ON GOLDSTONE FERMIONS

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Spontaneous violation of super-gauge symmetries is considered. One consequence must be the emergence of a particle with spin 1/2 and mass zero.

In two recent papers [1, 2] Wess and Zumino have generalized the super-gauge symmetry of dual model theory to apply to fields in 4-dimensional space-time. The most unusual feature of super-gauge symmetries is that they combine fermions with bosons in the same multiplet. This means that some of the conserved currents which generate these symmetries must be fermionic. With respect to the Lorentz group, these currents, $J_{\mu\alpha}$, will transform like the product of a vector and a Dirac spinor. Their components are a mixture of spins 1/2 and 3/2.

The unitary irreducible representations [3] of the super-gauge symmetry are characterized by three quantum numbers: the mass M, "spin" J, and parity η. If M is non-vanishing, then the (spin) parity content is $(J-\frac{1}{2})^{\eta}$, $J^{i\eta}$, $J^{-i\eta}$ and $(J+\frac{1}{2})^{\eta}$. If $M = 0$, then the helicity content is J, $J+\frac{1}{2}$ and $-J$, $-(J+\frac{1}{2})$.

The purpose of this note is to consider what happens if the vacuum is degenerate, i.e. not a super-gauge singlet. It is natural to expect, on the one hand, a lifting of the mass degeneracy in the previously irreducible multiplets and, on the other, a Goldstone phenomenon. In fact, the immediate implication would be that the system must contain a zero-mass fermion of spin 1/2 because of the following general argument. The action of an infinitesimal super-gauge transformation, ϵ_α, on the Dirac spinor field, ψ_α, is expressed formally by

$$\delta\psi_\alpha(x) = [\psi_\alpha(x), \int d_3x' \bar{\epsilon}^\beta J_{0\beta}(x')]. \tag{1}$$

Of particular relevance is the vacuum expectation value of this equation, which can be expressed in the form

$$\partial_\mu \langle T^* \bar{\epsilon}^\beta J_{\mu\beta}(x)\, \psi_\alpha(0)\rangle = -\delta_4(x)\langle\delta\psi_\alpha(0)\rangle. \tag{2}$$

This equation is equivalent to (1) if the vacuum is translation invariant and if the current is local and conserved. Now, if the vacuum is not super-gauge invariant, then $\langle\delta\psi_\alpha\rangle$ will not vanish; instead it will have the form

$$\langle\delta\psi_\alpha\rangle = \langle F\rangle\epsilon_\alpha = \langle F\rangle C_{\alpha\beta}\bar{\epsilon}^\beta\,, \tag{3}$$

where $C_{\alpha\beta}$ denotes the charge conjugation matrix [4]. The non-vanishing coefficient $\langle F\rangle$ which appears here is to be interpreted as the vacuum expectation value of some scalar field to which ψ is related by super-gauge transformations. Thus we have the Ward identity,

$$\partial_\mu\langle T^* J_{\mu\alpha}(x)\,\psi_\beta(0)\rangle = C_{\alpha\beta}\langle F\rangle\delta_4(x). \tag{4}$$

Define the momentum-space transform of this amplitude and resolve it into invariant components,

$$\begin{aligned}\int dx \exp(ikx)\langle T^* J_{\mu\alpha}(x)\,\psi_\beta(0)\rangle &= M_{\mu\alpha\beta}(k)\\ &= M_1(k^2)k_\mu C_{\alpha\beta} + (M_2(k^2)\eta_{\mu\nu} + M_3(k^2)k_\mu k_\nu)(\gamma_\nu C)_{\alpha\beta}\\ &\quad + M_4(k^2)\,k_\nu(\sigma_{\mu\nu}C)_{\alpha\beta}.\end{aligned} \tag{5}$$

It follows from (4) that the invariant components are constrained by the equations

$$\begin{aligned}&k^2 M_1(k^2) = i\langle F\rangle\\ &M_2(k^2) + k^2 M_3(k^2) = 0.\end{aligned} \tag{6}$$

Table 1

	$\langle A\rangle$	$\langle B\rangle$	$\langle F\rangle$	$\langle G\rangle$	m_A^2	m_B^2	m_ψ
$\mathrm{I}_\pm$	$-m/g$	$\pm(m/g)\sqrt{g\lambda/m^2-1}$	0	0	$g\lambda-m^2$	$g\lambda-m^2$	$\pm\sqrt{g\lambda-m^2}$
II	$-m/g$	0	$\frac{m^2}{4g}-\frac{\lambda}{4}$	0	$\frac{1}{2}(g\lambda-m^2)$	$-\frac{1}{2}(g\lambda-m^2)$	0
$\mathrm{III}_\pm$	$-(m/g)\,[1\pm\sqrt{1-g\lambda/m^2}]$	0	0	0	$m^2-g\lambda$	$m^2-g\lambda$	$\pm\sqrt{m^2-g\lambda}$

The second of these equations serves only to eliminate M_2 from the decomposition (5). The first equation gives the explicit form of $M_1(k^2)$, viz. a simple zero-mass pole with residue $\mathrm{i}\langle F\rangle$. This indicates that the intermediate states which contribute to $M_{\mu\alpha\beta}$ must include a massless particle of spin $\frac{1}{2}$: a Goldstone fermion.

The notion of a Goldstone fermion is highly attractive since it might provide a fundamental theory of the neutrino. We have tested the idea in the context of a simple Lagrangian model but have, however, come upon a dilemma, at least so far as the tree approximation goes. The model we consider is that of Wess and Zumino, which employs a multiplet of eight real components: two scalars, A, F, two pseudoscalars, B, G, and a Majorana spinor, ψ. They transform according to

$$\begin{aligned} &\delta A=\bar{\epsilon}\psi, \qquad \delta B=\bar{\epsilon}\gamma_5\psi,\\ &\delta\psi=(F+G\gamma_5)\epsilon-\tfrac{1}{2}\mathrm{i}\partial\!\!\!/(A+B\gamma_5)\epsilon,\\ &\delta F=-\tfrac{1}{2}\mathrm{i}\bar{\epsilon}\,\partial\!\!\!/\psi, \qquad \delta G=-\tfrac{1}{2}\mathrm{i}\bar{\epsilon}\gamma_5\partial\!\!\!/\psi. \end{aligned} \tag{7}$$

The Lagrangian (which is invariant up to a 4-divergence) is given by

$$L=L_0+L_m+L_g+L_\lambda, \tag{8}$$

where

$$\begin{aligned} &L_0=\tfrac{1}{2}(\partial_\mu A)^2+\tfrac{1}{2}(\partial_\mu B)^2+\bar{\psi}\mathrm{i}\partial\!\!\!/\psi+2(F^2+G^2)\\ &L_m=2m(AF+BG-\tfrac{1}{2}\bar{\psi}\psi)\\ &L_g=g[(A^2-B^2)F+2ABG-\bar{\psi}(A-B\gamma_5)\psi]\\ &L_\lambda=\lambda F. \end{aligned} \tag{9}$$

The conserved Noether current is given by (suppressing the Dirac index):

$$\begin{aligned} J_\mu=&\ \partial\!\!\!/(A-B\gamma_5)\gamma_\mu\psi+\mathrm{i}m(A+B\gamma_5)\gamma_\mu\psi\\ &+\tfrac{1}{2}\mathrm{i}g(A+B\gamma_5)^2\gamma_\mu\psi-\tfrac{1}{2}\lambda\gamma_\mu\gamma_5\psi. \end{aligned} \tag{10}$$

The algebraic variables F and G are redundant. They can be eliminated by using the corresponding Euler-Lagrange equations.‡

$$\begin{aligned} &0=4F+2mA+g(A^2-B^2)+\lambda\\ &0=4G+2mB+2gAB. \end{aligned} \tag{11}$$

When this is done the Lagrangian assumes the form

$$\begin{aligned} L'=&\ \tfrac{1}{2}(\partial_\mu A)^2+\tfrac{1}{2}(\partial_\mu B)^2\\ &+\bar{\psi}(\mathrm{i}\partial\!\!\!/-m)\psi-g\bar{\psi}(A-B\gamma_5)\psi-V(A,B), \end{aligned} \tag{12}$$

where the potential V is given by

$$\begin{aligned} V=&\ \tfrac{1}{2}m^2(A+\lambda/2m)^2+\tfrac{1}{2}m^2B^2\\ &+\tfrac{1}{4}g\lambda(A^2-B^2)+\tfrac{1}{2}mgA(A^2+B^2)\\ &+\tfrac{1}{8}g^2(A^2+B^2)^2. \end{aligned} \tag{13}$$

One can now proceed in the standard fashion to find the extrema of V. The ground states selected in this way must then be tested for stability by computing the particle masses.

There are three solutions. We list in table 1 the values of $\langle A\rangle$, $\langle B\rangle$, $\langle F\rangle$ and $\langle G\rangle$ as well as m_A^2, m_B^2 and m_ψ.

‡ These equations may be used to eliminate F and G from the transformation laws (7) which then become *non-linear.* In other words, the Lagrangian (12) is invariant (up to a 4-divergence) under a non-linear realization of the supergauge symmetry.

The only stable solutions[‡] are $\mathrm{I}_{\pm}$ (for $m^2 < g\lambda$) and $\mathrm{III}_{\pm}$ (for $m^2 > g\lambda$). Both of these are symmetry preserving, i.e. $\langle F \rangle = 0$. The Goldstone solution II is unstable since either A or B is a tachyon. We therefore conclude that the Lagrangian (8) cannot support a vacuum asymmetry, at least in the tree approximation here considered.

There are two possible escapes from this dilemma. One, we may need to go beyond the tree approximation [5] [‡‡]; or alternatively, that we have chosen the wrong super-gauge multiplet (and the wrong Lagrangian) to implement the notion of Goldstone fermions. The particle may in fact belong not to the multiplet with $J = 0$, but to either of the multiplets with $J = \frac{1}{2}$ and $J = 1$.

Note added. We have been informed by Professor B. Zumino that the problem of emergence of Goldstone germions has also been considered by himself and Professor J. Iliopoulos in a forthcoming CERN preprint.

‡ The two solutions denoted I_+ and I_- are equivalent. One can be brought into the other by a field redefinition. Likewise for III_+ and III_-. The solution I does not violate parity but rather imposes a redefinition of the parity operator. For the special value of λ given by $\lambda = m^2/g$, note, however, the existence of the degenerate solution $\langle A \rangle \neq 0$, but $\langle B \rangle = \langle F \rangle = \langle G \rangle = m_A = m_B = m_\psi = 0$.

‡‡ It is well known that in order to generate Goldstone solutions *in a tree approximation,* one must arrange the sequence of signs in the potential in a favourable manner. In the case considered in the text, the super-gauge symmetry has already fixed all the signs for us.

References

[1] J. Wess and B. Zumino, CERN preprint TH.1753 (1973), to appear in Nucl. Phys.
[2] J. Wess and B. Zumino, CERN preprint TH.1794 (1973), to appear in Nucl. Phys.
[3] A. Salam and J. Strathdee, ICTP, Trieste, preprint IC/74/16.
[4] For the notational conventions used here see: A. Salam and J. Strathdee, ICTP Triests, preprint IC/74/11.
[5] S. Coleman and E. Weinberg, Phys. Rev. D7 (1973) 1888.