Conservation Principles in AQUAL

Clara Bradley* and James Owen Weatherall†
Department of Logic and Philosophy of Science
University of California, Irvine

(Dated: January 23, 2024)

We consider conservation of momentum in AQUAL, a field-theoretic extension to Modified Newtonian Dynamics (MOND). We show that while there is a sense in which momentum is conserved, it is only if momentum is attributed to the gravitational field, and thus Newton's third law fails as usually understood. We contrast this situation with that of Newtonian gravitation on a field theoretic formulation. We then briefly discuss the situation in TeVeS, a relativistic theory that has AQUAL as a classical limit.

I. INTRODUCTION

Modified Newtonian Dynamics (MOND) is a heterodox but intriguing proposal to accommodate dark matter phenomenology, particularly at galaxy and cluster scales, by modifying Newtonian dynamics [1]. First introduced by Milgrom in 1983 [2], MOND has long been controversial, for a range of reasons. Here we focus on just one of those controversies; we set aside both other problems for MOND and the empirical support for it.

Among the immediate criticisms of MOND is that it fails to satisfy standard conservation principles, including conservation of momentum and Newton's third law [3]. This feature of the theory represented a major departure from essentially all modern physical theorizing, and even Milgrom took it seriously. The year after introducing MOND, Milgrom and Bekenstein introduced a second theory, called AQUAL ("A QUAdratic Lagrangian"), that they argued reproduced MOND's characteristic predictions in central force problems while also satisfying the conservation principles [4]. Since then, concerns about conservation principles for MOND and its successors have faded into the background, with little attention paid to whether the solution offered by AQUAL is satisfactory.

Here we investigate conservation in AQUAL in more detail, drawing on recent work on conservation of energy and momentum in the field-theoretic formulation of Newtonian gravitation on which AQUAL is based [5]. In section II, we briefly review AQUAL and MOND. Then, in III, we discuss the two arguments that Bekenstein and Milgrom offer for momentum conservation in AQUAL. As we show, AQUAL does conserve momentum, but only if the gravitational field carries momentum. It follows from the form of the AQUAL field equation that momentum is non-local in that it can propagate instantaneously. In section IV, we compare this situation with Newtonian gravitation.

Of course, neither MOND nor AQUAL is relativistic. There is no universally accepted relativistic version of MOND; but one frequently discussed candidate theory is TeVeS (Tensor-Vector-Scalar theory), introduced by Bekenstein in 2004 [6], which recovers MOND-like behavior in the sense that it has AQUAL as a non-relativistic limit. In section V, we discuss what the arguments of this paper look like in TeVeS. We argue that there, too, conservation holds, but it has a different character from general relativity, because energy-momentum must be attributed to the "gravitational" sector.

II. MOND AND AQUAL

For present purposes, MOND can be viewed as a modification of Newtonian gravitational theory according to which the magnitude of the force F exerted by a body of mass M on another body of mass m at a distance r is:

$$F = \frac{GMm}{\mu(\frac{a}{a_0})r^2}, \tag{1}$$

where a is the magnitude of the acceleration of the body of mass m, a_0 is a new constant of nature with dimensions of acceleration, and $\mu(x)$ is an unspecified function with the properties that for $x \gg 1$, $\mu(x) \approx 1$, and for $x \ll 1$, $\mu(x) \approx x$. Thus in the regime where the body experiences large accelerations, relative to a_0, the force is approximately the same as in Newtonian gravitation, whereas in the small acceleration ("deep MOND") regime, the force has a new acceleration dependence. In the case where all acceleration is assumed to be due to gravitation, one can replace a with $\bar{g}$, the MOND acceleration field due to gravity at each point, and write:

$$\bar{g}\mu(\frac{\bar{g}}{a_0}) = \frac{GMm}{r^2} = g \tag{2}$$

where g is the Newtonian gravitational field magnitude.

Investigating Eq. (1) shows that MOND does not satisfy conservation of momentum or Newton's third law of motion. Consider two point masses, m_1 and m_2, at some distance r and with accelerations of magnitudes a_1 and a_2, respectively. Then the magnitude of the force on each body is given by $F_i = \frac{Gm_1m_2}{\mu(\frac{a_i}{a_0})r^2}$, for $i = 1, 2$. If Newton's third law held, it would follow that $\mu(\frac{a_1}{a_0}) = \mu(\frac{a_2}{a_0})$. But also, since $F = ma$ for both bodies, Newton's

* cbradle1@uci.edu
† weatherj@uci.edu

third law would imply $a_1 = \frac{m_2}{m_1} a_2$. Thus we conclude $\mu((\frac{m_2}{m_1})\frac{a_2}{a_0}) = \mu(\frac{a_2}{a_0})$. Assuming $a_2 \neq 0$, for this equality to hold for all values of m_1, m_2, μ must be constant. But this contradicts the requirements on μ set by MOND.

To restore conservation of momentum, Bekenstein and Milgrom [4] introduced AQUAL. To describe AQUAL, it will be convenient to suppose we are working with a classical spacetime structure (M, t_a, h^{ab}, ∇), where M is $\mathbb{R}^4$; t_a is a "temporal metric", i.e., a closed one-form on M; h^{ab} is a "spatial metric", which is a symmetric tensor field such that (1) $h^{ab}t_b = \mathbf{0}$ and (2) h^{ab} induces a flat Riemannian metric on surfaces normal to t_a; and ∇ is a flat, torsion-free covariant derivative operator relative to which both metrics are constant [see 7, §4 for discussion]. Poisson's equation may then be written as $\nabla_a \nabla^a \varphi = 4\pi G\rho$, where φ is a gravitational potential, $g^a = \nabla^a\varphi$ is the gravitational acceleration field, and indices are raised using h^{ab}. It follows via Hamilton's principle from the Lagrangian density,

$$\mathcal{L} = -\rho\varphi - \frac{1}{8\pi G}(\nabla\varphi)^2, \tag{3}$$

where $(\nabla\varphi)^2$ is shorthand for $h^{ab}(\nabla_a\varphi\nabla_b\varphi)$ and, in the first instance, we assume that the mass density ρ is fixed (so that only φ is varied in the extremization problem).

AQUAL is introduced by modifying Eq. (3):

$$\mathcal{L} = -\rho\varphi - \frac{1}{8\pi G} a_0^2 \mathcal{F}[\frac{(\nabla\varphi)^2}{a_0^2}] \tag{4}$$

Here $\mathcal{F}$ is some function such that $\frac{\partial \mathcal{F}(x^2)}{\partial x^2} = \mu(x)$, where $\mu(x)$ is again an unspecified function assumed to have the properties described above. If we assume again that ρ is fixed, then the Euler-Lagrange equations yield the following analogue of Poisson's equation,

$$\nabla_a[\mu(\frac{||\nabla\varphi||}{a_0})\nabla^a\varphi] = 4\pi G\rho, \tag{5}$$

where we have invoked the assumed properties of $\mathcal{F}$ to re-introduce $\mu(x)$ and where $||\nabla\varphi|| = \sqrt{(\nabla\varphi)^2}$. Eq. (5) is the fundamental field equation for the gravitational potential in AQUAL.

III. CONSERVATION OF MOMENTUM IN AQUAL

Bekenstein and Milgrom offer two arguments that momentum is conserved in AQUAL.

A. Argument from Noether's Theorem

They write that "the conservation laws follow from the symmetry of the Lagrangian under spacetime translations and space rotations" [4, p. 9] via Noether's firs theorem. They do not explicitly provide the argument; since field-theoretic approaches to Newtonian gravitation are not often treated in a Lagrangian framework, it will be helpful to present it in more detail. We will focus on linear momentum conservation.

Following the Newtonian case [5], we first define a mass-momentum tensor as:

$$\tilde{T}^a{}_b = \frac{\delta L}{\delta(\nabla_a\varphi)}\nabla_b\varphi + \delta^a{}_b L,$$

where $\delta^a{}_b$ is the identity and L is the "saturated Lagrangian" for some field configuration φ, i.e., the field on spacetime that results from evaluating the Lagrangian functional $\mathcal{L}$ at φ and its derivative. From Noether's theorem and the invariance of the Lagrangian density $\mathcal{L}$ under spatial translations, it follows that

$$\nabla_a \tilde{T}^a{}_b = [\frac{\delta L}{\delta\varphi} - \nabla_a(\frac{\delta L}{\delta\nabla_a\varphi})]\nabla_b\varphi,$$

which vanishes for fields φ that solve the Euler-Lagrange equations, because the Euler-Lagrange equations assert the quantity in brackets vanishes. Thus we find that the mass-momentum tensor as defined is the conserved current associated with spatial translations.

In AQUAL, the mass-momentum tensor becomes:

$$\begin{aligned}\tilde{T}^a{}_b = \frac{1}{4\pi G}\mu(\frac{||\nabla\varphi||}{a_0})\nabla^a\varphi\nabla_b\varphi \\ + \delta^a{}_b(-\rho\varphi - \frac{1}{8\pi G}a_0^2 F[\frac{(\nabla\varphi)^2}{a_0^2}]),\end{aligned} \tag{6}$$

which is guaranteed to be divergence-free by the arguments already given. Thus we find a sense in which momentum is conserved in AQUAL as a consequence of Noether's theorem.

Even so, the conservation law reflected in Eq. (6) is not what one might have expected. It says that the mass-momentum tensor associated with the *gravitational field* is locally conserved, but it says nothing about the momentum of matter. The conservation principles that MOND was shown to violate, meanwhile, concerned the momentum of massive bodies under the influence of gravitation. The behavior of massive matter does not enter into Eq. (6) because we have assumed that the mass density is fixed—an unrealistic assumption. If we relax it, we find that $\tilde{T}^a{}_b$ is no longer 0. Instead:

$$\nabla_a \tilde{T}^a_b = \frac{\delta L}{\delta\rho}\nabla_b\rho = -\varphi\nabla_b\rho = -\nabla_b(\varphi\rho) + \rho\nabla_b\varphi \tag{7}$$

What does this show us? The AQUAL Lagrangian does not describe matter dynamics. Any conservation principle derived from this Lagrangian alone, assuming only that the gravitational potential is a dynamical variable, will establish only that the momentum of the gravitational field is conserved. On the other hand, if we suppose that mass density is not constant, then we should expect momentum to be exchanged between the gravitational field and momentum. If one added a suitable kinetic term to the Lagrangian, one would expect that the total momentum, for matter and the gravitational field, would be divergence-free, but that neither would be conserved alone. This does not establish the conservation principles one expects in Newtonian gravitation.

B. Direct Argument

Bekenstein and Milgrom devote more space to their second argument. We know from the foregoing that the matter mass-momentum, $\hat{T}^a{}_b$, must satisfy $\nabla_a \hat{T}^a{}_b = \nabla_b(\varphi\rho) - \rho\nabla_b\varphi$ in order for total mass-momentum $\tilde{T}^a{}_b + \hat{T}^a{}_b$ to be divergence-free. Thus, if we interpret $h^{ab}(\nabla_n \hat{T}^n{}_b) = F^a$ as a force density, we can directly compute the total change of momentum associated with some isolated system of bodies at a time t by integrating F^a over a compact volume V containing those bodies:

$$\dot{P}^a = \int_V F^a dV = \int_V -\rho\nabla^a\varphi dV \tag{8}$$

where we have neglected the total derivative, since that will give rise to a boundary term. They then argue that if $\dot{P}^a = \mathbf{0}$, one can conclude that total momentum is conserved and Newton's third law holds, since the total change in momentum attributed to material bodies vanishes.

To evaluate this integral, we substitute for ρ using the modified Poisson equation to find:

$$\begin{aligned}\dot{P}^a &= -\frac{1}{4\pi G}\int_V \nabla^a\varphi\nabla_n\left(\mu\nabla^n\varphi\right)\, dV \\ &= -\frac{1}{4\pi G}\int_V \left(\nabla^a\varphi\nabla^n\varphi\nabla_n\mu + \mu\nabla^a\varphi\nabla_n\nabla^n\varphi\right)\, dV\end{aligned} \tag{9}$$

where we have suppressed the argument of $\mu(||\nabla\varphi||/a_0)$.

Eq. (9) can be rewritten as:

$$4\pi G\dot{P}^a = -\int_S \mu n_b \nabla^a\varphi\nabla^b\varphi dS + \frac{{a_0}^2}{2}\int n^a \mathcal{F} dS, \tag{10}$$

where S is the boundary of the region V, n_a is the unit normal to S, and we suppress the argument of $\mathcal{F}((\nabla\varphi)^2/a_0)$. To see this, integrate Eq. (9) by parts and recall that ∇ is torsion-free to find that the right hand side of Eq. (9) is the difference of a divergence and the gradient of $\frac{2}{a_0}\mathcal{F}((\nabla\varphi)^2/a_0)$. Invoking the divergence and gradient theorems then yields Eq. (10).

Bekenstein and Milgrom proceed to argue that the right hand side of Eq. (10) is zero. To do so, they consider the limit as the volume V becomes arbitrarily large. The details depend on additional assumptions about the potential φ, which need not concern us here. What matters is that they conclude that the integrands of both surface integrals increase as $\frac{1}{r^3}$, while the surfaces increase as r^2, from which it follows that as $r \to \infty$, both integrals vanish.

Suppose we grant the assumptions needed for this argument to go through. It follows that the change in total momentum associated with *all of space* vanishes, and thus total momentum is conserved. But this is not the question with which we began. We wished to know whether the momentum associated with massive bodies in an isolated system was conserved. To answer this question, we calculated the change of momentum for a fixed volume V enclosing the system, as in Eq. (8). If the answer were "yes", it would follow that the result would not depend on the volume V chosen, so long as it enclosed the system. But what Bekenstein and Milgrom show is that (a) this integral *does* depend on the volume chosen; and (b) that the total momentum is conserved only when one includes the contribution to momentum from the gravitational field over all of space, i.e., it is *not* conserved among the bodies in the isolated system. Thus, one finds a sense in which momentum is conserved in AQUAL, but only if some momentum is attributed to the gravitational field. Newton's third law is not recovered after all, at least not for the gravitating bodies.

IV. COMPARISON WITH THE NEWTONIAN CASE

It is useful to compare the foregoing with the Newtonian field theory case. Some of the same considerations arise there. For instance, the Newtonian mass-momentum tensor also has a contribution from the gravitational field, and in general only the sum of the gravitational momentum and momentum due to matter is conserved. This is because gravitational influences are mediated by the gravitational field. The arguments in Sec. III A were intended to show only that Noether's theorem does not show that AQUAL has the sort of conservation principles that critics worried about.

Even the arguments in Sec. III B look similar in Newtonian gravitation. Running identical arguments, with the same assumptions, one would conclude that $\dot{P}^a$, computed for larger and larger surfaces, goes as $1/r^2$ as $r \to \infty$. Thus it, too, vanishes in the limit, but only in the limit, with contributions from all of space. More generally, it is not the case that for every solution of Poisson's equation one gets local conservation of momentum. For example, consider the case of a background acceleration field at all points in space. So one might conclude that the situation in AQUAL is no different from Newtonian gravitation after all.

But of course in ordinary (non-field-theoretic) Newtonian gravitation, momentum *is* conserved among bodies in an isolated system and Newton's third law holds. How does this come about in field theoretic formulations? Here is one argument. There exists a class of formal solutions to Poisson's equations, for compactly supported mass density ρ, with Green's functions that vanish at spatial infinity. Relative to an arbitrarily chosen origin, these may be written:

$$\varphi(\mathbf{r}) = G \int \frac{\rho(\mathbf{r}')}{||\mathbf{r} - \mathbf{r}'||} d^3\mathbf{r}' \tag{11}$$

where $\mathbf{r}$ and $\mathbf{r}'$ are vectors based at an arbitrarily chosen origin and G is Newton's constant.

For such fields, $\dot{P}^a$, computed over a sphere V centered at the origin and containing all of the support of ρ, becomes

$$\int_V \rho(\mathbf{r})\nabla\varphi(\mathbf{r})d^3\mathbf{r} = G \int_V \rho(\mathbf{r})\rho(\mathbf{r}')\frac{-(\mathbf{r} - \mathbf{r}')}{||\mathbf{r} - \mathbf{r}'||^3} d^3\mathbf{r}' d^3\mathbf{r} \tag{12}$$

Now observe that the integrand is antisymmetric in the position variables, but the order of integration can be swapped. Thus we find that $\dot{P}^a = -\dot{P}^a$, and so $\dot{P}^a = 0$, independent of the radius of the volume over which the integral is taken.

Could similar argument go through in AQUAL? Though one cannot derive a Green's function without specifying a form for μ, heuristically one would expect not. The reason is that the the corresponding integrand would be expected to depend on $1/\mu$, which would generically break the anti-symmetry between $\mathbf{r}$ and $\mathbf{r}'$.

V. CONCLUSION

We have assessed the claim that AQUAL restores the conservation principles that MOND lacked. We showed that the arguments of Bekenstein and Milgrom do establish a sense of momentum conservation in AQUAL, but we also showed that they do not recover conservation of momentum between bodies in an isolated system or Newton's third law; instead, conservation of momentum is restored only if one attributes momentum to the gravitational field over all of space. In this sense, momentum conservation is non-local in AQUAL.

We conclude by discussing two possible responses to these arguments. The first is that there is nothing surprising about AQUAL here. After all, it is a field theory, and we know that in other field theories, such as electromagnetism, the fields carry momentum. This is a fair remark, but there are two points to emphasize. First, it is still the case that AQUAL does not restore the Newtonian conservation principles; it requires a novel interpretation of the gravitational field. Second, in electromagnetism, the momentum-carrying fields satisfy hyperbolic equations, so that their momentum propagates at a finite speed. This recovers a sense in which momentum propagation is local, even though fields carry momentum. This is not the case in AQUAL. Momentum propagates instantaneously, just as in Newtonian gravitation. And yet not all momentum changes can be attributed to the bodies.

The second response is to say that AQUAL is a non-relativistic theory, and some of the odd features of the theory are likely to go away in a fully relativistic formulation. It is true that in relativistic versions of MOND, such as Bekenstein's TeVeS [6], the same pathologies do not arise, and indeed, one would not expect quite the same behavior, since even in general relativity, momentum is not conserved among (distant) bodies.

Nonetheless, there are hints of AQUAL in full TeVeS. That theory is a bimetric theory, with a dynamical Lorentzian metric and a conformally transformed "physical metric" that couples to matter. Thus we have two different definitions of the stress-energy tensor for matter fields, and two different possible conservation laws. Applying standard arguments, one can show that the physical stress-energy tensor (the functional derivative of the matter Lagrangian density with respect to the physical metric) is divergence-free relative to the physical metric's Levi-Civita derivative operator. But from the perspective of the background metric, neither this stress-energy nor the "background" stress-energy is divergence-free; instead, in both cases, conservation is restored only if one includes additional terms depending on the scalar field generating the conformal transformation between the metrics. Thus we see a certain sense in which a "gravitational stress-energy" contributes to stress-energy conservation. It is this behavior that ultimately gives rise to the modified Poisson equation in the classical limit.

ACKNOWLEDGMENTS

This material is partially based upon work produced for the project "New Directions in Philosophy of Cosmology", funded by the John Templeton Foundation under grant number 61048.

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