The analysis of Harold Puthoff applied to the Natario warp drive spacetime: Can the spacetime metric engineering be really used for superluminal interstellar spaceflight??

Fernando Loup [*]
Residencia de Estudantes Universitas Lisboa Portugal

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Abstract

Warp Drives are solutions of the Einstein Field Equations that allows superluminal travel within the framework of General Relativity. There are at the present moment two known solutions: The Alcubierre warp drive discovered in 1994 and the Natario warp drive discovered in 2001. However the major drawback concerning warp drives is the huge amount of negative energy density able to sustain the warp bubble.In order to perform an interstellar space travel to a "nearby" star at 20 light-years away in a reasonable amount of time a ship must attain a speed of about 200 times faster than light.However the negative energy density at such a speed is directly proportional to the factor 10^{48} which is 1.000.000.000.000.000.000.000.000 times bigger in magnitude than the mass of the planet Earth!!. With the correct form of the shape function the Natario warp drive can overcome this obstacle at least in theory.Other drawbacks that affects the warp drive geometry are the collisions with hazardous inter-stellar matter(asteroids,comets,interstellar dust etc)that will unavoidably occurs when a ship travels at superluminal speeds and the problem of the Horizons(causally disconnected portions of spacetime).The geometrical features of the Natario warp drive are the required ones to overcome these obstacles also at least in theory.

Some years ago the American physicist Harold Puthoff published a very interesting work in the Journal of the British Interplanetary Society.He theorized about the possibility of the modification of the space-time geometry by arbitrary advanced civilizations able to generate the so-called metric engineering and such a modification would be supposed to "allow" the propulsion of spaceships at superluminal veloci-ties.However Puthoff used only diagonalized metrics for his analysis and he even quotes the Schwarzschild metric.In this work we reproduce the Puthoff analysis for the Natario warp drive spacetime and due to the fact that the Natario warp drive is a non-diagonalized metric due to the presence of both the shift and Natario vectors our results are different than the ones obtained by Puthoff.However his idea of a spacetime metric engineering able to distort the spacetime geometry "allowing" superluminal interstellar spaceflight is perfectly possible.

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[*]spacetimeshortcut@yahoo.com

1   Introduction:

The Warp Drive as a solution of the Einstein field equations of General Relativity that allows superluminal travel appeared first in 1994 due to the work of Alcubierre.([1]) The warp drive as conceived by Alcubierre worked with an expansion of the spacetime behind an object and contraction of the spacetime in front.The departure point is being moved away from the object and the destination point is being moved closer to the object.The object do not moves at all[1].It remains at the rest inside the so called warp bubble but an external observer would see the object passing by him at superluminal speeds(pg 8 in [1])(pg 1 in [2]).

Later on in 2001 another warp drive appeared due to the work of Natario.([2]).This do not expands or contracts spacetime but deals with the spacetime as a "strain" tensor of Fluid Mechanics(pg 5 in [2]). Imagine the object being a fish inside an aquarium and the aquarium is floating in the surface of a river but carried out by the river stream.The warp bubble in this case is the aquarium whose walls do not expand or contract. An observer in the margin of the river would see the aquarium passing by him at a large speed but inside the aquarium the fish is at the rest with respect to his local neighborhoods.

However there are 3 major drawbacks that compromises the warp drive physical integrity as a viable tool for superluminal interstellar travel.

The first drawback is the quest of large negative energy requirements enough to sustain the warp bubble. In order to travel to a "nearby" star at 20 light-years at superluminal speeds in a reasonable amount of time a ship must attain a speed of about 200 times faster than light.However the negative energy density at such a speed is directly proportional to the factor 10^{48} which is 1.000.000.000.000.000.000.000.000 times bigger in magnitude than the mass of the planet Earth!!!(see [7],[8] and [9]).

Another drawback that affects the warp drive is the quest of the interstellar navigation:Interstellar space is not empty and from a real point of view a ship at superluminal speeds would impact asteroids,comets,interstellar space dust and photons.(see [5],[7] and [8]).

The last drawback raised against the warp drive is the fact that inside the warp bubble an astronaut cannot send signals with the speed of the light to control the front of the bubble because an Horizon(causally disconnected portion of spacetime)is established between the astronaut and the warp bubble.(see [5],[7] and [8]).

We can demonstrate that the Natario warp drive can "easily" overcome these obstacles as a valid candidate for superluminal interstellar travel(see [7],[8] and [9]).

In this work we cover only the Natario warp drive and we avoid comparisons between the differences of the models proposed by Alcubierre and Natario since these differences were already deeply covered by the existing available literature.(see [5],[6] and [7])However we use the Alcubierre shape function to define its Natario counterpart.

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[1]do not violates Relativity

Some years ago the American physicist Harold Puthoff published a very interesting work in the Journal of the British Interplanetary Society.[10]

He theorized about the possibility of the modification of the spacetime geometry by arbitrary advanced civilizations able to generate the so-called metric engineering and such a modification would be supposed to "allow" the propulsion of spaceships at superluminal velocities(see pg 85 in [10]).

However Puthoff used only diagonalized metrics for his analysis and he even quotes the Schwarzschild metric.(see pg 83 in [10]).

In this work we reproduce the Puthoff analysis for the Natario warp drive spacetime and due to the fact that the Natario warp drive is a non-diagonalized metric due to the presence of both the shift and Natario vectors our results are different than the ones obtained by Puthoff.

However his idea of a spacetime metric engineering able to distort the spacetime geometry "allowing" superluminal interstellar spaceflight is perfectly possible.

This work is organized as follows:

- Section 2)-Introduces the Natario warp drive continuous shape function able to low the negative energy density requirements when a ship travels with a speed of 200 times faster than light.
  The negative energy density for such a speed is directly proportional to the factor 10^{48} which is 1.000.000.000.000.000.000.000.000 times bigger in magnitude than the mass of the planet Earth!!!.

- Section 3)-Introduces the mathematical structure for the equation of the Natario warp drive spacetime metric outlining its non-diagonalized features due to the presence of both the shift and Natario vectors.

- Section 4)-Introduces the Puthoff analysis for the Natario warp drive spacetime.The results obtained by ourselves are radically different than the ones obtained originally by Puthoff.

Natario in its warp drive uses the spherical coordinates rs and θ.In order to simplify our analysis we consider motion in the $x - axis$ or the equatorial plane rs where $\theta = 0$ $\sin(\theta) = 0$ and $\cos(\theta) = 1$.(see pgs 4,5 and 6 in [2]).$1 + 1$ spacetime.

We adopt here the Geometrized system of units in which $c = G = 1$ for geometric purposes and the International System of units for energetic purposes

2 The Natario warp drive continuous shape function

Introducing here $f(rs)$ as the Alcubierre shape function that defines the Alcubierre warp drive spacetime we can construct the Natario shape function $n(rs)$ that defines the Natario warp drive spacetime using its Alcubierre counterpart.Below is presented the equation of the Alcubierre shape function.[2].

$$f(rs) = \frac{1}{2}[1 - tanh[@(rs - R)]] \tag{1}$$

$$rs = \sqrt{(x - xs)^2 + y^2 + z^2} \tag{2}$$

According with Alcubierre any function $f(rs)$ that gives 1 inside the bubble and 0 outside the bubble while being $1 > f(rs) > 0$ in the Alcubierre warped region is a valid shape function for the Alcubierre warp drive.(see eqs 6 and 7 pg 4 in [1] or top of pg 4 in [2]).

In the Alcubierre shape function xs is the center of the warp bubble where the ship resides. R is the radius of the warp bubble and @ is the Alcubierre parameter related to the thickness.According to Alcubierre these can have arbitrary values.We outline here the fact that according to pg 4 in [1] the parameter @ can have arbitrary values.rs is the path of the so-called Eulerian observer that starts at the center of the bubble $xs = R = rs = 0$ and ends up outside the warp bubble $rs > R$.

According to Natario(pg 5 in [2]) any function that gives 0 inside the bubble and $\frac{1}{2}$ outside the bubble while being $0 < n(rs) < \frac{1}{2}$ in the Natario warped region is a valid shape function for the Natario warp drive.

The Natario warp drive continuous shape function can be defined by:

$$n(rs) = \frac{1}{2}[1 - f(rs)] \tag{3}$$

$$n(rs) = \frac{1}{2}[1 - [\frac{1}{2}[1 - tanh[@(rs - R)]]]] \tag{4}$$

This shape function gives the result of $n(rs) = 0$ inside the warp bubble and $n(rs) = \frac{1}{2}$ outside the warp bubble while being $0 < n(rs) < \frac{1}{2}$ in the Natario warped region.

Note that the Alcubierre shape function is being used to define its Natario shape function counterpart.

For the Natario shape function introduced above it is easy to figure out when $f(rs) = 1$(interior of the Alcubierre bubble) then $n(rs) = 0$(interior of the Natario bubble) and when $f(rs) = 0$(exterior of the Alcubierre bubble)then $n(rs) = \frac{1}{2}$(exterior of the Natario bubble).

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[2]$tanh[@(rs + R)] = 1$,$tanh(@R) = 1$ for very high values of the Alcubierre thickness parameter $@ >> |R|$

Another Natario warp drive valid shape function can be given by:

$$n(rs) = [\frac{1}{2}][1 - f(rs)^{WF}]^{WF} \tag{5}$$

Its derivative square is :

$$n'(rs)^2 = [\frac{1}{4}]WF^4[1 - f(rs)^{WF}]^{2(WF-1)}[f(rs)^{2(WF-1)}]f'(rs)^2 \tag{6}$$

The shape function above also gives the result of $n(rs) = 0$ inside the warp bubble and $n(rs) = \frac{1}{2}$ outside the warp bubble while being $0 < n(rs) < \frac{1}{2}$ in the Natario warped region(see pg 5 in [2]).

Note that like in the previous case the Alcubierre shape function is being used to define its Natario shape function counterpart. The term WF in the Natario shape function is dimensionless too:it is the warp factor.It is important to outline that the warp factor $WF >> |R|$ is much greater than the modulus of the bubble radius.

For the second Natario shape function introduced above it is easy to figure out when $f(rs) = 1$(interior of the Alcubierre bubble) then $n(rs) = 0$(interior of the Natario bubble) and when $f(rs) = 0$(exterior of the Alcubierre bubble)then $n(rs) = \frac{1}{2}$(exterior of the Natario bubble).

- Numerical plot for the second shape function with @ $= 50000$ and warp factor with a value $WF = 200$

rs	$f(rs)$	$n(rs)$	$f'(rs)^2$	$n'(rs)^2$
$9,99970000000E + 001$	1	0	$2,650396620740E - 251$	0
$9,99980000000E + 001$	1	0	$1,915169647489E - 164$	0
$9,99990000000E + 001$	1	0	$1,383896564748E - 077$	0
$1,00000000000E + 002$	$0,5$	$0,5$	$6,250000000000E + 008$	$3,872591914849E - 103$
$1,00001000000E + 002$	0	$0,5$	$1,383896486082E - 077$	0
$1,00002000000E + 002$	0	$0,5$	$1,915169538624E - 164$	0
$1,00003000000E + 002$	0	$0,5$	$2,650396470082E - 251$	0

- Numerical plot for the second shape function with @ $= 75000$ and warp factor with a value $WF = 200$

rs	$f(rs)$	$n(rs)$	$f'(rs)^2$	$n'(rs)^2$
$9,99980000000E + 001$	1	0	$5,963392481410E - 251$	0
$9,99990000000E + 001$	1	0	$1,158345097767E - 120$	0
$1,00000000000E + 002$	$0,5$	$0,5$	$1,406250000000E + 009$	$8,713331808411E - 103$
$1,00001000000E + 002$	0	$0,5$	$1,158344999000E - 120$	0
$1,00002000000E + 002$	0	$0,5$	$5,963391972940E - 251$	0

- Numerical plot for the second shape function with @ $= 100000$ and warp factor with a value $WF = 200$

rs	$f(rs)$	$n(rs)$	$f'(rs)^2$	$n'(rs)^2$
$9,99990000000E + 001$	1	0	$7,660678807684E - 164$	0
$1,00000000000E + 002$	$0,5$	$0,5$	$2,500000000000E + 009$	$1,549036765940E - 102$
$1,00001000000E + 002$	0	$0,5$	$7,660677936765E - 164$	0

The plots in the previous page demonstrate the important role of the thickness parameter @ in the warp bubble geometry wether in both Alcubierre or Natario warp drive spacetimes.For a bubble of 100 meters radius $R = 100$ the regions where $1 > f(rs) > 0$(Alcubierre warped region) and $0 < n(rs) < \frac{1}{2}$(Natario warped region) becomes thicker or thinner as @ becomes higher.

Then the geometric position where both Alcubierre and Natario warped regions begins with respect to R the bubble radius is $rs = R - \epsilon < R$ and the geometric position where both Alcubierre and Natario warped regions ends with respect to R the bubble radius is $rs = R + \epsilon > R$

As large as @ becomes as smaller ϵ becomes too.

Note from the plots of the previous page that we really have two warped regions:

- 1)-The geometrized warped region where $1 > f(rs) > 0$(Alcubierre warped region) and $0 < n(rs) < \frac{1}{2}$ (Natario warped region).

- 2)-The energized warped region where the derivative squares of both Alcubierre and Natario shape functions are not zero.

The parameter @ affects both energized warped regions wether in Alcubierre or Natario cases but is more visible for the Alcubierre shape function because the warp factor WF in the Natario shape functions squeezes the energized warped region into a very small thickness.

The negative energy density for the Natario warp drive is given by(see pg 5 in [2])

$$\rho = T_{\mu\nu}u^\mu u^\nu = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(n'(rs))^2\cos^2\theta + \left(n'(rs) + \frac{r}{2}n''(rs)\right)^2\sin^2\theta\right] \tag{7}$$

Converting from the Geometrized System of Units to the International System we should expect for the following expression:

$$\rho = -\frac{c^2}{G}\frac{vs^2}{8\pi}\left[3(n'(rs))^2\cos^2\theta + \left(n'(rs) + \frac{rs}{2}n''(rs)\right)^2\sin^2\theta\right]. \tag{8}$$

Rewriting the Natario negative energy density in cartezian coordinates we should expect for[3]:

$$\rho = T_{\mu\nu}u^\mu u^\nu = -\frac{c^2}{G}\frac{v_s^2}{8\pi}\left[3(n'(rs))^2(\frac{x}{rs})^2 + \left(n'(rs) + \frac{r}{2}n''(rs)\right)^2(\frac{y}{rs})^2\right] \tag{9}$$

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[3]see Appendix D

6

In the equatorial plane($1+1$ dimensional spacetime with $rs = x - xs$,$y = 0$ and center of the bubble $xs = 0$):

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{c^2}{G}\frac{v_s^2}{8\pi}\left[3(n'(rs))^2\right] \tag{10}$$

Note that in the above expressions the warp drive speed vs appears raised to a power of 2. Considering our Natario warp drive moving with $vs = 200$ which means to say 200 times light speed in order to make a round trip from Earth to a nearby star at 20 light-years away in a reasonable amount of time(in months not in years) we would get in the expression of the negative energy the factor $c^2 = (3 \times 10^8)^2 = 9 \times 10^{16}$ being divided by $6,67 \times 10^{-11}$ giving $1,35 \times 10^{27}$ and this is multiplied by $(6 \times 10^{10})^2 = 36 \times 10^{20}$ coming from the term $vs = 200$ giving $1,35 \times 10^{27} \times 36 \times 10^{20} = 1,35 \times 10^{27} \times 3,6 \times 10^{21} = 4,86 \times 10^{48}$!!!

A number with 48 zeros!!!The planet Earth have a mass[4] of about $6 \times 10^{24}kg$

This term is $1.000.000.000.000.000.000.000.000$ times bigger in magnitude than the mass of the planet Earth!!!or better:The amount of negative energy density needed to sustain a warp bubble at a speed of 200 times faster than light requires the magnitude of the masses of $1.000.000.000.000.000.000.000.000$ planet Earths!!!

Note that if the negative energy density is proportional to 10^{48} this would render the warp drive impossible but fortunately the square derivative of the Natario shape function possesses values of 10^{-102} ameliorating the factor 10^{48} making the warp drive negative energy density more "affordable".

Note also that when we define the Natario shape function as being (pg 5 in [2]) any function $n(rs)$ that gives 0 inside the bubble and $\frac{1}{2}$ outside the bubble while being $0 < n(rs) < \frac{1}{2}$ in the Natario warped region we are defining the shape function $n(rs)$ for a coordinate system frame based on the spaceship placed inside the bubble at the rest while the remaining Universe outside the bubble passes by the ship with a speed vs.

On the other hand we need to define a Natario shape function $o(rs)$ that gives $\frac{1}{2}$ inside the bubble 0 outside the bubble and $\frac{1}{2} > o(rs) > 0$ in the Natario warped region which represents the Natario shape function $o(rs)$ defined for a coordinate system frame based on the distant observer in the rest of the Universe outside the bubble watching the bubble passing by him with a speed vs.

The Natatio shape function for a coordinate system frame based on a distant observer at the rest outside the bubble is given by:

$$o(rs) = \frac{1}{2} - n(rs) \tag{11}$$

From above it is easy to see that when $n(rs) = 0$,$o(rs) = \frac{1}{2}$ (inside the bubble) and when $n(rs) = \frac{1}{2}$,$o(rs) = 0$ (outside the bubble).

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[4]see Wikipedia:The free Encyclopedia

3 The equation of the Natario warp drive spacetime metric

The warp drive spacetime according to Natario for the coordinates rs and θ is defined by the following equation:(see Appendix A for details)

$$ds^2 = [1 - (X^{rs})^2 - (X^{\theta})^2]dt^2 + 2[X^{rs}drs + X^{\theta}rsd\theta]dt - drs^2 - rs^2d\theta^2 \tag{12}$$

The expressions for X^{rs} and X^{θ} are given by:(see pg 5 in [2],see also Appendix B for details)

$$X^{rs} = -2v_s n(rs)\cos\theta \tag{13}$$

$$X^{rs} = 2v_s n(rs)\cos\theta \tag{14}$$

$$X^{\theta} = v_s(2n(rs) + (rs)n'(rs))\sin\theta \tag{15}$$

$$X^{\theta} = -v_s(2n(rs) + (rs)n'(rs))\sin\theta \tag{16}$$

Looking both the equation of the Natario warp drive and the equation of the Natario vector nX(pg 2 and 5 in [2]):

$$ds^2 = [1 - (X^{rs})^2 - (X^{\theta})^2]dt^2 + 2[X^{rs}drs + X^{\theta}rsd\theta]dt - drs^2 - rs^2d\theta^2 \tag{17}$$

$$nX = X^{rs}drs + X^{\theta}rsd\theta \tag{18}$$

We can see that the Natario vector is completely inserted twice in the non-diagonalized components of the metric of the Nayario warp drive equation which gives:

$$g_{01} = g_{10} = X^{rs} = 2v_s n(rs)\cos\theta \tag{19}$$

$$g_{02} = g_{20} = X^{\theta}rs = -v_s(2n(rs) + (rs)n'(rs))rs\sin\theta \tag{20}$$

Since we have two sets of non-diagonalized components in the Natario warp drive equation and each set possesses equal components of the Natario vector nX this is the reason why the Natario vector nX appears twice in the Natario warp drive equation.

The diagonalized components of the metric of the Natario warp drive equation are given by:

$$g_{00} = 1 - (X^{rs})^2 - (X^{\theta})^2 = 1 - (2v_s n(rs)\cos\theta)^2 - (-v_s(2n(rs) + (rs)n'(rs))\sin\theta)^2 \tag{21}$$

The term $(-v_s(2n(rs) + (rs)n'(rs))\sin\theta)^2 = (v_s(2n(rs) + (rs)n'(rs))\sin\theta)^2$

$$g_{00} = 1 - (X^{rs})^2 - (X^{\theta})^2 = 1 - (2v_s n(rs)\cos\theta)^2 - (v_s(2n(rs) + (rs)n'(rs))\sin\theta)^2 \tag{22}$$

$$g_{11} = -1 \tag{23}$$

$$g_{22} = -rs^2 \tag{24}$$

Considering a valid $n(rs)$ as a Natario shape function being $n(rs) = \frac{1}{2}$ for large rs(outside the warp bubble) and $n(rs) = 0$ for small rs(inside the warp bubble) while being $0 < n(rs) < \frac{1}{2}$ in the walls of the warp bubble also known as the Natario warped region(pg 5 in [2]):

We can see that the Natario warp drive equation given in the previous page satisfies the Natario requirements for a warp bubble defined by:

any Natario vector nX generates a warp drive spacetime if $nX = 0$ and $X = vs = 0$ for a small value of rs defined by Natario as the interior of the warp bubble and $nX = -vs(t)dx$ or $nX = vs(t)dx$ with $X = vs$ for a large value of rs defined by Natario as the exterior of the warp bubble with $vs(t)$ being the speed of the warp bubble.(pg 4 in [2])

The statement above can be explained in the following way:

Consider again the Natario vector nX(pg 2 and 5 in [2]) defined below as:

$$nX = X^{rs}drs + X^{\theta}rsd\theta \tag{25}$$

The components of the Natario vector nX are X^{rs} and X^{θ}.These are the shift vectors.Then a Natario vector is constituted by one or more shift vectors.

When the Natario shape function $n(rs) = 0$ inside the bubble then $X^{rs} = 2v_sn(rs)\cos\theta = 0$ and $X^{\theta} = -v_s(2n(rs) + (rs)n'(rs))\sin = 0$.Then inside the bubble both shift vectors are zero resulting in a zero Natario vector.

When the Natario shape function $n(rs) = \frac{1}{2}$ outside the bubble then $X^{rs} = 2v_sn(rs)\cos\theta = v_s\cos\theta$ and $X^{\theta} = -v_s(2n(rs) + (rs)n'(rs))\sin\theta = -v_s\sin\theta$.Then outside the bubble both shift vectors are not zero resulting in a not zero Natario vector.

Natario in its warp drive uses the spherical coordinates rs and θ.In order to simplify our analysis we consider motion in the $x - axis$ or the equatorial plane rs where $\theta = 0 \sin(\theta) = 0$ and $\cos(\theta) = 1$.(see pgs 4,5 and 6 in [2]).

The Natario warp drive equation and the Natario vector nX in the equatorial plane $1 + 1$ spacetime now becomes:

$$ds^2 = [1 - (X^{rs})^2]dt^2 + 2[X^{rs}drs]dt - drs^2 \tag{26}$$

$$nX = X^{rs}drs \tag{27}$$

Note that the Natario vector nX is still inserted twice in the Natario warp drive equation due to the 2 remaining non-diagonalized components which are:

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$$g_{01} = g_{10} = X^{rs} = 2v_s n(rs) \tag{28}$$

When the Natario shape function $n(rs) = 0$ inside the bubble then the shift vector $X^{rs} = 2v_s n(rs) = 0$ .Then inside the bubble the shift vector $X^{rs} = 0$ is zero resulting in a zero Natario vector.

When the Natario shape function $n(rs) = \frac{1}{2}$ outside the bubble then the shift vector $X^{rs} = 2v_s n(rs) = v_s$ .Then outside the bubble both shift and Natario vectors are not zero and the shift vector is equal to the bubble speed vs $X^{rs} = vs$.

The above statements explain the Natario affirmation of $X = 0$ inside the bubble and $X = vs$ outside the bubble.(pg 4 in [2])

The diagonalized components of the metric of the Natario warp drive equation are given by:

$$g_{00} = 1 - (X^{rs})^2 = 1 - (2v_s n(rs))^2 \tag{29}$$

$$g_{11} = -1 \tag{30}$$

The results above are valid for a ship frame coordinates system.

The Natario warp drive equation and the Natario vector nX in the equatorial plane $1 + 1$ spacetime for a remote frame coordinates system now becomes:

$$ds^2 = [1 - (X^{rs})^2]dt^2 + 2[X^{rs}drs]dt - drs^2 \tag{31}$$

$$nX = X^{rs}drs \tag{32}$$

Note that the Natario vector nX is still inserted twice in the Natario warp drive equation due to the 2 remaining non-diagonalized components which are:

$$g_{01} = g_{10} = X^{rs} = 2v_s o(rs) \tag{33}$$

When the Natario shape function $o(rs) = \frac{1}{2}$ inside the bubble then the shift vector $X^{rs} = 2v_s o(rs) = vs$ .Then inside the bubble the shift and Natario vectors are not zero and the shift vector is equal to the bubble speed vs $X^{rs} = vs$.

When the Natario shape function $o(rs) = 0$ outside the bubble then the shift vector $X^{rs} = 2v_s o(rs) = 0$ .Then outside the bubble both shift and Natario vectors are zero.

The diagonalized components of the metric of the Natario warp drive equation are given by:

$$g_{00} = 1 - (X^{rs})^2 = 1 - (2v_s o(rs))^2 \tag{34}$$

$$g_{11} = -1 \tag{35}$$

We have two different physical situations here considering two different coordinate frames:

- 1)-Coordinates frame based on the ship inside the bubble

Note that when we define the Natario shape function as being (pg 5 in [2]) any function $n(rs)$ that gives 0 inside the bubble and $\frac{1}{2}$ outside the bubble while being $0 < n(rs) < \frac{1}{2}$ in the Natario warped region we are defining the shape function $n(rs)$ for a coordinate system frame based on the spaceship placed inside the bubble at the rest while the remaining Universe outside the bubble passes by the ship with a speed vs.

When the Natario shape function $n(rs) = 0$ inside the bubble then the shift vector $X^{rs} = 2v_s n(rs) = 0$ .Then inside the bubble the shift vector is zero resulting in a zero Natario vector.$X^{rs} = 0$
When the Natario shape function $n(rs) = \frac{1}{2}$ outside the bubble then the shift vector $X^{rs} = 2v_s n(rs) = v_s$ .Then outside the bubble both shift and Natario vectors are not zero and the shift vector is equal to the bubble speed vs.$X^{rs} = vs$

- 2)-Coordinates frame based on the remote observer outside the bubble

We need to define a Natario shape function $o(rs)$ that gives $\frac{1}{2}$ inside the bubble 0 outside the bubble and $\frac{1}{2} > o(rs) > 0$ in the Natario warped region which represents the Natario shape function $o(rs)$ defined for a coordinate system frame based on the distant observer in the rest of the Universe outside the bubble watching the bubble passing by him with a speed vs.

When the Natario shape function $o(rs) = \frac{1}{2}$ inside the bubble then the shift vector $X^{rs} = 2v_s o(rs) = vs$ .Then inside the bubble the shift and Natario vectors are not zero zero and the shift vector is equal to the bubble speed vs.$X^{rs} = vs$
When the Natario shape function $o(rs) = 0$ outside the bubble then the shift vector $X^{rs} = 2v_s o(rs) = 0$ .Then outside the bubble both shift and Natario vectors are zero.$X^{rs} = 0$

4 The analysis of Harold Puthoff applied to the Natario warp drive spacetime: Can the spacetime metric engineering be really used for superluminal interstellar spaceflight??

Some years ago Harold Puthoff published a work [10] about the possibility of the modification of the spacetime geometry by arbitrary advanced civilizations able to generate the so-called metric engineering and such a modification would be supposed to "allow" the propulsion of spaceships at superluminal velocities(see pg 85 in [10]).

However Puthoff used only diagonalized metrics for his analysis and he even quotes the Schwarzschild metric.(see pg 83 in [10]).

Puthoff also introduced a coefficient for the metric engineering that would make an observer subluminal in an interior reference frame(a ship frame) appears superluminal when seen by an observer in an exterior reference frame (a remote observer outside the spacetime distortion watching the spaceship passing by him).

The Puthoff metric engineering coefficient(pmc) is given by the following expression:(see pg 85 in [10])

$$pmc = \sqrt{\frac{g_{00}}{-g_{11}}} \tag{36}$$

The analysis of Puthoff may be correct for diagonalized metrics however the Natario warp drive is a non-diagonalized metric due to the presence of both the shift and the Natario vectors.

We are about the develop the Puthoff analysis for the Natario warp drive in a $1 + 1$ spacetime:

- 1)-spacetime metric tensor components for the Natario warp drive in a ship frame based coordinates system:

$$g_{01} = g_{10} = X^{rs} = 2v_s n(rs) \tag{37}$$

$$g_{00} = 1 - (X^{rs})^2 = 1 - (2v_s n(rs))^2 \tag{38}$$

$$g_{11} = -1 \tag{39}$$

- 2)–spacetime metric tensor components for the Natario warp drive in a remote frame based coordinates system:

$$g_{01} = g_{10} = X^{rs} = 2v_s o(rs) \tag{40}$$

$$g_{00} = 1 - (X^{rs})^2 = 1 - (2v_s o(rs))^2 \tag{41}$$

$$g_{11} = -1 \tag{42}$$

- A)-Puthoff analysis for the Natario warp drive in a ship frame based coordinates system:

In a ship frame based coordinates system the Natario shape function $n(rs)$ is 0 inside the bubble and $\frac{1}{2}$ outside the bubble while being $0 < n(rs) < \frac{1}{2}$ in the Natario warped region.The shift and Natario vectors X and nX are 0 inside the bubble and the shift vector is equal to vs outside the bubble.

This is due to the fact that the ship is at the rest in its own reference frame then $X = 0$ and the observer inside the bubble sees the remaining Universe outside the bubble passing by him with a relative speed vs.Then the observer inside the bubble sees the shift and Natario vectors outside the bubble with non-null values and the shift vector with a value $X = vs$.

- A1)-measures for the spacetime metric components taken inside the bubble.$n(rs) = 0$.ship frame based coordinates system:

$$g_{01} = g_{10} = X^{rs} = 2v_s n(rs) = 0 \tag{43}$$

$$g_{00} = 1 - (X^{rs})^2 = 1 - (2v_s n(rs))^2 = 1 \tag{44}$$

- A2)-measures for the spacetime metric components taken outside the bubble.$n(rs) = \frac{1}{2}$.ship frame based coordinates system:

$$g_{01} = g_{10} = X^{rs} = 2v_s n(rs) = vs \tag{45}$$

$$g_{00} = 1 - (X^{rs})^2 = 1 - (2v_s n(rs))^2 = 1 - vs^2 \tag{46}$$

- A3)-measures for the spacetime metric components taken in the Natario warped region.$0 < n(rs) < \frac{1}{2}$.ship frame based coordinates system:

$$g_{01} = g_{10} = X^{rs} = 2v_s n(rs) < vs \tag{47}$$

$$g_{00} = 1 - (X^{rs})^2 = 1 - (2v_s n(rs))^2 < 1 - vs^2 \tag{48}$$

Examining the Appendix C we have an artistic presentation of a spaceship inside a Natario warp bubble.The ship is at the rest inside the bubble while the rest of the Universe passes by the ship with a relative speed vs.So inside the bubble the shift vector $X^{rs} = 0$ and outside the bubble the shift vector $X^{rs} = vs$ and in the Natario warped region (the blue circle) $0 < X^{rs} < vs$.

Remember that we the adopted the geometrized system of units for geometrical purposes so we use the units $c = G = 1$.

Then we have 3 different types of bubble velocity vs:

- 1)-bubble at subluminal velocity $vs < 1$

- 2)-bubble at luminal velocity $vs = 1$

- 3)-bubble at superluminal velocity $vs > 1$

- A1.1)-measures for the spacetime metric components taken inside the bubble.$n(rs) = 0$.ship frame based coordinates system:bubble at subluminal velocity $vs < 1$

$$g_{01} = g_{10} = X^{rs} = 2v_s n(rs) = 0 \tag{49}$$

$$g_{00} = 1 - (X^{rs})^2 = 1 - (2v_s n(rs))^2 = 1 \tag{50}$$

- A2.1)-measures for the spacetime metric components taken outside the bubble.$n(rs) = \frac{1}{2}$.ship frame based coordinates system:bubble at subluminal velocity $vs < 1$

$$g_{01} = g_{10} = X^{rs} = 2v_s n(rs) = vs < 1 \tag{51}$$

$$g_{00} = 1 - (X^{rs})^2 = 1 - (2v_s n(rs))^2 = 1 - vs^2 < 1 \tag{52}$$

$$g_{00} = 1 - (X^{rs})^2 = 1 - (2v_s n(rs))^2 = 1 - vs^2 > 0 \tag{53}$$

- A3.1)-measures for the spacetime metric components taken in the Natario warped region.$0 < n(rs) < \frac{1}{2}$.ship frame based coordinates system:bubble at subluminal velocity $vs < 1$

$$g_{01} = g_{10} = X^{rs} = 2v_s n(rs) < vs \tag{54}$$

$$g_{01} = g_{10} = X^{rs} = 2v_s n(rs) < 1 \tag{55}$$

$$g_{00} = 1 - (X^{rs})^2 = 1 - (2v_s n(rs))^2 < 1 - vs^2 \tag{56}$$

$$g_{00} = 1 - (X^{rs})^2 = 1 - (2v_s n(rs))^2 < 1 \tag{57}$$

$$g_{00} = 1 - (X^{rs})^2 = 1 - (2v_s n(rs))^2 > 0 \tag{58}$$

Note that in this case inside the bubble we recover the flat spacetime $g_{00} = 1$ while outside the bubble $g_{00} < 1$ however $g_{00} > 0$.It is very important to outline that in this case outside the bubble $g_{00} > 0$ always.Note that the conditions outside the bubble are the same ones for the Natario warped region.

Note that we found for a spacetime obtained by metric engineering (the Natario warp drive) a value $g_{00} < 1$ but $g_{00} > 0$ while Puthoff found a result of $g_{00} > 1$ for a metric engineered spacetime.(see table 1 pg 84 in [10])

And remember that we are working with a non-diagonalized metric but at subluminal speeds.

- A1.2)-measures for the spacetime metric components taken inside the bubble.$n(rs) = 0$.ship frame based coordinates system:bubble at luminal velocity $vs = 1$

$$g_{01} = g_{10} = X^{rs} = 2v_s n(rs) = 0 \tag{59}$$

$$g_{00} = 1 - (X^{rs})^2 = 1 - (2v_s n(rs))^2 = 1 \tag{60}$$

- A2.2)-measures for the spacetime metric components taken outside the bubble.$n(rs) = \frac{1}{2}$.ship frame based coordinates system:bubble at luminal velocity $vs = 1$

$$g_{01} = g_{10} = X^{rs} = 2v_s n(rs) = vs = 1 \tag{61}$$

$$g_{00} = 1 - (X^{rs})^2 = 1 - (2v_s n(rs))^2 = 1 - vs^2 = 0 \tag{62}$$

- A3.2)-measures for the spacetime metric components taken in the Natario warped region.$0 < n(rs) < \frac{1}{2}$.ship frame based coordinates system:bubble at luminal velocity $vs = 1$

$$g_{01} = g_{10} = X^{rs} = 2v_s n(rs) < vs \tag{63}$$

$$g_{01} = g_{10} = X^{rs} = 2v_s n(rs) < 1 \tag{64}$$

$$g_{00} = 1 - (X^{rs})^2 = 1 - (2v_s n(rs))^2 < 1 - vs^2 \tag{65}$$

$$g_{00} = 1 - (X^{rs})^2 = 1 - (2v_s n(rs))^2 < 1 \tag{66}$$

$$g_{00} = 1 - (X^{rs})^2 = 1 - (2v_s n(rs))^2 > 0 \tag{67}$$

Note that in this case inside the bubble we also recover the flat spacetime $g_{00} = 1$. Note that the conditions of the Natario warped region $g_{00} < 1$ $g_{00} > 0$ and the conditions inside the bubble are equal to the ones of the previous case..

But outside the bubble $g_{00} = 0$.It is very important to outline that in this case outside the bubble $g_{00} = 0$ always.

A $g_{00} = 0$ in the Schwarzschild metric results in an Event Horizon.Exactly what happens here with a warp bubble at luminal speeds. In the end of the warped region and in the beginning of the region outside the bubble (end of the blue circle in the Appendix C) an Event Horizon forms when the bubble moves at luminal seeds.

Note that we found for a spacetime obtained by metric engineering (the Natario warp drive) a value $g_{00} < 1$ but $g_{00} > 0$ for the Natario warped region and we found an Event Horizon for the region outside the bubble $g_{00} = 0$ while Puthoff found a result of $g_{00} > 1$ for a metric engineered spacetime.(see table 1 pg 84 in [10])

And remember that we are working with a non-diagonalized metric but at luminal speeds.

- A1.3)-measures for the spacetime metric components taken inside the bubble.$n(rs) = 0$.ship frame based coordinates system:bubble at superluminal velocity $vs > 1$

$$g_{01} = g_{10} = X^{rs} = 2v_s n(rs) = 0 \tag{68}$$

$$g_{00} = 1 - (X^{rs})^2 = 1 - (2v_s n(rs))^2 = 1 \tag{69}$$

- A2.3)-measures for the spacetime metric components taken outside the bubble.$n(rs) = \frac{1}{2}$.ship frame based coordinates system:bubble at suoerluminal velocity $vs > 1$

$$g_{01} = g_{10} = X^{rs} = 2v_s n(rs) = vs > 1 \tag{70}$$

$$g_{00} = 1 - (X^{rs})^2 = 1 - (2v_s n(rs))^2 = 1 - vs^2 < 1 \tag{71}$$

$$g_{00} = 1 - (X^{rs})^2 = 1 - (2v_s n(rs))^2 = 1 - vs^2 < 0 \tag{72}$$

- A3.3)-measures for the spacetime metric components taken in the Natario warped region.$0 < n(rs) < \frac{1}{2}$.ship frame based coordinates system:bubble at superluminal velocity $vs > 1$

$$g_{01} = g_{10} = X^{rs} = 2v_s n(rs) < vs \tag{73}$$

$$g_{01} = g_{10} = X^{rs} = 2v_s n(rs) <= 1 \tag{74}$$

$$g_{00} = 1 - (X^{rs})^2 = 1 - (2v_s n(rs))^2 < 1 - vs^2 \tag{75}$$

$$g_{00} = 1 - (X^{rs})^2 = 1 - (2v_s n(rs))^2 < 1 \tag{76}$$

$$g_{00} = 1 - (X^{rs})^2 = 1 - (2v_s n(rs))^2 = 0 \tag{77}$$

$$g_{00} = 1 - (X^{rs})^2 = 1 - (2v_s n(rs))^2 < 0 \tag{78}$$

Like in the previous cases inside the bubble we also recover the flat spacetime $g_{00} = 1$.

Outside the bubble in this physical situation we have a striking feature:$g_{00} < 0$.
Now g_{00} becomes negative.Then the term $\sqrt{g_{00}}$ in the Puthoff metric engineering coefficient(pmc) given by the following expression:(see pg 85 in [10])

$$pmc = \sqrt{\frac{g_{00}}{-g_{11}}} \tag{79}$$

becomes imaginary.The Puthoff analysis cannot be applied to the Natario warp drive at superluminal speeds because the metric is non-diagonalized.

But the most remarkable physical situation for the Natario warp bubble at superluminal speeds $(vs > 1)$ occurs in the Natario warped region:

We already know that inside the bubble the shift vector $X^{rs} = 0$ and outside the bubble the shift vector $X^{rs} = vs$ and in the Natario warped region $0 < X^{rs} < vs$ and in the superluminal case $vs > 1$. Assuming a continuous growth of X^{rs} from 0 to vs and since in this case $vs > 1$ then in a certain moment $X^{rs} = 1$ and in consequence $g_{00} = 1 - (X^{rs})^2 = 0$.

An Event Horizon $g_{00} = 0$ is established but now inside the Natario warped region.(In the contours of the blue circle in Appendix C).

The point inside the Natario warped region where the Event Horizon occurs is the point where the Natario shape function have the following value:

$$g_{00} = 1 - (X^{rs})^2 = 1 - (2v_s n(rs))^2 = 0 \tag{80}$$

$$1 = (2v_s n(rs))^2 \dashrightarrow 1 = 2v_s n(rs) \tag{81}$$

$$n(rs) = \frac{1}{2vs} \tag{82}$$

Remember that the Natario shape function $n(rs)$ in the Natario warped region possesses the values $0 < n(rs) < \frac{1}{2}$ and $\frac{1}{2vs}$ is still far from $\frac{1}{2}$ like a luminal speed 1 is still far from a superluminal speed $vs > 1$. Then in the part of the Natario warped region where $\frac{1}{2vs} < n(rs) < \frac{1}{2}$ which means to say the part of the Natario warped region beyond the Event Horizon from the point of view of a ship frame coordinates inside the bubble where $g_{00} = 1 - (X^{rs})^2 = 1 - (2v_s n(rs))^2 < 0$ we have again negative values for g_{00} and with a $g_{00} < 0$ the term $\sqrt{g_{00}}$ in the Puthoff metric engineering coefficient becomes again imaginary.

Then in the Natario warped region for the superluminal case we have two relevant physical situations:

- 1)-$g_{00} = 0$.Existence of the Event Horizon.

- 2)-$g_{00} < 0$.A g_{00} negative makes the term $\sqrt{g_{00}}$ imaginary

The Puthoff analysis cannot be applied to the Natario warp drive at superluminal speeds because the metric is non-diagonalized.

- B)-Puthoff analysis for the Natario warp drive in a remote frame based coordinates system outside the bubble:

In a remote frame based coordinates system the Natario shape function $o(rs)$ is 0 outside the bubble and $\frac{1}{2}$ inside the bubble while being $0 < o(rs) < \frac{1}{2}$ in the Natario warped region.The shift and Natario vectors X and nX are 0 outside the bubble and the shift vector is equal to vs inside the bubble.

This is due to the fact that the ship is passing by a remote observer at the rest outside the bubble watching the ship moving with a speed vs relative to him.Then the remote observer measures a shift vector $X = vs$ for the ship inside the bubble while measuring a shift vector $X = 0$ for him and in its local neighborhoods outside the bubble.

- B1)-measures for the spacetime metric components taken outside the bubble.$o(rs) = 0$.remote frame based coordinates system:

$$g_{01} = g_{10} = X^{rs} = 2v_s o(rs) = 0 \tag{83}$$

$$g_{00} = 1 - (X^{rs})^2 = 1 - (2v_s o(rs))^2 = 1 \tag{84}$$

- B2)-measures for the spacetime metric components taken inside the bubble.$o(rs) = \frac{1}{2}$.remote frame based coordinates system:

$$g_{01} = g_{10} = X^{rs} = 2v_s o(rs) = vs \tag{85}$$

$$g_{00} = 1 - (X^{rs})^2 = 1 - (2v_s o(rs))^2 = 1 - vs^2 \tag{86}$$

- B3)-measures for the spacetime metric components taken in the Natario warped region.$0 < o(rs) < \frac{1}{2}$.remote frame based coordinates system:

$$g_{01} = g_{10} = X^{rs} = 2v_s o(rs) < vs \tag{87}$$

$$g_{00} = 1 - (X^{rs})^2 = 1 - (2v_s o(rs))^2 < 1 - vs^2 \tag{88}$$

It is easy to see that this physical situation for the remote frame observer outside the bubble is exactly the opposite of the situation of the ship frame observer inside the bubble previously discussed.The region inside the bubble for a ship frame observer corresponds to the region outside the bubble for a remote frame observer and the region outside the bubble for a ship frame observer corresponds to the region inside the bubble for a remote frame observer.

It is also easy to see that in this case of the remote frame coordinates system when $vs = 1$ the Event Horizon will appear and when $vs > 1$ $g_{00} < 0$ makings the term $\sqrt{g_{00}}$ imaginary.

18

5   Conclusion:

In this work we demonstrated the analysis of Harold Puthoff applied to the Natario warp drive spacetime.

In section 2 we presented two Natario shape functions and while one of them makes the Natario warp drive impossible to be physically achieved due to high negative energy density requirements the other makes the Natario warp drive perfectly possible to be achieved because this shape function have a form that allows low and "affordable" negative energy density requirements.Then the form of the shape functions affects the behavior of the Natario warp drive spacetime specially in the Natario warped region.(See [8] and [9])

We also introduced in section 3 the mathematical structure of the equation of the Natario warp drive spacetime metric outlining the importance of the shift and Natario vectors presented in this non-diagonalized metric with the shift vector being the non-diagonalized component in the spacetime metric tensor for the Natario warp drive.

In section 4 we discuss the Puthoff analysis presented in [10] applied to the Natario warp drive spacetime.While Puthoff uses only diagonalized metrics for his analysis and he even quotes the Schwarzschild metric.(see pg 83 in [10]) and also uses a result of $g_{00} > 1$ for a metric engineered spacetime(see table 1 pg 84 in [10]) designed to satisfy the Puthoff coefficient with the term $\sqrt{g_{00}}$ in the Puthoff metric engineering coefficient(see pg 85 in [10]) being always real we found that for the Natario warp drive spacetime at subluminal speeds the term $g_{00} < 1$ but $g_{00} > 0$.At luminal speeds outside the bubble $g_{00} = 0$ and an Event Horizon is formed.At superluminal speeds the Event Horizon also appears but inside the middle layers of the Natario warped region and in the outermost layers of the Natario warped region the term $g_{00} < 0$ making the term $\sqrt{g_{00}}$ imaginary.

Of course our results are very different than the ones obtained by Puthoff:This is due to the fact that the Natario warp drive spacetime is a non-diagonalized metric and the presence of both shift and Natario vectors alters the result of the analysis.

About the possibility of the modification of the spacetime geometry by arbitrary advanced civilizations able to generate the so-called metric engineering in order for such a modification "allow" the propulsion of spaceships at superluminal velocities(see pg 85 in [10]) as described by Puthoff considering the Natario warp drive spacetime then the advanced civilizations must discover a way to generate reasonable outputs of negative energy density.While mathematically we can get arbitrarily low amounts of negative energy density as described in [8] we now know that the total amount of negative energy density able to sustain a warp bubble with a spaceship inside must be a fraction of the total spaceship mass as described in [9].

The Event Horizon described here resembles the Horizon problem generated by causally disconnected portions of spacetime as described in [5],[7] and [8].Any advanced civilization in order to have a Natario warp drive able to achieve superluminal speeds must overcome the Horizon problem using a solution perhaps similar to the one described in [7] and [8].

With the problems of negative energy density and Horizons solved then the modification of the space-time geometry by arbitrary advanced civilizations able to generate the so-called metric engineering in order for such a modification "allow" the propulsion of spaceships at superluminal velocities(see pg 85 in [10]) as described by Puthoff is perfectly possible.

But unfortunately although we can discuss mathematically how to generate the negative energy density to sustain a warp drive we dont know how to generate the shape function that distorts the spacetime geometry creating the warp drive effect.So unfortunately all the discussions about warp drives are still under the domain of the mathematical conjectures.

However we are confident to affirm that the Natario warp drive will survive the passage of the Century XXI and will arrive to the Future.The Natario warp drive as a valid candidate for faster than light interstellar space travel will arrive to the the Century $XXIV$ on-board the future starships up there in the middle of the stars helping the human race to give his first steps in the exploration of our Galaxy

Live Long And Prosper

6 Appendix A:mathematical demonstration of the Natario warp drive equation for a constant speed vs

The warp drive spacetime according to Natario is defined by the following equation but we changed the metric signature from $(-, +, +, +)$ to $(+, -, -, -)$(pg 2 in [2])

$$ds^2 = dt^2 - \sum_{i=1}^{3}(dx^i - X^i dt)^2 \qquad (89)$$

where X^i is the so-called shift vector.This shift vector is the responsible for the warp drive behavior defined as follows(pg 2 in [2]):

$$X^i = X, Y, Z \curvearrowright i = 1, 2, 3 \qquad (90)$$

The warp drive spacetime is completely generated by the Natario vector nX(pg 2 in [2])

$$nX = X^i \frac{\partial}{\partial x^i} = X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y} + Z\frac{\partial}{\partial z}, \qquad (91)$$

Defined using the canonical basis of the Hodge Star in spherical coordinates as follows(pg 4 in [2]):

$$e_r \equiv \frac{\partial}{\partial r} \sim dr \sim (rd\theta) \wedge (r\sin\theta d\varphi) \qquad (92)$$

$$e_\theta \equiv \frac{1}{r}\frac{\partial}{\partial \theta} \sim rd\theta \sim (r\sin\theta d\varphi) \wedge dr \qquad (93)$$

$$e_\varphi \equiv \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi} \sim r\sin\theta d\varphi \sim dr \wedge (rd\theta) \qquad (94)$$

Redefining the Natario vector nX as being the rate-of-strain tensor of fluid mechanics as shown below(pg 5 in [2]):

$$nX = X^r \mathbf{e}_r + X^\theta \mathbf{e}_\theta + X^\varphi \mathbf{e}_\varphi \qquad (95)$$

$$nX = X^r dr + X^\theta rd\theta + X^\varphi r\sin\theta d\varphi \qquad (96)$$

$$ds^2 = dt^2 - \sum_{i=1}^{3}(dx^i - X^i dt)^2 \qquad (97)$$

$$X^i = r, \theta, \varphi \curvearrowright i = 1, 2, 3 \qquad (98)$$

We are interested only in the coordinates r and θ according to pg 5 in [2])

$$ds^2 = dt^2 - (dr - X^r dt)^2 - (rd\theta - X^\theta dt)^2 \qquad (99)$$

$$(dr - X^r dt)^2 = dr^2 - 2X^r drdt + (X^r)^2 dt^2 \qquad (100)$$

$$(rd\theta - X^\theta dt)^2 = r^2 d\theta^2 - 2X^\theta r d\theta dt + (X^\theta)^2 dt^2 \tag{101}$$

$$ds^2 = dt^2 - (X^r)^2 dt^2 - (X^\theta)^2 dt^2 + 2X^r dr dt + 2X^\theta r d\theta dt - dr^2 - r^2 d\theta^2 \tag{102}$$

$$ds^2 = [1 - (X^r)^2 - (X^\theta)^2] dt^2 + 2[X^r dr + X^\theta r d\theta] dt - dr^2 - r^2 d\theta^2 \tag{103}$$

making $r = rs$ we have the Natario warp drive equation:

$$ds^2 = [1 - (X^{rs})^2 - (X^\theta)^2] dt^2 + 2[X^{rs} drs + X^\theta rs d\theta] dt - drs^2 - rs^2 d\theta^2 \tag{104}$$

According with the Natario definition for the warp drive using the following statement(pg 4 in [2]):any Natario vector nX generates a warp drive spacetime if $nX = 0$ and $X = vs = 0$ for a small value of rs defined by Natario as the interior of the warp bubble and $nX = -vs(t)dx$ or $nX = vs(t)dx$ with $X = vs$ for a large value of rs defined by Natario as the exterior of the warp bubble with $vs(t)$ being the speed of the warp bubble

The expressions for X^{rs} and X^θ are given by:(see pg 5 in [2]).See also Appendix B.

$$nX \sim -2v_s n(rs) \cos\theta \mathbf{e}_{rs} + v_s(2n(rs) + (rs)n'(rs)) \sin\theta \mathbf{e}_\theta \tag{105}$$

$$nX \sim 2v_s n(rs) \cos\theta \mathbf{e}_{rs} - v_s(2n(rs) + (rs)n'(rs)) \sin\theta \mathbf{e}_\theta \tag{106}$$

$$nX \sim -2v_s n(rs) \cos\theta drs + v_s(2n(rs) + (rs)n'(rs)) \sin\theta rs d\theta \tag{107}$$

$$nX \sim 2v_s n(rs) \cos\theta drs - v_s(2n(rs) + (rs)n'(rs)) \sin\theta rs d\theta \tag{108}$$

But we already know that the Natario vector nX is defined by(pg 2 and 5 in [2]):

$$nX = X^{rs} drs + X^\theta rs d\theta \tag{109}$$

Hence we should expect for:

$$X^{rs} = -2v_s n(rs) \cos\theta \tag{110}$$

$$X^{rs} = 2v_s n(rs) \cos\theta \tag{111}$$

$$X^\theta = v_s(2n(rs) + (rs)n'(rs)) \sin\theta \tag{112}$$

$$X^\theta = -v_s(2n(rs) + (rs)n'(rs)) \sin\theta \tag{113}$$

Looking both the equation of the Natario warp drive and the equation of the Natario vector nX(pg 2 and 5 in [2]):

$$ds^2 = [1 - (X^{rs})^2 - (X^\theta)^2]dt^2 + 2[X^{rs}drs + X^\theta rsd\theta]dt - drs^2 - rs^2 d\theta^2 \qquad (114)$$

$$nX = X^{rs}drs + X^\theta rsd\theta \qquad (115)$$

We can see that the Natario vector is completely inserted twice in the non-diagonalized components of the metric of the Nayario warp drive equation which gives:

$$g_{01} = g_{10} = X^{rs} = 2v_s n(rs)\cos\theta \qquad (116)$$

$$g_{02} = g_{20} = X^\theta rs = -v_s(2n(rs) + (rs)n'(rs))rs\sin\theta \qquad (117)$$

Since we have two sets of non-diagonalized components in the Natario warp drive equation and each set possesses equal components of the Natario vector nX this is the reason why the Natario vector nX appears twice in the Natario warp drive equation.

The diagonalized components of the metric of the Natario warp drive equation are given by:

$$g_{00} = 1 - (X^{rs})^2 - (X^\theta)^2 = 1 - (2v_s n(rs)\cos\theta)^2 - (-v_s(2n(rs) + (rs)n'(rs))\sin\theta)^2 \qquad (118)$$

The term $(-v_s(2n(rs) + (rs)n'(rs))\sin\theta)^2 = (v_s(2n(rs) + (rs)n'(rs))\sin\theta)^2$

$$g_{00} = 1 - (X^{rs})^2 - (X^\theta)^2 = 1 - (2v_s n(rs)\cos\theta)^2 - (v_s(2n(rs) + (rs)n'(rs))\sin\theta)^2 \qquad (119)$$

$$g_{11} = -1 \qquad (120)$$

$$g_{22} = -rs^2 \qquad (121)$$

Considering a valid $n(rs)$ as a Natario shape function being $n(rs) = \frac{1}{2}$ for large rs(outside the warp bubble) and $n(rs) = 0$ for small rs(inside the warp bubble) while being $0 < n(rs) < \frac{1}{2}$ in the walls of the warp bubble also known as the Natario warped region(pg 5 in [2]):

We can see that the Natario warp drive equation given above satisfies the Natario requirements for a warp bubble defined by:

any Natario vector nX generates a warp drive spacetime if $nX = 0$ and $X = vs = 0$ for a small value of rs defined by Natario as the interior of the warp bubble and $nX = -vs(t)dx$ or $nX = vs(t)dx$ with $X = vs$ for a large value of rs defined by Natario as the exterior of the warp bubble with $vs(t)$ being the speed of the warp bubble.(pg 4 in [2])

The statement above can be explained in the following way:

Consider again the Natario vector nX(pg 2 and 5 in [2]) defined below as:

$$nX = X^{rs}drs + X^{\theta}rsd\theta \qquad (122)$$

The components of the Natario vector nX are X^{rs} and X^{θ}.These are the shift vectors.Then a Natario vector is constituted by one or more shift vectors.

When the Natario shape function $n(rs) = 0$ inside the bubble then $X^{rs} = 2v_sn(rs)\cos\theta = 0$ and $X^{\theta} = -v_s(2n(rs) + (rs)n'(rs))\sin = 0$.Then inside the bubble both shift vectors are zero resulting in a zero Natario vector.

When the Natario shape function $n(rs) = \frac{1}{2}$ outside the bubble then $X^{rs} = 2v_sn(rs)\cos\theta = v_s\cos\theta$ and $X^{\theta} = -v_s(2n(rs) + (rs)n'(rs))\sin\theta = -v_s\sin\theta$.Then outside the bubble both shift vectors are not zero resulting in a not zero Natario vector.

Natario in its warp drive uses the spherical coordinates rs and θ.In order to simplify our analysis we consider motion in the $x - axis$ or the equatorial plane rs where $\theta = 0$ $\sin(\theta) = 0$ and $\cos(\theta) = 1$.(see pgs 4,5 and 6 in [2]).

The Natario warp drive equation and the Natario vector nX in the equatorial plane $1 + 1$ spacetime now becomes:

$$ds^2 = [1 - (X^{rs})^2]dt^2 + 2[X^{rs}drs]dt - drs^2 \qquad (123)$$

$$nX = X^{rs}drs \qquad (124)$$

Note that the Natario vector nX is still inserted twice in the Natario warp drive equation due to the 2 remaining non-diagonalized components which are:

$$g_{01} = g_{10} = X^{rs} = 2v_sn(rs) \qquad (125)$$

When the Natario shape function $n(rs) = 0$ inside the bubble then the shift vector $X^{rs} = 2v_sn(rs) = 0$ .Then inside the bubble the shift vector is zero resulting in a zero Natario vector.$X^{rs} = 0$

When the Natario shape function $n(rs) = \frac{1}{2}$ outside the bubble then the shift vector $X^{rs} = 2v_sn(rs) = v_s$ .Then outside the bubble both shift and Natario vectors are not zero and the shift vector is equal to the bubble speed vs.$X^{rs} = v_s$

The above statements explain the Natario affirmation of $X = 0$ inside the bubble and $X = vs$ outside the bubble.(pg 4 in [2])

The diagonalized components of the metric of the Natario warp drive equation are given by:

$$g_{00} = 1 - (X^{rs})^2 = 1 - (2v_sn(rs))^2 \qquad (126)$$

$$g_{11} = -1 \qquad (127)$$

The results above are valid for a ship frame coordinates system.

The Natario warp drive equation and the Natario vector nX in the equatorial plane $1 + 1$ spacetime for a remote frame coordinates system now becomes:

$$ds^2 = [1 - (X^{rs})^2]dt^2 + 2[X^{rs}drs]dt - drs^2 \tag{128}$$

$$nX = X^{rs}drs \tag{129}$$

Note that the Natario vector nX is still inserted twice in the Natario warp drive equation due to the 2 remaining non-diagonalized components which are:

$$g_{01} = g_{10} = X^{rs} = 2v_s o(rs) \tag{130}$$

When the Natario shape function $o(rs) = \frac{1}{2}$ inside the bubble then the shift vector $X^{rs} = 2v_s o(rs) = vs$ .Then inside the bubble the shift and Natario vectors are not zero and the shift vector is equal to the bubble speed $vs.X^{rs} = vs$

When the Natario shape function $o(rs) = 0$ outside the bubble then the shift vector $X^{rs} = 2v_s o(rs) = 0$ .Then outside the bubble both shift and Natario vectors are zero.$X^{rs} = 0$

The diagonalized components of the metric of the Natario warp drive equation are given by:

$$g_{00} = 1 - (X^{rs})^2 = 1 - (2v_s o(rs))^2 \tag{131}$$

$$g_{11} = -1 \tag{132}$$

We have two different physical situations here considering two different coordinate frames:

- 1)-Coordinates frame based on the ship inside the bubble

Note that when we define the Natario shape function as being (pg 5 in [2]) any function $n(rs)$ that gives 0 inside the bubble and $\frac{1}{2}$ outside the bubble while being $0 < n(rs) < \frac{1}{2}$ in the Natario warped region we are defining the shape function $n(rs)$ for a coordinate system frame based on the spaceship placed inside the bubble at the rest while the remaining Universe outside the bubble passes by the ship with a speed vs.

When the Natario shape function $n(rs) = 0$ inside the bubble then the shift vector $X^{rs} = 2v_s n(rs) = 0$ .Then inside the bubble the shift vector is zero resulting in a zero Natario vector.$X^{rs} = 0$
When the Natario shape function $n(rs) = \frac{1}{2}$ outside the bubble then the shift vector $X^{rs} = 2v_s n(rs) = v_s$ .Then outside the bubble both shift and Natario vectors are not zero and the shift vector is equal to the bubble speed vs.$X^{rs} = v_s$

- 2)-Coordinates frame based on the remote observer outside the bubble

We need to define a Natario shape function $o(rs)$ that gives $\frac{1}{2}$ inside the bubble 0 outside the bubble and $\frac{1}{2} > o(rs) > 0$ in the Natario warped region which represents the Natario shape function $o(rs)$ defined for a coordinate system frame based on the distant observer in the rest of the Universe outside the bubble watching the bubble passing by him with a speed vs.

When the Natario shape function $o(rs) = \frac{1}{2}$ inside the bubble then the shift vector $X^{rs} = 2v_s o(rs) = vs$ .Then inside the bubble the shift and Natario vectors are not zero zero and the shift vector is equal to the bubble speed vs.$X^{rs} = vs$
When the Natario shape function $o(rs) = 0$ outside the bubble then the shift vector $X^{rs} = 2v_s o(rs) = 0$ .Then outside the bubble both shift and Natario vectors are zero.$X^{rs} = 0$

7 Appendix B:differential forms,Hodge star and the mathematical demonstration of the Natario vectors $nX = -vsdx$ and $nX = vsdx$ for a constant speed vs

This appendix is being written for novice or newcomer students on Warp Drive theory still not acquainted with the methods Natario used to arrive at the final expression of the Natario Vector nX

The Canonical Basis of the Hodge Star in spherical coordinates can be defined as follows(pg 4 in [2]):

$$e_r \equiv \frac{\partial}{\partial r} \sim dr \sim (rd\theta) \wedge (r\sin\theta d\varphi) \sim r^2 \sin\theta(d\theta \wedge d\varphi) \tag{133}$$

$$e_\theta \equiv \frac{1}{r}\frac{\partial}{\partial\theta} \sim rd\theta \sim (r\sin\theta d\varphi) \wedge dr \sim r\sin\theta(d\varphi \wedge dr) \tag{134}$$

$$e_\varphi \equiv \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi} \sim r\sin\theta d\varphi \sim dr \wedge (rd\theta) \sim r(dr \wedge d\theta) \tag{135}$$

From above we get the following results

$$dr \sim r^2 \sin\theta(d\theta \wedge d\varphi) \tag{136}$$

$$rd\theta \sim r\sin\theta(d\varphi \wedge dr) \tag{137}$$

$$r\sin\theta d\varphi \sim r(dr \wedge d\theta) \tag{138}$$

Note that this expression matches the common definition of the Hodge Star operator * applied to the spherical coordinates as given by(pg 8 in [4]):

$$*dr = r^2 \sin\theta(d\theta \wedge d\varphi) \tag{139}$$

$$*rd\theta = r\sin\theta(d\varphi \wedge dr) \tag{140}$$

$$*r\sin\theta d\varphi = r(dr \wedge d\theta) \tag{141}$$

Back again to the Natario equivalence between spherical and cartezian coordinates(pg 5 in [2]):

$$\frac{\partial}{\partial x} \sim dx = d(r\cos\theta) = \cos\theta dr - r\sin\theta d\theta \sim r^2\sin\theta\cos\theta d\theta\wedge d\varphi + r\sin^2\theta dr\wedge d\varphi = d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \tag{142}$$

Look that

$$dx = d(r\cos\theta) = \cos\theta dr - r\sin\theta d\theta \tag{143}$$

Or

$$dx = d(r\cos\theta) = \cos\theta dr - \sin\theta rd\theta \tag{144}$$

27

Applying the Hodge Star operator * to the above expression:

$$*dx = *d(r\cos\theta) = \cos\theta(*dr) - \sin\theta(*rd\theta) \tag{145}$$

$$*dx = *d(r\cos\theta) = \cos\theta[r^2\sin\theta(d\theta \wedge d\varphi)] - \sin\theta[r\sin\theta(d\varphi \wedge dr)] \tag{146}$$

$$*dx = *d(r\cos\theta) = [r^2\sin\theta\cos\theta(d\theta \wedge d\varphi)] - [r\sin^2\theta(d\varphi \wedge dr)] \tag{147}$$

We know that the following expression holds true(see pg 9 in [3]):

$$d\varphi \wedge dr = -dr \wedge d\varphi \tag{148}$$

Then we have

$$*dx = *d(r\cos\theta) = [r^2\sin\theta\cos\theta(d\theta \wedge d\varphi)] + [r\sin^2\theta(dr \wedge d\varphi)] \tag{149}$$

And the above expression matches exactly the term obtained by Natario using the Hodge Star operator applied to the equivalence between cartezian and spherical coordinates(pg 5 in [2]).

Now examining the expression:

$$d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \tag{150}$$

We must also apply the Hodge Star operator to the expression above

And then we have:

$$*d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \tag{151}$$

$$*d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \sim \frac{1}{2}r^2 * d[(\sin^2\theta)d\varphi] + \frac{1}{2}\sin^2\theta * [d(r^2)d\varphi] + \frac{1}{2}r^2\sin^2\theta * d[(d\varphi)] \tag{152}$$

According to pg 10 in [3] the term $\frac{1}{2}r^2\sin^2\theta * d[(d\varphi)] = 0$

This leaves us with:

$$\frac{1}{2}r^2 * d[(\sin^2\theta)d\varphi] + \frac{1}{2}\sin^2\theta * [d(r^2)d\varphi] \sim \frac{1}{2}r^2 2\sin\theta\cos\theta(d\theta \wedge d\varphi) + \frac{1}{2}\sin^2\theta 2r(dr \wedge d\varphi) \tag{153}$$

Because and according to pg 10 in [3]:

$$d(\alpha + \beta) = d\alpha + d\beta \tag{154}$$

$$d(f\alpha) = df \wedge \alpha + f \wedge d\alpha \tag{155}$$

$$d(dx) = d(dy) = d(dz) = 0 \tag{156}$$

28

From above we can see for example that

$$*d[(\sin^2\theta)d\varphi] = d(\sin^2\theta) \wedge d\varphi + \sin^2\theta \wedge dd\varphi = 2sin\theta \cos\theta(d\theta \wedge d\varphi) \tag{157}$$

$$*[d(r^2)d\varphi] = 2rdr \wedge d\varphi + r^2 \wedge dd\varphi = 2r(dr \wedge d\varphi) \tag{158}$$

And then we derived again the Natario result of pg 5 in [2]

$$r^2\sin\theta\cos\theta(d\theta \wedge d\varphi) + r\sin^2\theta(dr \wedge d\varphi) \tag{159}$$

Now we will examine the following expression equivalent to the one of Natario pg 5 in [2] except that we replaced $\frac{1}{2}$ by the function $f(r)$:

$$*d[f(r)r^2\sin^2\theta d\varphi] \tag{160}$$

From above we can obtain the next expressions

$$f(r)r^2 * d[(\sin^2\theta)d\varphi] + f(r)\sin^2\theta * [d(r^2)d\varphi] + r^2\sin^2\theta * d[f(r)d\varphi] \tag{161}$$

$$f(r)r^2 2sin\theta\cos\theta(d\theta \wedge d\varphi) + f(r)\sin^2\theta 2r(dr \wedge d\varphi) + r^2\sin^2\theta f'(r)(dr \wedge d\varphi) \tag{162}$$

$$2f(r)r^2 sin\theta\cos\theta(d\theta \wedge d\varphi) + 2f(r)r\sin^2\theta(dr \wedge d\varphi) + r^2\sin^2\theta f'(r)(dr \wedge d\varphi) \tag{163}$$

Comparing the above expressions with the Natario definitions of pg 4 in [2]):

$$e_r \equiv \frac{\partial}{\partial r} \sim dr \sim (rd\theta) \wedge (r\sin\theta d\varphi) \sim r^2\sin\theta(d\theta \wedge d\varphi) \tag{164}$$

$$e_\theta \equiv \frac{1}{r}\frac{\partial}{\partial\theta} \sim rd\theta \sim (r\sin\theta d\varphi) \wedge dr \sim r\sin\theta(d\varphi \wedge dr) \sim -r\sin\theta(dr \wedge d\varphi) \tag{165}$$

$$e_\varphi \equiv \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi} \sim r\sin\theta d\varphi \sim dr \wedge (rd\theta) \sim r(dr \wedge d\theta) \tag{166}$$

We can obtain the following result:

$$2f(r)\ cos\theta[r^2 sin\theta(d\theta \wedge d\varphi)] + 2f(r)\ sin\theta[r\sin\theta(dr \wedge d\varphi)] + f'(r)r\sin\theta[r\sin\theta(dr \wedge d\varphi)] \tag{167}$$

$$2f(r)\ cos\theta e_r - 2f(r)\ sin\theta e_\theta - rf'(r)\sin\theta e_\theta \tag{168}$$

$$*d[f(r)r^2\sin^2\theta d\varphi] = 2f(r)\ cos\theta e_r - [2f(r) + rf'(r)]\sin\theta e_\theta \tag{169}$$

Defining the Natario Vector as in pg 5 in [2] with the Hodge Star operator * explicitly written :

$$nX = vs(t) * d\left(f(r)r^2\sin^2\theta d\varphi\right) \tag{170}$$

$$nX = -vs(t) * d\left(f(r)r^2\sin^2\theta d\varphi\right) \tag{171}$$

We can get finally the latest expressions for the Natario Vector nX also shown in pg 5 in [2]

$$nX = 2vs(t)f(r)\ cos\theta e_r - vs(t)[2f(r) + rf'(r)]\sin\theta e_\theta \tag{172}$$

$$nX = -2vs(t)f(r)\ cos\theta e_r + vs(t)[2f(r) + rf'(r)]\sin\theta e_\theta \tag{173}$$

With our pedagogical approaches

$$nX = 2vs(t)f(r)\ cos\theta dr - vs(t)[2f(r) + rf'(r)]r\sin\theta d\theta \tag{174}$$

$$nX = -2vs(t)f(r)\ cos\theta dr + vs(t)[2f(r) + rf'(r)]r\sin\theta d\theta \tag{175}$$
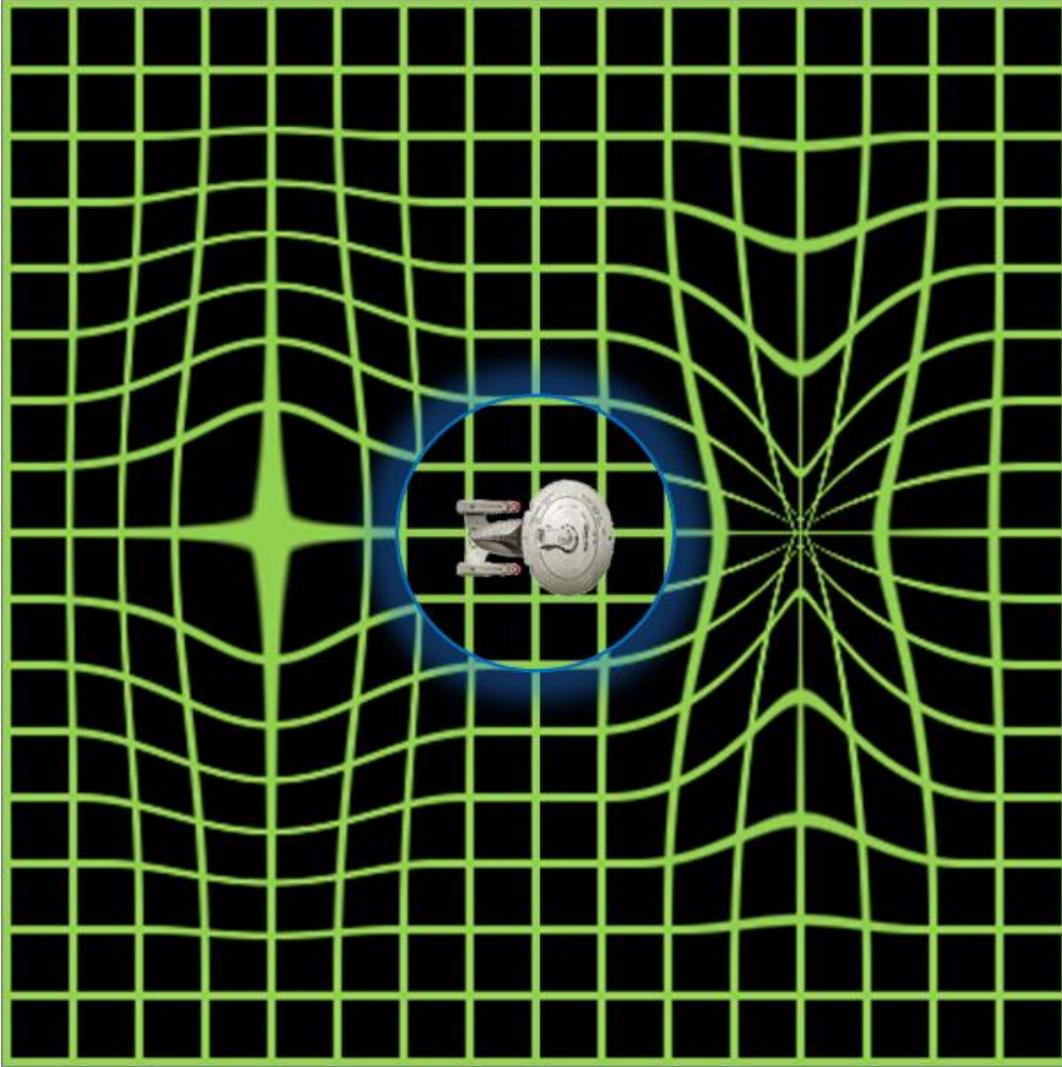
Figure 1: Artistic representation of the Natario Warp Bubble .(Source:Internet)

8   Appendix C:artistic presentation of the Natario warp bubble

According to the Natario definition for the warp drive using the following statement(pg 4 in [2]):

- 1)-Any Natario vector nX generates a warp drive spacetime if $nX = 0$ and $X = vs = 0$ for a small value of rs defined by Natario as the interior of the bubble and $nX = -vs(t)dx$ or $nX = vs(t)dx$ with $X = vs$ for a large value of rs defined by Natario as the exterior of the bubble with $vs(t)$ being the speed of the bubble(pg 5 in [2]).The blue region is the Natario warped region(bubble walls)

A given Natario vector nX generates a Natario warp drive Spacetime if and only if satisfies these conditions stated below:

- 1)-A Natario vector nX being $nX = 0$ for a small value of rs(interior of the bubble)

- 2)-A Natario vector $nX = -Xdx$ or $nX = Xdx$ for a large value of rs(exterior of the bubble)

- 3)-A shift vector X depicting the speed of the bubble being $X = 0$(interior of the bubble) while $X = vs$ seen by distant observers(exterior of the bubble).

The Natario vector nX is given by:

$$nX = -v_s(t)d\left[n(rs)rs^2\sin^2\theta d\varphi\right] \sim -2v_s n(rs)\cos\theta drs + v_s(2n(rs) + rsn'(rs))rs\sin\theta d\theta \qquad (176)$$

$$nX = -v_s(t)d\left[n(rs)rs^2\sin^2\theta d\varphi\right] \sim 2v_s n(rs)\cos\theta drs - v_s(2n(rs) + rsn'(rs))rs\sin\theta d\theta \qquad (177)$$

This holds true if we set for the Natario vector nX a continuous Natario shape function being $n(rs) = \frac{1}{2}$ for large rs(outside the bubble) and $n(rs) = 0$ for small rs(inside the bubble) while being $0 < n(rs) < \frac{1}{2}$ in the walls of the bubble(pg 5 in [2])

The Natario vector $nX = -vs(t)dx = 0$ vanishes inside the bubble because inside the bubble there are no motion at all because $dx = 0$ or $n(rs) = 0$ or $X = 0$ while being $nX = -vs(t)dx \neq 0$ or $nX - vs(t)dx \neq 0$ not vanishing outside the bubble because $n(rs)$ do not vanish.Then an external observer would see the bubble passing by him with a speed defined by the shift vector $X = -vs(t)$ or $X = vs(t)$.

The "spaceship" above lies in the interior of the bubble at the rest $X = vs = 0$ but the observer outside the bubble sees the "spaceship" passing by him with a speed $X = vs$.

See also pgs 7,8 and 9 in [2] for more graphical presentations of the Natario warp bubble

9 Appendix D:The Natario warp drive negative energy density in Cartezian coordinates

The negative energy density according to Natario is given by(see pg 5 in [2])[5]:

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(n'(rs))^2\cos^2\theta + \left(n'(rs) + \frac{r}{2}n''(rs)\right)^2\sin^2\theta\right] \qquad (178)$$

In the bottom of pg 4 in [2] Natario defined the x-axis as the polar axis.In the top of page 5 we can see that $x = rs\cos(\theta)$ implying in $\cos(\theta) = \frac{x}{rs}$ and in $\sin(\theta) = \frac{y}{rs}$

Rewriting the Natario negative energy density in cartezian coordinates we should expect for:

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(n'(rs))^2(\frac{x}{rs})^2 + \left(n'(rs) + \frac{r}{2}n''(rs)\right)^2(\frac{y}{rs})^2\right] \qquad (179)$$

Considering motion in the equatorial plane of the Natario warp bubble (x-axis only) then $[y^2 + z^2] = 0$ and $rs^2 = [(x - xs)^2]$ and making $xs = 0$ the center of the bubble as the origin of the coordinate frame for the motion of the Eulerian observer then $rs^2 = x^2$ because in the equatorial plane $y = z = 0$.

Rewriting the Natario negative energy density in cartezian coordinates in the equatorial plane we should expect for:

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(n'(rs))^2\right] \qquad (180)$$

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[5]$n(rs)$ is the Natario shape function.Equation written in the Geometrized System of Units $c = G = 1$

10 Epilogue

- "The only way of discovering the limits of the possible is to venture a little way past them into the impossible."-Arthur C.Clarke[6]

- "The supreme task of the physicist is to arrive at those universal elementary laws from which the cosmos can be built up by pure deduction. There is no logical path to these laws; only intuition, resting on sympathetic understanding of experience, can reach them"-Albert Einstein[78]

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[6]special thanks to Maria Matreno from Residencia de Estudantes Universitas Lisboa Portugal for providing the Second Law Of Arthur C.Clarke

[7]"Ideas And Opinions" Einstein compilation, ISBN $0-517-88440-2$, on page 226."Principles of Research" ([Ideas and Opinions],pp.224-227), described as "Address delivered in celebration of Max Planck's sixtieth birthday (1918) before the Physical Society in Berlin"

[8]appears also in the Eric Baird book Relativity in Curved Spacetime ISBN $978-0-9557068-0-6$

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