weighted average of two or more (distinct) states with respect to some positive normalized measure; if it cannot, then ω is called *pure*. A subalgebra $\mathcal{B}$ of $\mathcal{S}$ will be said to *have beable status for the state* ω if $\omega|_\mathcal{B}$—the restriction of ω to $\mathcal{B}$—is either a mixture of dispersion-free states on $\mathcal{B}$ or itself dispersion-free. As a check on the adequacy of this definition, we get the following intuitively expected result.[27]

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[26]This idea deliberately parallels the idea of a 'quasidistributive lattice' introduced in Bell and Clifton (1995, Thm. 1). I should also note that results similar to 1. $\Leftrightarrow$ 2. of Thm. 4 have been proven by Misra (1967, Thm. 1) and Plymen (1968, Thm. 4.2) in the context of $C^\star$- and von Neumann algebras, respectively.

[27]Here, I follow Segal's (1947, p. 933) argument almost to the letter.

Theorem 5 *Every commutative subalgebra of $\mathcal{S}$ has beable status for every state on $\mathcal{S}$.*

Proof. Let $\mathcal{C}$ be a commutative subalgebra of $\mathcal{S}$ and ω any of $\mathcal{S}$'s states. Since $\mathcal{C}$ is commutative, it is isomorphic to the set of real-valued functions $\mathcal{C}(X)$ on some compact Hausdorff space X. Defining $\phi(A(x)) = \omega(A)$ for every $A \in \mathcal{C}$, ϕ is a state on $\mathcal{C}(X)$. By the Riesz-Markov representation theorem,[28] ϕ must take the form

$$\phi(A(x)) = \int_X A(x) \mathrm{d}\mu_\phi(x), \text{ for any } A(x) \in \mathcal{C}(X), \tag{29}$$

where μ_ϕ is some positive normalized (completely additive) measure on X. But since for any $x \in X$ the map $\langle A \rangle_x = A(x)$ defines a dispersion-free state on $\mathcal{C}$, Eqn. 29 exhibits $\omega|_{\mathcal{C}}$ as a mixture of dispersion-free states on $\mathcal{C}$ (if $\omega|_{\mathcal{C}}$ is not already one of those states itself, which would correspond to the case where the complement of some point in X has μ_ϕ-measure zero). *QED.*

At this point, it is instructive to look at Bohm's (1952) theory which supplies a concrete example of a commutative subalgebra with beable status for every state. For simplicity, consider the space of states of a single spinless particle in one-dimension given by the Hilbert space $L_2(\mathcal{R})$ of all (measurable) square-integrable, complex-valued functions on $\mathcal{R}$. The position $\hat{x}$ of the particle and all self-adjoint functions thereof are the only true beables in Bohm's theory. Of course, Segalgebras cannot contain unbounded observables. But any assignment of a value to some unbounded self-adjoint operator which is a function of position, such as $\hat{x}$ itself, is equivalent to assigning a corresponding set of values to self-adjoint operators which *are* bounded functions of $\hat{x}$, such as its spectral projections. Thus, we lose no generality by characterizing Bohm's theory as granting beable status to all *bounded* self-adjoint operator-valued functions of $\hat{x}$.

Such functions form a commutative Segalgebra with beable status for every wavefunction in $L_2(\mathcal{R})$. To see this, let f be any (measurable) essentially bounded, complex-valued function on $\mathcal{R}$ and define the bounded self-adjoint operator $\hat{O}_f$ by

$$\hat{O}_f(\psi(x)) = f(x)\psi(x) \text{ for each } \psi(x) \in L_2(\mathcal{R}). \tag{30}$$

---

[28]See Rudin (1974, Thm. 2.14).

The set of all such operators is obviously commutative, and it is well-known that they form a $C^\star$-subalgebra of the $C^\star$-algebra of all bounded operators on $L_2(\mathcal{R})$.[29] If f is some bounded function of x, $\hat{O}_f$ is the corresponding operator-valued function of $\hat{x}$. In fact, the operators $\{\hat{O}_f\}$ capture all the bounded operators which are functions of $\hat{x}$, since any such function would have to commute with all the $\hat{O}_f$'s, and it is known that they form a maximal commutative set of bounded operators on $L_2(\mathcal{R})$.[30] The subset of $\{\hat{O}_f\}$ where the f's are real-valued functions (almost everywhere) is therefore the commutative sub-Segalgebra $\mathcal{S}_{\hat{x}}$ of all bounded observables that are functions of $\hat{x}$. (In particular, the spectral projections of $\hat{x}$ correspond to characteristic functions whose characteristic sets in $\mathcal{R}$ have nonzero measure.)

Now for every $\hat{O}_f \in \mathcal{S}_{\hat{x}}$, define $\delta_r(\hat{O}_f) = f(r)$—so δ_r is the Dirac delta distribution at the point r. Clearly δ_r qualifies as a state on $\mathcal{S}_{\hat{x}}$ (note how liberal the algebraic definition of a state is!) and is dispersion-free. Furthermore, for any $\psi(x) \in L_2(\mathcal{R})$ and any $\hat{O}_f \in \mathcal{S}_{\hat{x}}$ we have

$$\int_{\mathcal{R}} \psi^*(x)\hat{O}_f\psi(x)dx = \int_{\mathcal{R}} f(x)|\psi(x)|^2 dx = \int_{\mathcal{R}} \delta_x(\hat{O}_f)d\rho_\psi(x), \qquad (31)$$

so that every state of $L_2(\mathcal{R})$ is indeed a mixture of dispersion-free (Dirac) states on $\mathcal{S}_{\hat{x}}$.

To show that it is not necessary that subalgebras with beable status for some state be commutative, I need the following result about the ideals determined by states on Segalgebras.[31]

Theorem 6 *If ω is a state on a Segalgebra $\mathcal{S}$, the set*

$$\mathcal{I}_\omega = \{A \in \mathcal{S}|\omega(A^2) = 0\} \qquad (32)$$

is a proper closed ideal in $\mathcal{S}$ on which ω is dispersion-free.

Proof. As in the proof of Thm. 2, extend ω to a state τ on the $C^\star$-algebra $\mathcal{S} + i\mathcal{S}$. Fix an arbitrary $A \in \mathcal{I}_\omega$, so $\omega(A^2) = 0$. Replacing C by A in the Schwartz inequality (Eqn. 14) yields $\tau(AD) = 0$ for all $D \in \mathcal{S} + i\mathcal{S}$. In particular, with D replaced by I we see that $\tau(A) = \omega(A) = 0$, so that ω is dispersion-free on the set $\mathcal{I}_\omega$.

[29]See Geroch (1985, Ch. 49) and Kadison and Ringrose (1983, Sec. 4.1).

[30]Kadison and Ringrose (1983, p. 308).

[31]Once again, this result just adapts standard $C^\star$-algebraic arguments to the present Segalgebraic context—see Kadison (1975, pp. 103-4).

Now let B be any element of $\mathcal{S}$. We need to show that both $A \circ B$ and $A \bullet B$ lie in $\mathcal{I}_\omega$. With D replaced by B^2A in the argument above, $\tau(AB^2A) = 0$ and therefore $\omega(\Re(AB^2A)) = 0$. Using Eqns. 9 and the properties of the symmetric and antisymmetric product,

$$
\begin{aligned}
\Re([AB][BA]) &= [AB] \circ [BA] & (33) \\
&= [A \circ B - i(A \bullet B)] \circ [B \circ A - i(B \bullet A)] & (34) \\
&= [A \circ B - i(A \bullet B)] \circ [A \circ B + i(A \bullet B)] & (35) \\
&= (A \circ B)^2 + (A \bullet B)^2. & (36)
\end{aligned}
$$

Therefore since ω acts positively on squares, $\omega((A \circ B)^2) = \omega((A \bullet B)^2) = 0$, as required.

Finally, note that $\mathcal{I}_\omega$ *is* a real closed linear subspace of $\mathcal{S}$ not containing the identity. For example, to get closure under vector sums, assuming $A, B \in \mathcal{I}_\omega$ implies

$$
\begin{aligned}
\omega([A+B]^2) &= \omega([A+B] \circ [A+B]) & (37) \\
&= \omega(A^2) + \omega(B^2) + 2\omega(A \circ B) & (38) \\
&= 2\omega(A)\omega(B) = 0 & (39)
\end{aligned}
$$

using the fact that ω is dispersion-free on A and B and Thm. 2. *QED.*

If one wants to include some pair of noncommuting observables A and B in a subalgebra with beable status for some state ω, the 'trick' is simply to make sure $A \bullet B$ lies inside the ideal $\mathcal{I}_\omega$ determined by that state. If so, then it won't matter that dispersion-free states map antisymmetric products of noncommuting observables to zero, because $A \bullet B$ will *also* get assigned value zero in the state ω! In short, to have beable status in some state it is enough for a subalgebra to be quasicommutative with respect to the ideal determined by the state. It is also necessary, as shown by the following theorem.

Theorem 7 *Let $\mathcal{B}$ be a subalgebra of a Segalgebra $\mathcal{S}$ and ω a state on $\mathcal{S}$. Then $\mathcal{B}$ has beable status for ω if and only if $\mathcal{B}$ is $\mathcal{I}_{\omega|_\mathcal{B}}$-quasicommutative.*

Proof. 'Only if'. $\mathcal{B}$'s $\mathcal{I}_{\omega|_\mathcal{B}}$-quasicommutativity is easily inferred from 2. of Thm. 4. Thus if $A \in \mathcal{B}$ with $A \notin \mathcal{I}_{\omega|_\mathcal{B}}$, then $\omega|_\mathcal{B}(A^2) \neq 0$. But by hypothesis, there is a collection of dispersion-free states $\{\langle \cdot \rangle_x | x \in X\}$ on $\mathcal{B}$ of which $\omega|_\mathcal{B}$

is a mixture. Therefore, for at least one $x_0 \in X$, $\langle A^2 \rangle_{x_0} \neq 0$ and $\langle A \rangle_{x_0} \neq 0$ as required.

'If'. By 1. of Thm. 4, $\mathcal{B}/\mathcal{I}_{\omega|_{\mathcal{B}}}$ is commutative. Define the map

$$\phi : \mathcal{B}/\mathcal{I}_{\omega|_{\mathcal{B}}} \to \mathcal{R} \text{ by } \phi(\hat{A}) = \omega|_{\mathcal{B}}(A). \tag{40}$$

Since the 'hat' map is surjective, this defines ϕ on all of $\mathcal{B}/\mathcal{I}_{\omega|_{\mathcal{B}}}$, and it is easy to check that ϕ is indeed *well*-defined. Furthermore, since the hat map is a homomorphism, ϕ is a state on $\mathcal{B}/\mathcal{I}_{\omega|_{\mathcal{B}}}$. By the commutativity of $\mathcal{B}/\mathcal{I}_{\omega|_{\mathcal{B}}}$ and Thm. 5, ϕ is a mixture of dispersion-free states $\langle \cdot \rangle_x$ ($x \in X$) on $\mathcal{B}/\mathcal{I}_{\omega|_{\mathcal{B}}}$. But for any $\langle \cdot \rangle_x$ on $\mathcal{B}/\mathcal{I}_{\omega|_{\mathcal{B}}}$, $\langle \hat{\cdot} \rangle_x$ is a dispersion-free state on $\mathcal{B}$. So since $\omega|_{\mathcal{B}}(\cdot) = \phi(\hat{\cdot})$ (Eqn. 40), $\omega|_{\mathcal{B}}$ is a mixture of dispersion-free states $\langle \hat{\cdot} \rangle_x$ ($x \in X$) on $\mathcal{B}$. *QED*.

The examples of noncommutative subalgebras of beables that I shall consider make use of the following result.

Theorem 8 *For any state ω on a Segalgebra $\mathcal{S}$, the definite set of ω defined by*

$$\mathcal{D}_\omega = \{ A \in \mathcal{S} | \omega(A^2) = (\omega(A))^2 \} \tag{41}$$

has beable status for ω.

Proof. To see that $\mathcal{D}_\omega$ is a subalgebra of $\mathcal{S}$, it is easiest to use the fact that $A \in \mathcal{D}_\omega$ if and only if $A - \omega(A)I \in \mathcal{I}_\omega$ and invoke the properties of closed ideals. To illustrate, let $A, B \in \mathcal{D}_\omega$. Then $(A - \omega(A)I) \circ (B + \omega(B)I) \in \mathcal{I}_\omega$ and $(A + \omega(A)I) \circ (B - \omega(B)I) \in \mathcal{I}_\omega$, thus

$$\frac{1}{2}[(A - \omega(A)I) \circ (B + \omega(B)I) + (A + \omega(A)I) \circ (B - \omega(B)I)] \tag{42}$$

$$= A \circ B - \omega(A)\omega(B)I \in \mathcal{I}_\omega. \tag{43}$$

But since $\omega(A)\omega(B) = \omega(A \circ B)$ by Thm. 2, this means $A \circ B \in \mathcal{D}_\omega$.

The beable status of $\mathcal{D}_\omega$ for ω follows trivially from 2. of Thm. 4 and Thm. 7. Thus if A lies in $\mathcal{D}_\omega$ but not in $\mathcal{I}_\omega$, then of course there *is* a dispersion-free state on $\mathcal{D}_\omega$ mapping A to a nonzero value—ω itself! *QED*.

In the case where ω is represented by a state vector $|v\rangle$ in a Hilbert space representation of the Segalgebra $\mathcal{S}$ (so $\omega(A) = \langle v|A|v \rangle$ for all $A \in \mathcal{S}$), ω's definite set consists of all those (bounded) self-adjoint operators on the Hilbert

space that share the eigenstate $|v\rangle$—a highly noncommutative set if the space has more than two dimensions. This is the orthodox (Dirac-von Neumann) 'eigenstate-eigenvalue link' approach to assigning definite values to observables.

Definite sets can be used to build subalgebras with beable status for a state ω that are *not* just subalgebras of ω's own definite set. The next result specifies a general class of examples of this sort, containing Thm. 8 as a degenerate case (when $\omega = \omega_x$ for all $x \in X$).

Theorem 9 *Let $\mathcal{S}$ be a Segalgebra, ω a state on $\mathcal{S}$ and ω_x ($x \in X$) any family of states satisfying $\bigcap_{x \in X} \mathcal{I}_{\omega_x} \subseteq \mathcal{I}_\omega$. Then $\mathcal{B}_{\{\omega_x\}} = \bigcap_{x \in X} \mathcal{D}_{\omega_x}$ has beable status for ω.*

Proof. Since $\mathcal{B}_{\{\omega_x\}}$ is the intersection of a collection of subalgebras of $\mathcal{S}$ (Thm. 8), it is itself a subalgebra. To establish beable status for ω, all we need to show (by 3. of Thm. 4 and Thm. 7) is that $\mathcal{B}_{\{\omega_x\}} \bullet \mathcal{B}_{\{\omega_x\}} \subseteq \mathcal{I}_{\omega mega}$. So suppose $A, B \in \mathcal{B}_{\{\omega_x\}}$. Then both A and B lie in the definite sets of all the ω_x's. This is equivalent to both $A - \omega_x(A)I$ and $B - \omega_x(B)I$ lying in $\mathcal{I}_{\omega_x}$ for all $x \in X$, which implies

$$(A - \omega_x(A)I) \bullet (B - \omega_x(B)I) = A \bullet B \in \mathcal{I}_{\omega_x} \tag{44}$$

for all $x \in X$. But by hypothesis, $\bigcap_{x \in X} \mathcal{I}_{\omega_x} \subseteq \mathcal{I}_\omega$, therefore $A \bullet B \subseteq \mathcal{I}_\omega$. QED.

To get a concrete example of Thm. 9, consider again the case where ω is represented by a state vector $|v\rangle$ in a Hilbert space representation H. Let R be any bounded self-adjoint operator on H with discrete spectrum and eigenprojections $\{R_i\}$ (counting only those for which $\omega(R_i) > 0$), and consider the (renormalized) orthogonal projections $\{|v_{R_i}\rangle\}$ of the state vector $|v\rangle$ onto the eigenspaces of R. Since any observable with eigenvalue 0 in all the states $\{|v_{R_i}\rangle\}$ must have eigenvalue 0 in the state $|v\rangle$ (the latter lying in the span of the former), the conditions of Thm. 9 are satisfied so that $\mathcal{B}_{\{|v_{R_i}\rangle\}}$ has beable status in the state $|v\rangle$. This is the modal (i.e. state-dependent) method adopted by Bub (1997)[32] for building a set of beables out of the state of a system and a 'preferred observable' R.[33] It is easy to see that a

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[32] See also Bub and Clifton (1996).
[33] Bub's method turns out to include the Kochen-Dieks modal interpretation (discussed in Clifton 1995) as a special case—see Bub 1997, Sec. 6.3 for the argument.

'Bub-definite' subalgebra $\mathcal{B}_{\{|v_{R_i}\rangle\}}$ will not be a subalgebra of $\mathcal{D}_{|v\rangle}$ unless $|v\rangle$ is an eigenstate of R; in fact, the Bub-definite subalgebra in that case *coincides* with the definite set of $|v\rangle$. Also Bub-definite subalgebras will not generally be commutative: the main exceptions are when H is two-dimensional, and when R is nondegenerate with all its eigenvalues given nonzero probability by $|v\rangle$.

6 Maximal Beable Subalgebras

Bub-definite subalgebras have the extra feature that their projections form a maximal determinate sublattice of the ortholattice of projections on H. This means that any enlargement of the projection lattice of a Bub-definite subalgebra $\mathcal{B}_{\{|v_{R_i}\rangle\}}$ generates an ortholattice from which it is impossible to recover the probabilities prescribed by $|v\rangle$ as a measure over the two-valued homomorphisms on the enlarged lattice.[34] Since I have stopped short of making any a priori assumptions about the lattice structure of the projections in Segalgebras, it is natural to ask whether Bub-definite subalgebras are still maximal *as Segalgebras*. Indeed, the general question of when (if ever) Segalgebras of beables are 'as big as one can possibly get' for a given state is philosophically interesting in its own right, since an answer would seem to set a limit on how far a simple realism of possessed values in quantum mechanics can be pushed.

Call a subalgebra $\mathcal{B}$ of $\mathcal{S}$ with beable status for a state ω a *maximal beable subalgebra for ω* if it is not properly contained in any other subalgebra of $\mathcal{S}$ with beable status for ω. An easy application of Zorn's lemma shows that a maximal beable subalgebra for any given state always exists. The following result gives an explicit (but not completely general) characterization of maximal beable subalgebras that covers the case of Bub-definite subalgebras.

Theorem 10 *Let $\mathcal{S}(H)$ be the Segalgebra of all bounded self-adjoint operators on a Hilbert space H and let $|v\rangle$ be any state vector in H. Then if $|v_x\rangle$ $(x \in X)$ is any family of vectors in H satisfying:*

1. *the members of $\{|v_x\rangle\}$ are mutually orthonormal,*

2. *$|v\rangle$ is in the closed span of $\{|v_x\rangle\}$, and*

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[34]Bub (1997, Sec. 4.3).

3. $|v\rangle$ is not orthogonal to any member of $\{|v_x\rangle\}$,

$\mathcal{B}_{\{|v_x\rangle\}}$ *is a maximal beable subalgebra of $\mathcal{S}(H)$ for the state $|v\rangle$. If H is finite-dimensional, the converse also holds, i.e. if $\mathcal{B} \subseteq \mathcal{S}(H_n)$ is a maximal beable subalgebra for $|v\rangle \in H_n$, then there exists a family of vectors $|v_x\rangle$ ($x \in X$) in H_n satisfying 1.–3. such that $\mathcal{B} = \mathcal{B}_{\{|v_x\rangle\}}$ (and this family is unique up to phases).*

Proof. For the first claim, assuming 2. the beable status of $\mathcal{B}_{\{|v_x\rangle\}}$ for $|v\rangle$ follows exactly as discussed for Bub-definite subalgebras at the end of the last section. So all that remains is to prove maximality. Note first that because the members of $\{|v_x\rangle\}$ are mutually orthogonal by 1., each is an eigenvector (with eigenvalue 0 or 1) of all the one-dimensional projection operators $P_{|v_x\rangle}$ ($x \in X$). Consequently, the set $\{P_{|v_x\rangle}\}$ is contained in $\mathcal{B}_{\{|v_x\rangle\}}$. Now consider the subalgebra $\mathcal{T}$ generated by $\mathcal{B}_{\{|v_x\rangle\}}$ and any $A \notin \mathcal{B}_{\{|v_x\rangle\}}$. Our task is to show that $\mathcal{T}$ cannot have beable status for $|v\rangle$—so suppose (for reductio ad absurdem) that $\mathcal{T}$ does. Then for any $x \in X$, $A \bullet P_{|v_x\rangle} \in \mathcal{T}$ and so beable status for $|v\rangle$ requires $A \bullet P_{|v_x\rangle} \in \mathcal{I}_{|v\rangle}$, i.e. $A \bullet P_{|v_x\rangle}|v\rangle = |0\rangle$ (Thms. 4, 7). Since $P_{|v_x\rangle}|v\rangle = c|v_x\rangle$ (with $c \neq 0$, by 3.), using the definition of the antisymmetric product (Eqn. 6) yields

$$cA|v_x\rangle = P_{|v_x\rangle}(A|v\rangle) \ (= c'|v_x\rangle, \text{ for some } c') \tag{45}$$

which shows that A has $|v_x\rangle$ as an eigenvector. Since this is true for any $x \in X$, A lies in the definite sets of all the states $\{|v_x\rangle\}$, and therefore $A \in \mathcal{B}_{\{|v_x\rangle\}}$ contrary to hypothesis.

For the converse claim, suppose $\mathcal{B} \subseteq \mathcal{S}(H_n)$ is a maximal beable subalgebra for the state $|v\rangle \in H_n$. Consider the subspace of H_n given by

$$S = \{|w\rangle \in H_n : \ (A \bullet B)|w\rangle = |0\rangle \text{ for all } A, B \in \mathcal{B}\} \tag{46}$$

which is nontrivial since $\mathcal{B}$'s beable status for $|v\rangle$ requires that $|v\rangle \in S$. We first show that S is invariant under $\mathcal{B}$, i.e. $|w\rangle \in S$ implies $C|w\rangle \in S$ for any $C \in \mathcal{B}$. (In fact, for this part of the argument the dimension of the Hilbert space needn't be finite.) To establish this, we need to show that if $|w\rangle \in S$, then $(A \bullet B)(C|w\rangle) = |0\rangle$ for any $A, B \in \mathcal{B}$. Using the fact that the $C^\star$ product of two operators X and Y is expressible as $XY = X \circ Y - iX \bullet Y$ (see the remarks following Eqn. 9), together with the

supposition that antisymmetric products formed in $\mathcal{B}$ map $|w\rangle$ to zero and the definition of the symmetric product (Eqn. 3), one calculates

$$
\begin{align}
(A \bullet B)C|w\rangle &= ((A \bullet B) \circ C)|w\rangle - i((A \bullet B) \bullet C)|w\rangle \tag{47} \\
&= ((A \bullet B) \circ C)|w\rangle \tag{48} \\
&= 1/2(A \bullet B)C|w\rangle + 1/2C(A \bullet B)|w\rangle \tag{49} \\
&= 1/2(A \bullet B)C|w\rangle. \tag{50}
\end{align}
$$

But Eqns. 47 and 50 are consistent only if $(A \bullet B)(C|w\rangle) = |0\rangle$, as required.

Now, by the definition of S, all the operators in $\mathcal{B}$ commute on S. And since S is invariant under $\mathcal{B}$, restricting the action of any self-adjoint operator in $\mathcal{B}$ to S induces a self-adjoint operator on the subspace S. Since H_n— and thus S—is finite-dimensional, it follows by a well-known result that the operators in $\mathcal{B}$ share at least one complete set of common eigenvectors $\{|v_y\rangle\}$ ($y \in Y$) *on the subspace S.* The set $\{|v_y\rangle\}$ clearly satisfies 1., and also 2. since $|v\rangle$ lies in S. We can also arrange for 3. to be satisfied—while preserving satisfaction of 1. and 2.—by just dropping from the set $\{|v_y\rangle\}$ any vectors orthogonal to $|v\rangle$. So we can conclude, then, that there is at least one set of vectors $\{|v_x\rangle\}$ ($x \in X$) satisfying 1.–3. which are common eigenvectors of all the beables in $\mathcal{B}$. If so, then clearly $\mathcal{B} \subseteq \mathcal{B}_{\{|v_x\rangle\}}$, and the hypothesis that $\mathcal{B}$ is maximal for $|v\rangle$ delivers the required conclusion that $\mathcal{B} = \mathcal{B}_{\{|v_x\rangle\}}$ for some set satisfying 1.-3. (For uniqueness of this set, it is easy to see that if $\mathcal{B}_{\{|v_x\rangle\}} = \mathcal{B}_{\{|v_y\rangle\}}$ for two sets of vectors satisfying 1.–3., then those sets must in fact generate the same rays in H_n.) *QED.*

For finite-dimensional H, Thm. 10 yields a complete picture of maximal subalgebras for any pure state on $\mathcal{S}$: they simply correspond 1-1 with sets of vectors satisfying 1.–3. of the theorem, which then end up being common eigenvectors for all the elements of the algebra. If the set contains only a single vector, we get the orthodox subalgebra; if the set is a basis for the Hilbert space, we get a commutative subalgebra; and if the set falls between these two extremes we get a subalgebra of Bub-definite type.

For infinite-dimensional H, the converse part of Thm. 10 proof breaks down at the point where the existence of a complete set of commuting eigenvectors on the finite-dimensional subspace S is invoked. For if S could no longer be assumed finite-dimensional, some of the elements of $\mathcal{B}$ might then have *no* eigenvectors in S, much less any common ones. However, I conjecture that in that case there still exists a family $\{\rho_x\}$ of *singular* pure states

on $\mathcal{S}(H)$ satisfying 1.-3. with respect to $|v\rangle$, where by that I mean states that are not representable by vectors in H, such as the Dirac states I invoked in the last section.[35] Furthermore, I suspect that if 1.–3. are satisfied by singular pure states $\{\rho_x\}$, $\mathcal{B}_{\{\rho_x\}}$ *is* a maximal beable subalgebra for $|v\rangle$—though obviously Thm. 10's maximality proof no longer works since it relies on assuming that the pure states satisfying 1.–3. are vector states in the ranges of one-dimensional projection operators.

I end this section with one last (rather suggestive) result that illustrates the possibility of a maximality argument without assuming the state is a vector state.

Theorem 11 *The definite set of a singular pure state ρ on $\mathcal{S}(H)$ is a maximal beable subalgebra for ρ.*

Proof. I need a nontrivial result due to Kadison and Singer (1959, Thm. 4). They show that the definite set of any pure state ρ on the $C^\star$-algebra $\mathcal{U}(H)$ of all bounded operators on a Hilbert space (defined as the set of all *self-adjoint* operators on H on which ρ is dispersion-free) is not properly contained in the definite set of any other state on $\mathcal{U}(H)$. It is not difficult to see that a state is pure on a Segalgebra $\mathcal{S}$ exactly when its extension to the $C^\star$-algebra generated by $\mathcal{S}$ is pure. So the Kadison-Singer maximality result also holds for the pure states on $\mathcal{S}(H)$, and in particular the singular pure states.

Now consider the definite set $\mathcal{D}_\rho$ of a singular pure state ρ on $\mathcal{S}(H)$. To show $\mathcal{D}_\rho$ is a maximal beable subalgebra for ρ, let $\mathcal{Q}$ denote the subalgebra generated by $\mathcal{D}_\rho$ together with any element $A \notin \mathcal{D}_\rho$. Note that $I - A \notin \mathcal{D}_\rho$, and either A or $I - A$ must lie outside of $\mathcal{I}_\rho$ (otherwise both would get value 0 in state ρ, yet their values must sum to 1). So there is a $B \in \mathcal{Q}$ such that $B \notin \mathcal{D}_\rho$ and $B \notin \mathcal{I}_\rho$. Now suppose $\mathcal{Q}$ has beable status for ρ. Then Thm. 4 (Part 2.) and Thm. 7 dictate that there is a dispersion-free state $\langle \cdot \rangle$ on $\mathcal{Q}$ such that $\langle B \rangle \neq 0$. Using a Hahn-Banach-type argument, one can show that any state of a sub-Segalgebra extends to a state on the whole algebra.[36] Therefore, $\langle \cdot \rangle$ extends to a state on $\mathcal{S}(H)$ that has a definite set properly containing ρ's—contradicting the Kadison-Singer maximality result. *QED.*

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[35]Note that 1.-3. make sense for families of states on $\mathcal{S}(H)$ whether or not they are singular. For orthogonality of two algebraic states is equivalent to the norm of their difference (as a linear functional) being 2 (Sakai 1971, p. 31), and 2. is equivalent to the assertion that any state orthogonal to all the ρ_x's is orthogonal to $|v\rangle$ (thanks to Laura Reutsche for this point).

[36]See Segal (1947, Lemma 2.2) or Kadison and Ringrose (1983, Thm. 4.3.13).

7 From Quasicommutative to Commutative

In this final section, I want to point out two ways of arguing for the commutativity of beables. The first way depends on rejecting the modal idea of letting a system's beables vary from one of its quantum states to the next, and the second way depends on the Reeh-Schlieder theorem of algebraic relativistic quantum field theory.

First, suppose one had reasons to demand that a quantum system possess a fixed set of beables for all its quantum states. For example, one could think that the idea of a physical magnitude having a definite value at one time but not at another is conceptually incoherent (though modalists would dispute this). Or one could think that the extra flexibility of having a state-dependent set of beables is not necessary; in particular, not needed to solve the measurement problem, since all measurement outcomes can be ensured simply by granting beable status to (essentially) a single observable, like position.[37]

Now it would be unreasonable to require a fixed set of beables to satisfy the statistics of *all* states, since they may not all be realizable in nature. An example is provided by Dirac delta states, which we saw a few sections back qualify as algebraic states. Obviously to actually prepare such a state would require infinite energy on pain of violating the uncertainty relation.[38] However, even without committing to definitive necessary and sufficient conditions for a state to count as physical, it seems reasonable to expect a system's physical states to make up a full set of states on the system's Segalgebra, as do the vector states of $\mathcal{S}(H)$.[39] Assuming this, we have the following converse to Thm. 5 establishing commutativity for nonmodal interpreters who want a state-independent ontology of a system's properties (as in Bohm's theory).

Theorem 12 *If a subalgebra of a Segalgebra $\mathcal{S}$ has beable status for every state in a full set of states on $\mathcal{S}$, then it is commutative.*

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[37]This is a view Bell seems to have held—cf. the second quotation of Section 1.

[38]A more sophisticated example of the fact that not all algebraicly defined states are necessarily physical is found in algebraic quantum field theory on curved spacetime. There, the expectation value of the stress-energy tensor is not defined for all algebraic states of the field, but only for the so-called 'Hadamard' states; see Wald (1994, Sec. 4.6).

[39]For the Hadamard states referred to in the previous note, Thm. 2.1 of Fulling et al. (1981) establishes that they are dense in any Hilbert space representation of a globally hyperbolic spacetime's Segalgebra of observables, from which it follows by an elementary argument that the set of Hadamard states is full too.

Proof. Let $\mathcal{B}$ be a subalgebra of $\mathcal{S}$ and Ω a full set of states on $\mathcal{S}$. If $\mathcal{B}$ has beable status for every state in Ω, Thms. 4 (Part 3.) and 7 dictate that $\mathcal{B} \bullet \mathcal{B} \subseteq \mathcal{I}_\omega$ for all $\omega \in \Omega$. But since Ω is a full set, it is easy to see that $\bigcap_{\omega \in \Omega} \mathcal{I}_\omega = \{0\}$, which forces $\mathcal{B}$ to be commutative. *QED.*

Thm. 12 also allows us to diagnose exactly what goes wrong in Misra's (1967) argument against hidden-variables (in fulfillment of a promise I made a few sections back). Without distinguishing 'observables' whose measurement outcomes are determined by hidden-variables from those which correspond to true beables of the system, Misra demands that the outcome of any measurement be determined by a dispersion-free state on the algebra of all *observables* of a system (1967, p. 856), which we've seen is only reasonable if all 'observables' are beables of the system. Misra also assumes that the hidden-variables must be adequate for recovering (after averaging) the expectation values of at least the physical states of a system, which he too assumes will be a full set. That this is a lethal combination of assumptions should be clear. If all 'observables' are treated as beables, and they are forced to satisfy the statistics of a full set of states of the system, then Thm. 12 dictates that all the *observables* of the system have to commute—which is absurd! Far from delivering a no-go theorem, this is simply one more confirmation of Bell's point that not all 'observables' can have beable status.

A second way to argue that beables must be commutative arises out of the algebraic approach to relativistic quantum field theory. In that approach, one associates with each bounded open region O in Minkowski spacetime M a $C^\star$-algebra $\mathcal{U}(O)$ whose Segalgebra represents all observables measurable in region O. In the 'concrete' approach, the algebras $\{\mathcal{U}(O)\}_{O \subseteq M}$ are taken to be von Neumann algebras of operators acting on some common Hilbert space consisting of states of the entire field on M. If the collection of algebras $\{\mathcal{U}(O)\}_{O \subseteq M}$ satisfies a number of very general assumptions involving locality, covariance, etc. (the details of which need not detain us here), then it becomes possible to prove the Reeh-Schlieder theorem whose main consequence is that every state vector of the field with bounded energy is a *separating vector* for all the local algebras $\{\mathcal{U}(O)\}_{O \subseteq M}$. This means that no nonzero operator A in any local algebra $\mathcal{U}(O)$ can annihilate such a state vector.[40]

Now consider this result in the context of my analysis of beable subal-

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[40]Haag (1992, Sec. II.5.3); see also Redhead (1995b) for an elementary discussion.

gebras of the observables of a system. Here the role of the system is played by the quantum field in some bounded open region O of spacetime, and the question is which of the observables in $\mathcal{U}(O)$'s Segalgebra can be granted beable status. As we've seen, a subalgebra $\mathcal{B}(O)$ will have beable status for a state ω of the field if and only if $\mathcal{B}(O)$ is $\mathcal{I}_{\omega|_{\mathcal{B}(O)}}$-quasicommutative. But if ω corresponds to a state vector $|v\rangle$ with bounded energy, then since that vector is separating for $\mathcal{U}(O)$, we have

$$A \in \mathcal{I}_{\omega|_{\mathcal{B}(O)}} \Leftrightarrow \langle v|A^2|v\rangle = 0 \Leftrightarrow \langle Av|Av\rangle = 0 \Leftrightarrow A|v\rangle = |0\rangle \Leftrightarrow A = 0. \quad (51)$$

It follows that:

Theorem 13 *Subalgebras of local beables selected from the Segalgebras of local observables in relativistic quantum field theory must be commutative in any state of the field with bounded energy.*

Notice that the orthodox approach to value definiteness reduces to absurdity in this context: since ω is dispersion-free on $A \in \mathcal{U}(O)$ exactly when

$$A - \omega(A)I \in \mathcal{I}_{\omega|_{\mathcal{B}(O)}} = \{0\}, \quad (52)$$

taking $\mathcal{B}(O)$ to be the definite set of $\omega|_{\mathcal{B}(O)}$ yields only multiples of the identity operator as beables! Of course, there is still plenty of room left for the realist to propose beable status for other more satisfactory sets of commuting local observables.

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