Three-Dimensional Geometry as Carrier of Information about Time*

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A geometry of curved empty space which evolves in time in accordance with Einstein's field equations may be termed a "geometrodynamical history." It is known that such a history can be specified by giving on a 3-dimensional space-like hypersurface ("initial surface") (1) the geometry intrinsic to this surface and (2) the extrinsic curvature of this surface (having to do with how the surface is imbedded, or is to be imbedded, in a yet-to-be-constructed 4-dimensional manifold). However, the intrinsic and extrinsic curvatures of the surface cannot be specified independently, but have to satisfy the initial value equations of Foures and Lichnerowicz (analogous to div $E=0$ and div $B=0$ in electromagnetism). An alternative way of specifying a history is outlined here in which the intrinsic geometry is given *freely* on each of *two* hypersurfaces, and nothing is specified as to the extrinsic curvature of either. In the special case in which the two so-specified 3-geometries are nearly alike—in a sense specified more precisely in the text—a procedure is outlined in order to find the following from Einstein's equations: (1) the invariant space-time interval between an arbitrary point on one surface and a nearby point on the other surface (and thus the 4-geometry interior to the thin sandwich); (2) the extrinsic curvature of the sandwich; hence (via the rest of Einstein's equations) (3) the entire enveloping 4-geometry or geometrodynamical history); and thus finally (4) the time-like separation of the original surfaces and their location in spacetime. In this sense two 3-geometries carry latent information about time.

EXTENDING to general relativity the Lagrangian formulation of mechanics, where one specifies a motion by giving x' at t' and x'' at t'', we consider this problem: Given the geometry ${}^{(3)}\mathcal{G}'$ intrinsic to one 3-space (here called surface) and ${}^{(3)}\mathcal{G}''$ intrinsic to another, find a 4-geometry ${}^{(4)}\mathcal{G}$—if there exists one—which (1) satisfies Einstein's equations—here for simplicity in the source-free form $R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=0$ and (2) reduces on some 3-surface σ' to ${}^{(3)}\mathcal{G}'$ and on another surface to ${}^{(3)}\mathcal{G}''$. In such a ${}^{(4)}\mathcal{G}$ the position of each ${}^{(3)}\mathcal{G}$ is well specified—and evidently by the two intrinsic geometries in question and nothing more.

We have translated into concrete terms this procedure for finding *time* from 3-dimensional *geometry*. We extend the work of Arnowitt, Deser, and Misner[1] (ADM) to obtain *a variational principle for general relativity in which the only quantities specified on the two boundaries are the intrinsic geometries* ${}^{(3)}\mathcal{G}'$ *and* ${}^{(3)}\mathcal{G}''$. Applying this principle to the case of two space-like surfaces of nearly identical geometry, we find *a prescription to determine from these two 3-geometries (1) their time-like separation and (2) their location in space-time*[2]*:*

(1) Specify nearly identical metrics, ${}^{(3)}g'_{ij}(x^1,x^2,x^3)$ and ${}^{(3)}g''_{ij}(x^1,x^2,x^3)$ (the conceptually simpler formulation A). Alternatively (the precise formulation B), give the metric ${}^{(3)}g_{ij}(x^1,x^2,x^3)$ and its rate of change with respect to a parameter x^0. In this case one considers a whole one-parameter family of 3-geometries ${}^{(3)}g_{ij}(x^0;\ x^1,x^2,x^3)$, but instead of picking out two of these (say, $x^0=0$ and $x^0=\Delta x^0=1$) at a finite separation one goes instead to the limit $\Delta x^0\rightarrow 0$.

(2) Fill in between these surfaces an as yet undermined 4-geometry which will allow one to give the separation (formulation A) between a point with coordinates $x^{i'}$ on σ' and a point with coordinates $x^{i''}$ on σ'':

$$ds^2={}^{(3)}g_{\mathrm{av}\,ij}(x^{i''}-x^{i'})(x^{j''}-x^{j'})+2\eta_i(x^{i''}-x^{i'})+(\eta_i\eta^i-\eta_0{}^2). \quad (1)$$

Here ${}^{(3)}g_{\mathrm{av}\,ij}$ is an appropriate average between ${}^{(3)}g'_{ij}$ and ${}^{(3)}g''_{ij}$; $\eta_0=\eta_0(x^1,x^2,x^3)$ gives the proper time interval (cm) between the two 3-surfaces, and $\eta_i(x^1,x^2,x^3)$ gives the sidewise shift between normals to the two 3-surfaces erected at points which have the same coordinate labels. In formulation B one deals, not directly with η_0 and η_i, but with the limits $(\Delta x^0\rightarrow 0)$ of $\eta_\alpha/\Delta x^0\rightarrow N_\alpha$ and writes

$$ds^2={}^{(3)}g_{ij}dx^idx^j+2N_idx^idx^o+(N_iN^i-N_0{}^2)(dx^o)^2. \quad (2)$$

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[1] R. Arnowitt, S. Deser, and C. W. Misner, Phys. Rev. **113**, 745 (1959); **116**, 1322 (1959); **117**, 1595 (1960); **118**, 1100 (1960).

[2] Contrast our conclusion with that of A. Peres and N. Rosen, Nuovo cimento **13**, 430 (1959), who, however, were working in the context of linear theory. We are indebted to Professor P. Bergmann and Professor C. W. Misner for discussions of this work.

In other words, ${}^{(3)}g_{ij}$ and $\partial{}^{(3)}g_{ij}/\partial x^0$ are *specified* on $x^0=0$, whereas varied choices are contemplated for the four functions $N_\alpha(x^1,x^2,x^3)$ (or for the η_α in formulation A), each choice giving rise between the surfaces to a different ${}^{(4)}\mathcal{G}$. That one of these choices is to be taken which is compatible with Einstein's field equations.

(3) A ${}^{(4)}\mathcal{G}$ which extremizes the action integral $I=\int L'(-{}^{(4)}g)^{\frac{1}{2}}d^4x$ subject to fixed ${}^{(3)}\mathcal{G}'$ and ${}^{(3)}\mathcal{G}''$ will satisfy Einstein's field equations—a result not restricted to the case of nearly similar geometries or even, so far as we can judge, to the same topologies on the two 3-surfaces.[3] The integrand in the action differs from the familiar four-dimensional scalar curvature invariant and also from the familiar $\Gamma\Gamma-\Gamma\Gamma$, in each case by a divergence:

$$I=\int L'(-{}^{(4)}g)^{\frac{1}{2}}d^4x=\int_{x^0=x^{0\prime}}^{x^0=x^{0\prime\prime}}\{\pi^{ij}\partial{}^{(3)}g_{ij}/\partial x^0 +N_0[{}^{(3)}g^{\frac{1}{2}}\,{}^{(3)}R-{}^{(3)}g^{-\frac{1}{2}}(\mathrm{Tr}\pi^2-\tfrac{1}{2}(\mathrm{Tr}\pi)^2)] +2N_i\pi^{ij}{}_{|j}\}d^4x. \quad (3)$$

Here $\pi^{ij}=-({}^{(3)}g)^{\frac{1}{2}}(K^{ij}-g^{ij}K)$ is the ADM field momentum in terms of the *extrinsic* curvature

$$K_{ij}\equiv-(1/2N_0)(\partial\,{}^{(3)}g_{ij}/\partial x^0-N_{i|j}-N_{j|i}), \quad (4)$$

of the 3-dimensional slices into which the 4-space has been subdivided. A vertical bar denotes covariant differentation within the intrinsic geometry of the 3-space.

(4) For the case of nearly identical geometries the volume element $(-{}^{(4)}g)^{\frac{1}{2}}d^4x$ becomes $({}^{(3)}g)^{\frac{1}{2}}\eta_0 d^3x$ (formulation A; the translation to formulation B is obvious) and the action becomes

$$I=\int\{\eta_0\,{}^{(3)}R+(1/4\eta_0)[\mathrm{Tr}\kappa^2-(\mathrm{Tr}\kappa)^2]\}\,{}^{(3)}g^{\frac{1}{2}}d^3x \quad (5)$$

with

$$\kappa_{ij}\equiv-2\eta_0K_{ij}\equiv g''_{ij}-g'_{ij}-\eta_{i|j}-\eta_{j|i}. \quad (6)$$

(5) Extremize with respect to the variable which enters purely algebraically and thus obtain *the proper time separation* between ${}^{(3)}\mathcal{G}'$ and ${}^{(3)}\mathcal{G}''$ as was to be found

$$\eta_0=\pm\tfrac{1}{2}[\mathrm{Tr}\kappa^2-(\mathrm{Tr}\kappa)^2]^{\frac{1}{2}}\,{}^{(3)}R^{-\frac{1}{2}}. \quad (7)$$

(6) This result is only usable when the sideways shift $\eta_i(x^1,x^2,x^3)$ is known. To find *it* from ${}^{(3)}\mathcal{G}'$ and ${}^{(3)}\mathcal{G}''$, substitute (7) into (5) and obtain the following action principle:

$$I=\pm\int\{{}^{(3)}R[\mathrm{Tr}\kappa^2-(\mathrm{Tr}\kappa)^2]\,{}^{(3)}g\}^{\frac{1}{2}}d^3x, \quad (8)$$

in which the three η_i are the functions to be varied. From this principle there result three coupled quasilinear equations of the second order for the η_i.

(7) Solve these equations: the analog of solving the equation $\nabla^2A^0=$ (given function) for the scalar potential in electromagnetism. The metric in both problems is 3-dimensional and positive definite. Appropriate boundary conditions on the η_i are essential.

(8) Put the solution for η_i back into (6) and (7), and at last obtain the time separation η_0 in terms of ${}^{(3)}\mathcal{G}'$ and ${}^{(3)}\mathcal{G}''$ (or in formulation B, find N_0 as a functional of ${}^{(3)}g_{ij}$ and $\partial{}^{(3)}g_{ij}/\partial x^0$) as was to be shown.

(9) The previously unknown *extrinsic* curvature, K_{ij}, of the sandwich in the enveloping—and still not known—4-space can now be found from (6). The steps so far generalize the procedure for finding the momentum p in particle mechanics from the coordinates x and $x+dx$ at two nearby times t and $t+dt$. From the way that the extrinsic curvature has been found, one is automatically assured that it will satisfy the four initial-value equations of Lichnerowicz[4] and Foures[5] (the analog of $\mathrm{div}\mathbf{H}=0$ and $\mathrm{div}\mathbf{E}=0$ in electromagnetism). Moreover Einstein's field equations plus regular *one*-surface data K_{ij} and ${}^{(3)}g_{ij}$ consistent with these four equations determine—according to Lichnerowicz and Foures—a 4-geometry which is unique (metric coefficients ${}^{(4)}g_{\alpha\beta}$ determinate up to a coordinate transformation) and free of singularity for a finite proper time into the past and future. Thus not only is the thickness of the thin sandwich from ${}^{(3)}\mathcal{G}'$ to ${}^{(3)}\mathcal{G}''$ determined by ${}^{(3)}\mathcal{G}'$ and ${}^{(3)}\mathcal{G}''$, but also its location in the enveloping ${}^{(4)}\mathcal{G}$ is determinate. This is the sense in which we discover a 3-geometry to be the carrier of information about time in general relativity.

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An account of the wider bearing of these considerations and their applications is being prepared for publication elsewhere.

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[4] A. Lichnerowicz, J. Math. Pure Appl. **23**, 37 (1944); *Théories relativistes de la gravitation et de l'electromagnetisme* (Masson et Cie, Paris, 1955); Helv. Phys. Acta Suppl. **4**, 176 (1956).

[5] Y. Foures (-Bruhat), Acta Math. **88**, 141 (1952); J. Rational Mech. Anal. **4**, 951 (1956).