Hyperbolic Motion in Curved Space Time

W. RINDLER
Department of Mathematics, Cornell University, Ithaca, New York
(Received May 2, 1960)

The differential equations of motion for a test particle moving with uniform acceleration in a curved space time are proposed. They are obtained by generalizing the differential-geometric characteristics of a rectangular hyperbola in Minkowski space time. The problem is proposed, though not solved, of deriving these equations of motion from the field equations of general relativity. However, it is suggested that they also hold independently of general relativity in cosmological space times based on the Robertson-Walker metric. The equations are solved in detail for the particular case of de Sitter space time, which is relevant to the steady-state theory. It is found, *inter alia*, that in this space time a particle moving radially with uniform acceleration ultimately moves with constant relative velocity through the substratum; that there is a critical first fundamental particle (galaxy) on its line of motion which it will never overtake; that, in turn, a light signal emitted at or after a certain critical time will not catch up with the accelerating particle; and that, if a particle with a given available acceleration α passes beyond a certain proper distance (the α horizon) it can no longer return to its place of origin. Possible applications to intergalactic rocketry are examined.

1. INTRODUCTION

IN the special theory of relativity the term "hyperbolic motion" is commonly applied to the motion of a test particle moving with constant proper acceleration along a straight line in a suitable Galilean frame of reference. (Proper acceleration = acceleration relative to the instantaneous Galilean rest frame.) Hyperbolic motion was first noted by Minkowski[1] and was further studied by Born,[2] who also coined its name. This name derives from the fact that the plot of distance against time is a rectangular hyperbola [see Eq. (9) below]. By the same terminology the classical motion with constant acceleration is "parabolic."

The problem discussed in the present paper is the generalization of the concept of hyperbolic motion to a general space time. The immediate motivation for this work was a paper by McMillan[3] which discusses some aspects of intergalactic rocket travel by use of the special relativistic formulas for hyperbolic motion. But it is evident that for such problems on the cosmological scale the special theory of relativity can, in general, only furnish answers that are at best approximate. One exception occurs in the cosmological theory of Milne[4] which is based strictly on special relativity and in which McMillan's calculations, if correctly interpreted, hold exactly. In general cosmological space times, however, a more general treatment is needed. Such expected effects as the inability of the space traveller to return to earth after crossing the cosmological horizon (or rather a nearer point, depending on the available acceleration) are, even qualitatively, beyond the scope of the special theory of relativity.

From a theoretical point of view the problem is by no means without interest. Indeed, on the basis of the general theory of relativity, the problem seems extraordinarily difficult. The intricate calculations necessary for obtaining the geodesic path law for *free* test particles from the field equations are well known.[5] To make allowance for the changes caused by a firing rocket in the surrounding field would seem to be a prodigious task, though it should theoretically be possible to derive the equations of motion in that way. The principle of equivalence by itself is certainly insufficient for the purpose, just as it is insufficient in the case of "free" motion: innumerable paths are consistent with it. But we can, nevertheless, proceed from the principle of equivalence with the help of a simplicity requirement, as was originally done also by Einstein[6] in order to justify his geodesic law. In this way we are led to unique differential equations for the path. It would be very surprising if these equations turned out, on deeper analysis, to be inconsistent with the field equations which themselves arise from the principle of equivalence by a simplicity requirement.

Even if the differential equations proposed below could be deduced from the field equations of general relativity, we should still have to merely postulate them in the case of cosmological theories not based on either general or special relativity. It is well known from the work of Robertson[7] and Walker[8] that all possible cosmological theories using homogeneous and isotropic world models share with general relativity the form of the line element and most of its properties [see Eq. (18) below], and I believe my results are applicable to such theories. I am not aware of alternative methods of deduction in any such theory, nor, on the other hand, of any conflict of the proposed equations with the axioms of such theories.

My equations turn out to be equivalent to one of the definitions of "uniform acceleration" proposed by Marder.[9] Marder also proposed another definition

[1] H. Minkowski, Physik. Z. **10**, 104 (1909).
[2] M. Born, Ann. Physik **30**, 1 (1909), Sec. 5.
[3] E. M. McMillan, Science **126**, 381 (1957).
[4] E. A. Milne, Nature **130**, 9 (1932); *Kinematic Relativity* (Oxford University Press, New York, 1948).
[5] V. Fock, *The Theory of Space, Time, and Gravitation* (Pergamon Press, New York, 1959), Chap. VI and references 28–42.
[6] A. Einstein, Ann. Physik **49**, 769 (1916), Sec. 13.
[7] H. P. Robertson, Astrophys. J. **82**, 284 (1935).
[8] A. G. Walker, Proc. London Math. Soc. **42**, 90 (1937).
[9] L. Marder, Proc. Cambridge Phil. Soc. **53**, 194 (1957).

which, however, I find does not reduce to hyperbolic motion in flat space time. I comment on Marder's work in an Appendix.

2. HYPERBOLIC MOTION IN MINKOWSKI SPACE TIME

For our later purpose it is necessary briefly to rederive the standard equations of hyperbolic motion in special relativity by a slightly novel method. The 4-velocity U^μ (Greek suffixes range from 1 to 4) and the 4-acceleration A^μ of a moving particle, which has coordinates $x^\mu=(x,y,z,t)$ in a given Galilean reference frame, are defined by the equations

$$U^\mu=dx^\mu/d\tau,\quad A^\mu=dU^\mu/d\tau=d^2x^\mu/d\tau^2, \tag{1}$$

where τ is proper time. Throughout this paper we shall assume that the units of length are chosen so as to make the speed of light unity. Then

$$d\tau^2=-(dx^2+dy^2+dz^2)+dt^2, \tag{2}$$

or

$$d\tau^2=g_{\mu\nu}dx^\mu dx^\nu,\quad (-g_{11}=-g_{22}=-g_{33}=g_{44}=1;\ g_{\mu\nu}=0,\quad \mu\neq\nu). \tag{3}$$

Primes will be used throughout to denote differentiation with respect to τ. Then from (2), as usual,

$$t'=(1-v^2)^{-\frac{1}{2}}=\gamma, \tag{4}$$

where v is the speed of the particle and γ its Lorentz factor. If $\mathbf{v}$ is the 3-velocity of the particle, we have, from (1) and (4),

$$U^\mu=\frac{dx^\mu}{dt}t'=\gamma(\mathbf{v},1),\quad A^\mu=\gamma\frac{d}{dt}(\gamma\mathbf{v},\gamma), \tag{5}$$

whence we can easily compute the squared magnitude $(A)^2$ of A^μ.[10] In the special case of a particle moving rectilinearly in a given Galilean frame we find

$$(A)^2=g_{\mu\nu}A^\mu A^\nu=-\gamma^6(dv/dt)^2. \tag{6}$$

If we denote by α the proper acceleration of the particle, we see from (6) that, in the rest frame, $(A)^2=-\alpha^2$. But $(A)^2$ is an invariant and thus in all Galilean reference frames we have

$$(A)^2=g_{\mu\nu}A^\mu A^\nu=-\alpha^2. \tag{7}$$

Now assume that the particle moves with constant proper acceleration α from rest at the origin at time zero along the positive x axis. On equating the right members of (6) and (7) we find, successively,

$$\frac{d}{dt}(\gamma v)=\alpha,\quad t=\frac{\gamma v}{\alpha},\quad v=\frac{\alpha t}{(1+\alpha^2t^2)^{\frac{1}{2}}}. \tag{8}$$

[10] W. Rindler, *Special Relativity* (Oliver and Boyd, Edinburgh, and Interscience Publishers Inc., New York, 1960), Sec. 23.

One further integration yields the equation of motion

$$\alpha x^2+2x-\alpha t^2=0, \tag{9}$$

which represents a rectangular hyperbola with asymptotes $(x+1/\alpha)=\pm t$ in the (x,t) plane.

Evidently, light signals emitted at the origin at or after $t=1/\alpha$ never reach the receding particle. Note also that any particle P which moves uniformly along the x axis meets the particle Q performing hyperbolic motion either not at all or twice, and then with the same relative speed on both occasions. This is obvious when we consider that in the rest frame of P the plot of Q's path is also a rectangular hyperbola and that the plot of P's path is a straight line perpendicular to the axis of the hyperbola.

Consider now the simplest of all modern cosmological models, namely that due to Milne.[4] According to Milne, all fundamental particles (galaxies) originally occupy a small volume in Minkowski space time and are then suddenly shot out in all directions with all speeds short of the speed of light. Each fundamental observer (i.e., one attached to a galaxy) sees himself at the center of a sphere of galaxies whose unattained boundary expands at the speed of light. Consider a rocket performing radial hyperbolic motion in this model. By reference to an (x,t) diagram it is graphically obvious that the rocket will eventually overtake every given fundamental particle on its line of motion, no matter how small α may be. Moreover, it will overtake the fundamental particles with ever increasing relative velocity, as follows easily from the second remark of the last paragraph. But these two properties are by no means characteristic of hyperbolic motion in all cosmological models, as we shall see.

The following formulas, easily deducible from Eqs. (4), (8), and (9), will be needed later. If τ denotes the proper time elapsed at the moving particle and τ and t vanish together, we find

$$\begin{aligned} &v=\tanh\alpha\tau, &&\gamma=\cosh\alpha\tau,\\ &t=(1/\alpha)\sinh\alpha\tau, &&x=(1/\alpha)(\cosh\alpha\tau-1). \end{aligned} \tag{10}$$

3. HYPERBOLIC MOTION IN GENERAL SPACE TIME

The term "hyperbolic motion" is very suggestive. For it draws attention to the purely geometric aspect of the world line of a uniformly accelerating particle, and this lends itself to immediate generalization. What characterizes a rectangular hyperbola in the (x,t) plane of Minkowski space time, as a curve in that 4-space, is that it is a timelike plane curve of constant curvature. (In Euclidean space with positive definite metric a curve so characterized would be a circle. We recall that Sommerfeld[11] referred to hyperbolic motion as "circular" motion.) And the properties of plane-ness and constant

[11] A. Sommerfeld, Ann. Physik 33, 670 (1910).

curvature in flat 4-space have well-defined generalizations in curved 4-space, which we shall now discuss.

Let us begin by recalling the following standard formulas[12] of the differential geometry of twisted curves in Euclidean 3-space:

$$\mathbf{t}=d\mathbf{r}/ds, \quad \kappa\mathbf{n}=d\mathbf{t}/ds, \quad \omega\mathbf{b}=d\mathbf{n}/ds+\kappa\mathbf{t}, \tag{11}$$

where κ, ω, s, $\mathbf{r}$, $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$ denote curvature, torsion, arc length, position vector, unit tangent, unit normal, unit binormal, respectively. (The last two equations are two of the three so-called *Serret-Frenet* formulas.) These formulas have analogs in the Riemannian geometry of n dimensions.[13] In particular, in the space times of relativity and in Robertson-Walker cosmological space time, referred to a metric

$$d\tau^2=g_{\mu\nu}dx^\mu dx^\nu,$$

we define the generalized velocity and acceleration vectors for a particle having world line $x^\mu=x^\mu(\tau)$ as follows:

$$U^\mu=\frac{dx^\mu}{d\tau}, \quad A^\mu=\frac{DU^\mu}{d\tau}=\frac{d^2x^\mu}{d\tau^2}+\Gamma_{\nu\sigma}{}^\mu\frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau}, \tag{12}$$

where $DU^\mu/d\tau$ denotes the absolute derivative of U^μ, and the $\Gamma_{\nu\sigma}{}^\mu$ are Christoffel symbols of the second kind.[14] Comparison of (12) with (11) shows that τ along the world line corresponds to arc length, and U^μ and A^μ correspond to $\mathbf{t}$ and $\kappa\mathbf{n}$, respectively. Consequently the magnitude of A^μ corresponds to the curvature of the world line, and, by reference to (7) and use of the equivalence principle, this magnitude is seen to be $i\alpha$, α being the proper acceleration. Analogously to (11) (*iii*) there is defined in Riemannian geometry[13] a torsion Ω (or second curvature) and unit binormal B^μ (or second normal) by the equations

$$\Omega B^\mu=\frac{D}{d\tau}\left(\frac{A^\mu}{i\alpha}\right)+i\alpha U^\mu.$$

Writing β^μ for $i\Omega B^\mu$, these equations become

$$\beta^\mu=\frac{D}{d\tau}\left(\frac{A^\mu}{\alpha}\right)-\alpha U^\mu. \tag{13}$$

In the differential geometry of twisted curves it is well known that the vanishing of the curvature of a curve implies that the curve is a straight line, and the vanishing of the torsion implies that it is a plane curve. Let us call the left members of Eqs. (11) (*ii*) and (11) (*iii*) the curvature vector and torsion vector, respectively. In the Euclidean 3-space in which these vectors are defined, the vanishing of their magnitudes implies the vanishing of each of their components. This is not the case in spaces with indefinite metric unless certain reality conditions are imposed.[15] The true generalization of the characteristics of straight lines and plane curves are obtained not by requiring the scalar curvature or torsion to vanish, respectively, but by requiring the curvature or torsion *vector* to vanish component-wise. The vanishing of the "curvature vector" A^μ is the well-known characteristic of a geodesic path. The analog of a plane (i.e., torsionless) curve is one whose "torsion vector" ΩB^μ vanishes, which is equivalent to β^μ vanishing. Thus the analog of a plane curve of constant curvature will be characterized by the differential equations

$$\beta^\mu=\frac{D}{d\tau}\left(\frac{A^\mu}{\alpha}\right)-\alpha U^\mu=0, \quad -\alpha^2=g_{\mu\nu}A^\mu A^\nu=\text{constant}. \tag{14}$$

It turns out that any curve so characterized will be timelike along its entirety if the initial direction is chosen timelike. This we shall show presently. At first sight it seems that we have too many equations to determine the path. However, the β^μ are not all independent; they automatically satisfy the two identities

$$g_{\mu\nu}U^\mu\beta^\nu=0, \quad g_{\mu\nu}A^\mu\beta^\nu=0, \tag{15}$$

as can easily be verified. (It must be remembered that $g_{\mu\nu}U^\mu U^\nu=1$, whence, by successive differentiation, $g_{\mu\nu}U^\mu A^\nu=0$ and $g_{\mu\nu}U^\mu DA^\nu/d\tau=-g_{\mu\nu}A^\mu A^\nu$.) Therefore only two of the four β^μ are independent. With (14) (*ii*) and the given metric of the space there are then exactly the right number of equations to determine the four unknowns $x^\mu(\tau)$.

The two equations (14) are completely equivalent to the equations

$$DA^\mu/d\tau=\alpha^2U^\mu, \quad \alpha=\text{constant}, \tag{16}$$

where the significance (14) (*ii*) of α is not *a priori* assumed (though it is implied). For, any curve satisfying (14) satisfies (16). And the converse is also true: multiplying (16) (*i*) by $g_{\mu\nu}U^\nu$ and using the parenthesized remark following (15), we find $-g_{\mu\nu}A^\mu A^\nu=\alpha^2$. This is constant by (16) (*ii*), whence (14) (*i*) follows.

[12] D. J. Struik, *Classical Differential Geometry* (Addison-Wesley Publishing Company, Reading, Massachusetts, 1950), Chap. I, Eqs. (2–7), (6–1).

[13] J. L. Synge and A. Schild, *Tensor Calculus* (University of Toronto Press, Toronto, 1949), Sec. 2.7.

[14] See reference 13, Sec. 2.5.

[15] It is shown in reference 10 that in Minkowski space time $(A)^2$ is related to the space curvature κ of the world line by the equation $-(A)^2=\gamma^6\dot{v}^2+\gamma^4v^4\kappa^2$, where $\dot{v}$ is the time derivative of the speed. If we drop the requirement $v<1$, it is easy to specify nongeodesic paths for which $(A)^2=0$. Examples are provided by any path satisfying $v=\cosh at$, $\kappa=a\,\mathrm{sech}^2 at$, $a=$constant. It is nevertheless true that, for any *timelike* portion of the path, the condition $(A)^2=0$ implies $A^\mu=0$. For A^μ, being orthogonal to U^μ, is necessarily spacelike over such a portion and the statement follows. But it is simpler to use the equations $A^\mu=0$ than the equivalent equations $(A)^2=0$, $g_{\mu\nu}dx^\mu dx^\nu>0$, for the free motion of a real particle. (The condition that the path be timelike is still necessary, but, as is well known, this is an automatic consequence of the equations $A^\mu=0$ if the initial direction is chosen timelike.) In the same way, the vanishing of the magnitude of β^μ implies $\beta^\mu=0$ only along timelike portions of the world line [see Eq. (15)(*i*)].

Equations (16) possess an obvious first integral,

$$U^\mu=(\cosh\alpha\tau)L^\mu+(\sinh\alpha\tau)M^\mu, \tag{17}$$

where $DL^\mu/d\tau=DM^\mu/d\tau=0$. Moreover, since at $\tau=0$ we have $U^\mu=L^\mu$ and $A^\mu=\alpha M^\mu$, it follows that L^μ and M^μ must be orthogonal unit vectors, timelike and spacelike, respectively, for a real particle path. It is also evident from (17) that the relevant world line must stay timelike if it is timelike initially, no matter what constant value we give to α.

We now assert that a particle having uniform acceleration in a general space time will have a timelike world line satisfying the differential equations (14) [or, equivalently, the differential equations (16)]. Firstly, these equations are certainly invariant. Secondly, in Minkowski space time they yield hyperbolic motion in the accepted sense. For if, in (17), we set $L^\mu=(0,0,0,1)$ and $M^\mu=(1,0,0,0)$ and integrate, we find

$$(x,y,z,t)=\alpha^{-1}(\cosh\alpha\tau-1,0,0,\sinh\alpha\tau),$$

which agrees with Eqs. (9) and (10). Thirdly, we have evidently chosen the simplest generalization from Minkowski space time to general space time. If we wish to avoid involving the curvature tensor, the generalization is unique. This completes our justification of the proposed equations.

Two important limiting cases may be noted. Although, as we have seen, the condition $\alpha=0$ is insufficient to ensure a geodesic path, any curve satisfying Eqs. (14) becomes a geodesic as $\alpha\to0$. For, on differentiating (17) absolutely, we find

$$A^\mu=\alpha[(\sinh\alpha\tau)L^\mu+(\cosh\alpha\tau)M^\mu],$$

and therefore in the limit, as $\alpha\to0$, we get $A^\mu=0$, the conditions for a geodesic path. If, on the other hand, we let $\alpha\to\infty$, the hyperbolic path becomes a null geodesic (light path). That this is true in Minowski space time is evident from Eq. (9). That it is also true in general space times can be seen as follows: reference to (17) shows that, as $\alpha\to\infty$, the direction of U^μ becomes $L^\mu+M^\mu$, i.e., a null direction; and since this direction is transported parallelly along the path, the path must be a geodesic.

As a simple illustration of hyperbolic motion consider the Schwarzschild metric

$$d\tau^2=-\xi^{-1}dr^2-r^2(d\theta^2+\sin^2\theta d\phi^2)+\xi dt^2,$$

where $\xi=1-2m/r$. It is easy to demonstrate the intuitively expected fact that a particle with fixed spatial coordinates in this metric executes hyperbolic motion. Setting $r,\theta,\phi=$constant, we find $U^\mu=(0,0,0,\xi^{-\frac{1}{2}})$. From (12) we then find $A^\mu=\Gamma_{44}{}^\mu\xi^{-1}$. The only nonvanishing $\Gamma_{44}{}^\mu$ for our metric[16] is $\Gamma_{44}{}^1=\xi m/r^2$, whence, from (14) (*ii*), $\alpha=m/\xi^{\frac{1}{2}}r^2$. The condition (14) (*i*) also holds. Evidently α equals the 3-force per unit mass experienced by the particle. The fact that the essential information comes from (14) (*ii*) and that (14) (*i*) is automatically satisfied is characteristic of all applications to axially symmetric situations.

[16] R. C. Tolman, *Relativity, Thermodynamics, and Cosmology* (Oxford University Press, New York, 1934), Eqs. (95.2).

4. HYPERBOLIC MOTION IN DE SITTER SPACE TIME

At the moment, the main interest of Eqs. (14) probably lies in the field of cosmology. We therefore propose to find the form which these equations take in the case of world models based on the Robertson-Walker metric, and to proceed to an integration of the equations only in the still more particular case of de Sitter space time. This is the space time relevant to the steady-state theory and as such constitutes an important special case. It also has the didactic merit that in it all the necessary integrations can be performed in terms of elementary functions.

The Robertson-Walker metric is defined by

$$\begin{aligned} d\tau^2&=-e^u[dr^2+r^2(d\theta^2+\sin^2\theta d\phi^2)]+dt^2,\\ e^u&=R^2(t)(1+kr^2/4)^{-2}, \end{aligned} \tag{18}$$

and it has been shown to be applicable to all homogeneous and isotropic world models.[7,8] It has the following significance: (*i*) t is a cosmic time coordinate; (*ii*) θ, ϕ are the usual angular measurements made on incoming light rays at the spatial origin $r=0$ (which can be identified with *any* fundamental particle); (*iii*) the world lines of fundamental particles are the geodesics $r,\theta,\phi=$constant, whence r is a "co-moving" radial coordinate; (*iv*) light tracks correspond to the null geodesics of the metric and, in particular, those through the spatial origin have the equations $\theta,\phi=$constant, and

$$dt/R(t)=\pm dr/(1+kr^2/4), \tag{19}$$

the positive sign evidently corresponding to light travelling in the direction of increasing r. Additional hypotheses are needed before a particular form can be assigned to the scale function $R(t)$ and a particular value (1, −1, or 0) to the curvature index k, and these are supplied by the various cosmological theories. Furthermore, the hypothesis of isotropy and homogeneity alone does not imply that free particles, other than the fundamental particles, have geodesic paths.

Tolman[17] gives the Christoffel symbols of the second kind for a metric somewhat more general than (18) above, taking $x^\mu=(r,\theta,\phi,t)$. For our purposes we shall need only those $\Gamma_{\nu\sigma}{}^\mu$ in which neither ν nor σ is 2 or 3. From Tolman's list we find that the only nonvanishing Γ's of this kind for the metric (18) are the following:

$$\Gamma_{11}{}^1=\tfrac{1}{2}\partial u/\partial r,\quad \Gamma_{11}{}^4=\tfrac{1}{2}e^u\partial u/\partial t,\quad \Gamma_{14}{}^1=\Gamma_{41}{}^1=\tfrac{1}{2}\partial u/\partial t. \tag{20}$$

We shall restrict our discussion to purely radial motions ($\theta,\phi=$constant).[18] From symmetry considera-

[17] See reference 16, Eqs. (98.5).

[18] In contrast to geodesic motions in a space (18), which are always radial with respect to a suitably chosen origin, hyperbolic

tions it is clear that all such motions satisfy Eqs. (14) (*i*) automatically. Thus the conditions for hyperbolic motion reduce to (14) (*ii*). From (12) (*ii*) and (20) we easily calculate the components of A^μ for a purely radial motion:

$$A^1=\frac{d^2r}{d\tau^2}+\frac{1}{2}\frac{\partial u}{\partial r}\left(\frac{dr}{d\tau}\right)^2+\frac{\partial u}{\partial t}\frac{dr}{d\tau}\frac{dt}{d\tau},$$
$$A^2=A^3=0, \qquad (21)$$
$$A^4=\frac{d^2t}{d\tau^2}+\frac{1}{2}e^u\frac{\partial u}{\partial t}\left(\frac{dr}{d\tau}\right)^2.$$

Reading off the metric coefficients $g_{\mu\nu}$ from (18) (*i*), and substituting the A^μ from (21), we find that Eq. (14) (*ii*) now takes the following form:

$$-e^u\left[\frac{d^2r}{d\tau^2}+\frac{1}{2}\frac{\partial u}{\partial r}\left(\frac{dr}{d\tau}\right)^2+\frac{\partial u}{\partial t}\frac{dr}{d\tau}\frac{dt}{d\tau}\right]^2+\left[\frac{d^2t}{d\tau^2}+\frac{1}{2}e^u\frac{\partial u}{\partial t}\left(\frac{dr}{d\tau}\right)^2\right]^2=-\alpha^2. \qquad (22)$$

This is true for a radial motion with constant proper acceleration α in the space time whose metric is (18). In a particular model we must substitute in (22) the particular function u.

We now specialize our discussion to de Sitter space time, which is characterized by the metric

$$d\tau^2=-e^{2pt}[dr^2+r^2(d\theta^2+\sin^2\theta d\phi^2)]+dt^2, \qquad (23)$$

where p is Hubble's constant. This is evidently a particular case of (18), with $R(t)=e^{pt}$ and $k=0$, i.e., $u=2pt$. For a radial path we deduce from (23) that

$$r'^2=e^{-2pt}(t'^2-1), \qquad (24)$$

where a prime, here and in the sequel, again denotes differentiation with respect to τ. Differentiating (24) logarithmically we get

$$r''/r'=-pt'+t't''/(t'^2-1). \qquad (25)$$

By means of this and the preceding relation, r can be eliminated from (22): taking out a factor r' from the first bracket and using (24) and (25), we find

$$t'^2(t'^2-1)[t''(t'^2-1)^{-1}+p]^2-[t''+p(t'^2-1)]^2={}^2\alpha. \qquad (26)$$

motions may or may not be so, depending on the initial conditions. It is well known that any two fundamental particles in a homogeneous and isotropic substratum determine a "linear equivalence" of fundamental particles (which can be repeatedly traversed by light signals). Every radial motion is confined to such a linear equivalence. Thus a motion containing three particles which do not belong to the same linear equivalence cannot be radial with respect to *any* origin. It is easily seen that there are hyperbolic motions of this type. One has only to recall that every substratum (18) is locally Minkowskian and that slow hyperbolic motions are locally parabolic: a spatially parabolic trajectory through an almost static and flat substratum can evidently not be duplicated by a light signal.

The substitution

$$t'=\cosh z \qquad (27)$$

reduces this differential equation to the form

$$z'+p\sinh z=\alpha, \qquad (28)$$

and thus we find

$$\int(\alpha-p\sinh z)^{-1}dz=\tau+\text{constant}. \qquad (29)$$

It is convenient at this stage to introduce the following notation, to which we shall adhere throughout:

$$\alpha^2+p^2=q^2, \quad q+p=S, \quad q-p=D, \qquad (30)$$

and we note that $SD=\alpha^2$. Integration of (29) now yields

$$\frac{S+\alpha\tanh(z/2)}{D-\alpha\tanh(z/2)}=Ae^{q\tau}, \qquad (31)$$

where A is an arbitrary constant of integration. From Eq. (4) and use of the the equivalence principle, it follows that t' represents the Lorentz factor γ of the accelerating particle relative to the substratum. We shall now suppose that the particle leaves the origin from rest when $t=\tau=0$. Then, from (27), $z=0$ at the beginning of the motion and, from (31), $A=S/D$. Thus, since from (27)

$$t'=\cosh z=\frac{1+\tanh^2(z/2)}{1-\tanh^2(z/2)},$$

we find from (31) that

$$\gamma=t'=q(Se^{2q\tau}+D)\Psi^{-1},$$
$$\Psi=pSe^{2q\tau}+2SDe^{q\tau}-pD. \qquad (32)$$

Making the substitution $E=e^{q\tau}$, integrating, changing back to $e^{q\tau}$, and finally adjusting the constant of integration, we obtain the relation

$$t=p^{-1}\ln(\Psi/2q^2e^{q\tau}). \qquad (33)$$

Next we propose to find r as a function of τ, by use of (24). For this purpose we obtain from (33) the relation

$$e^{-pt}=2q^2e^{q\tau}\Psi^{-1}, \qquad (34)$$

and from (27) the relation

$$(t'^2-1)^{\frac{1}{2}}=\alpha(Se^{2q\tau}-2pe^{q\tau}-D)\Psi^{-1}. \qquad (35)$$

Substituting these expressions in (24) and taking the positive square root for motion in the direction of increasing r, we get

$$r'=2q^2\alpha e^{q\tau}(Se^{2q\tau}-2pe^{q\tau}-D)\Psi^{-2}. \qquad (36)$$

We again make the substitution $E=e^{q\tau}$, integrate, change back to $e^{q\tau}$, and adjust the constant of integration; thus we finally obtain

$$r=\alpha(e^{q\tau}-1)^2\Psi^{-1}. \qquad (37)$$

The relative local velocity at which the particle moves through the substratum is given by

$$v=(1-\gamma^{-2})^{\frac{1}{2}}=(1/t')(t'^2-1)^{\frac{1}{2}},$$

which, by use of (32) and (35), becomes

$$v=\frac{\alpha}{q}\left(\frac{Se^{2q\tau}-2pe^{q\tau}-D}{Se^{2q\tau}+D}\right). \tag{38}$$

Now we can draw some conclusions. Note that τ is the time measured by a standard clock fixed to the accelerating particle while t is the cosmic time measured by standard clocks properly synchronized and fixed to the fundamental particles. At the beginning of the motion r, t, τ are all zero. Equation (33) now shows that t and τ become infinite together. From (38) we see that, as $\tau\rightarrow\infty$, v approaches the limiting value α/q, which means that the accelerating particle ultimately moves through the substratum with the constant relative velocity

$$v_\infty=\alpha/q=\alpha(p^2+\alpha^2)^{-\frac{1}{2}}. \tag{39}$$

The corresponding Lorentz factor γ_∞ can be obtained directly from (39), or by letting $\tau\rightarrow\infty$ in (32); we find

$$\gamma_\infty=q/p=p^{-1}(p^2+\alpha^2)^{\frac{1}{2}}. \tag{40}$$

From Eq. (37) it follows that the accelerating particle approaches asymptotically, but never overtakes, the fundamental particle with radial coordinate

$$r_\infty=\alpha/pS. \tag{41}$$

To investigate this phenomenon further, it is convenient to introduce the proper radial distance l. This represents the sum of the infinitesimal distance measurements made at some cosmic instant t by a chain of fundamental observers situated along the line θ, $\phi=$constant, between the origin and a given fundamental particle with coordinate r. In the case of de Sitter space time, we have, from (23),

$$l=e^{pt}r. \tag{42}$$

In terms of this variable the equation of motion of the accelerating particle simplifies. Substituting from (34) and (37) in (42), we find

$$l=(\alpha/q^2)(\cosh q\tau-1). \tag{43}$$

Note that l becomes infinite with τ, in contrast to r, which approaches a finite limit. Note also the similarity of Eq. (43) with the special relativistic formula (10) (*iv*). In fact, it is evident that when $p=0$ the metric (23) reduces to that of a flat static substratum and all our formulas must reduce to those of special relativity, as indeed they do.

At the beginning of the motion (i.e., at the cosmic instant $t=0$) the proper distance between the spatial origin and the fundamental particle with radial coordinate r_∞ is given by

$$l_{\text{crit}}=e^0r_\infty=\alpha/pS. \tag{44}$$

Suppose the rocket had originally come from this fundamental particle at r_∞. Evidently l_{crit} is a critical distance for rockets with an available proper acceleration α: having travelled that far from their place of origin and being then at rest relative to the substratum, they can no longer return. We shall call l_{crit} the distance of the "α horizon." For an actual return to its base, the outward going rocket must, of course, begin to decelerate long before it reaches that proper distance. We have seen that, as $\alpha\rightarrow\infty$, hyperbolic motion becomes geodesic motion with the constant velocity of light. The α horizon then becomes the event horizon,[19] from beyond which not even light signals can be sent to the origin. Letting $\alpha\rightarrow\infty$ in (44) we find that for the event horizon

$$l=1/p. \tag{45}$$

We can next show that a light signal emitted at the origin at or after the time

$$t=p^{-1}\ln(S/\alpha) \tag{46}$$

will not catch up with a particle released from rest at the origin at time $t=0$ and having constant proper acceleration α. This is analogous to the situation in Minkowski space time where a light signal sent out after $t=1/\alpha$ will not reach the receding particle. To prove our assertion, we note from (19) that the equation of motion of a light signal emitted at the origin at $t=0$ is

$$l=p^{-1}(e^{pt}-1). \tag{47}$$

Solving for t, we find the cosmic time t_s needed for the signal to travel a proper distance l, namely

$$t_s=p^{-1}\ln pl+O(l^{-1}). \tag{48}$$

On the other hand, from (43) we find

$$e^{q\tau}=2q^2l\alpha^{-1}+2+O(l^{-1}),$$

which, when substituted in (33), gives the cosmic time t_p which the particle takes to travel the distance l:

$$t_p=p^{-1}\ln pl+p^{-1}\ln(S/\alpha)+O(l^{-1}). \tag{49}$$

The difference between the times needed by the signal and by the particle to attain a given cosmic distance l is found by subtracting (48) from (49); if in this difference we let $l\rightarrow\infty$, we obtain (46). We may note that when p/α is small compared with unity, the critical time is approximately $1/\alpha$, as in special relativity.

For the sake of comparison we give brief mention to the case of unaccelerated motion in de Sitter space time. Tolman[20] has given the analysis of geodesic (i.e., unaccelerated) motion in a space time referred to the Robertson-Walker metric. In the present notation Tolman's formula (153.6) becomes

$$t'^2-1=A/R^2(t), \tag{50}$$

[19] W. Rindler, Monthly Notices Roy. Astron. Soc. **116**, 662 (1956).

[20] See reference 16, Sec. 153.

where A is an arbitrary constant. Thus in all models with unbounded expansion $\gamma = t' \to 1$, i.e., the particle ultimately comes to rest relative to the substratum. For de Sitter space time, using (24), we easily find from (50) that

$$r_\infty = \lim_{\tau\to\infty} r = \frac{1}{p}\left(\frac{\gamma_0 - 1}{\gamma_0 + 1}\right)^{\frac{1}{2}}, \tag{51}$$

where γ_0 is the Lorentz factor of the particle at the origin. This formula gives the radial coordinate at which the particle ultimately comes to rest. Thus the proper distance $l = r_\infty$ gives a horizon for *ballistic* missiles with an available take-off speed corresponding to a Lorentz factor γ_0. From such a distance the missile can no longer be shot back to the origin.

5. NUMERICAL CONSIDERATIONS

It will not be out of place here to compute a few of the numerical values involved in possible space travel. Present day space vehicles do not, of course, travel with constant proper acceleration, but schemes have been discussed for providing permanent thrust by the annihilation of matter carried on board or of hydrogen scooped up in flight. Something approximating to motion with uniform acceleration could then result. The terrestrial gravitational acceleration g would no doubt be the most comfortable intergalactic cruising acceleration. With it, we should reach the g horizon comparatively quickly, as measured by the proper time on the rocket. This proper time we find by equating the right members of (43) and (44), which gives

$$\cosh q\tau = q^2 p^{-1} S^{-1} + 1. \tag{52}$$

McMillan[3] has observed that if we measure time in years and distance in light years, the value of g is 1.03. As for p, the recent redetermination of Hubble's constant is not yet completed, but it seems to be agreed that the value $1/p = 10^{10}$ years is not likely to be out by more than a factor of 2. With these values for g and p we find from (52) that $\tau = 23.0$ years. Even if the error in $1/p$ is as much as a factor of 2, the corresponding error in τ is no more than ± 0.7 year. The proper time needed to reach the event horizon is only infinitesimally greater than that needed to reach the g horizon. From (43) we have

$$dl = (g/q)\sinh q\tau d\tau \approx q l d\tau.$$

But the dl relevant to our question is the difference between the right members of (45) and (44), and for l we can take either of these, whence

$$dl/l = (S-g)/g \approx p/g.$$

Combined with the preceding relation this gives $d\tau \approx p/gq \approx 10^{-10}$ years $\approx 3\times 10^{-3}$ second.

The cosmic time elapsed during the trip to the horizon is easily calculated from (33). It comes to 6.9×10^9 years. In this time the proper distances between the galaxies in the universe have just more than doubled. They double exactly in the cosmic time $(1/p)\ln 2$ which it takes a light signal to travel from the origin to the event horizon, and this cosmic time differs by less than one year from that taken by our g rocket. This follows from the remark after Eq. (49) above.

One fact brought out by such calculations is the vastness of the terrestrial acceleration in a cosmological context. A graphic view of this fact is that a radius of curvature of approximately one light year is extremely small compared with intergalactic distances of the order of millions of light years; or, a rectangular hyperbola with unit semiaxes and linear extension of the order of 10^6 units is hardly distinguishable from its asymptotes (light paths).

As long as $pe^{q\tau}$ is small compared with α, our main formulas (32), (33), (38), (43) do not differ appreciably from their counterparts 10(*i*)–(*iv*) in special relativity. Thus with $\alpha \approx 1$, when τ is less than 18, $pe^{q\tau}$ is still less than 1/100, and special relativistic formulas give fair approximations.

Unfortunately, the prospect of ever realizing the acceleration g intergalactically seems slight. Consider the two most likely sources of energy: (*i*) matter carried on board, and (*ii*) intergalactic hydrogen scooped up in flight. In either case the most efficient method of propulsion consists in converting the available mass totally into propellant radiation. In case (*i*), if the total rest mass of the vehicle at its proper time τ is M and radiative energy $E = -c^2 dM/d\tau$ is emitted per second (we temporarily adopt cgs units), the momentum gained per second by the vehicle is E/c relative to its rest frame. Consequently, if the process is regulated to produce constant proper acceleration α, we have, since Newton's laws apply in the rest frame,

$$-c\,dM/d\tau = (\alpha/c)M, \quad M_0/M = e^{\alpha\tau/c},$$

where M_0 is the initial rest mass. A ratio of M_0/M much larger than 10^3 would seem to be impracticable, and this makes the ratio $\alpha\tau/c \approx 7$. The maximum distance we could reach in, say, fifty years of proper time is then found from (43) to be less than 3500 light years, a cosmologically quite insignificant distance which would not even take us out of our own galaxy.

In the second method, viz., that of scooping up hydrogen, the rest mass of the column of hydrogen in front of the vehicle corresponds to the rest mass $M_0 - M$ annihilated in the first method. The steady-state theory predicts an intergalactic hydrogen concentration of the order of 10^{-29} g/cm^3 and thus a column one cm^2 across and extending from the earth to the event horizon (10^{10} light years) would contain no more than 0.1 gram of hydrogen. A square scoop of side 1 km would pick up no more than 10^3 tons in all that distance, and so this method would seem to offer no advantage over the first.

6. ACKNOWLEDGMENTS

I wish to thank Professor H. Bondi for reading and commenting on my manuscript, and for drawing my attention to Marder's paper. I also thank Professor R. J. Walker for his very stimulating interest.

APPENDIX. MARDER'S DEFINITIONS

Three years ago Marder[9] briefly proposed two definitions of "uniform acceleration in special and general relativity" which should be further clarified. In the notation of the present paper, Marder's first definition stipulates that (*i*) $g_{\mu\nu}A^\mu A^\nu$ remain constant and that (*ii*) A^μ remain in a plane determined by two orthogonal unit vectors L^μ and M^μ, of which the first is timelike and the second spacelike, and which are transported parallelly along the path. Marder finds that, if $L^\mu = U^\mu$ at the beginning of the motion, U^μ must then satisfy what is exactly Eq. (17) above. Hence, our definitions are equivalent. The simpler stipulation that U^μ, rather than A^μ, should remain in such a plane also implies Eq. (17) and would be analogous to the demand for the "plane-ness" of the path.

Marder's second definition is dynamical. The vector

$$C^\mu = -A^\mu - (m'/m)U^\mu, \tag{I}$$

where m is the rest mass of the moving particle, is to be transported parallelly along the path. Evidently, although this is not stated, C^μ is minus the four force per unit rest mass: $C^\mu = -(1/m)D(mU^\mu)/d\tau$. In these terms the definition seems "reasonable". But it does not reduce to hyperbolic motion in flat space time (except in the degenerate case of geodesic motion), it implies a very special change of rest mass, and it leads to infinite proper accelerations. Writing

$$\xi = -m'/m, \tag{II}$$

we have, from (I),

$$A^\mu = \xi U^\mu - C^\mu.$$

According to the definition, $DC^\mu/d\tau = 0$, whence

$$DA^\mu/d\tau = \xi' U^\mu + \xi A^\mu. \tag{III}$$

Multiplying by U_μ and A_μ in turn, and using the relations given after (15), we find, respectively,

$$\alpha^2 = \xi', \quad \text{(IV)} \quad \text{and} \quad \alpha\alpha' = \xi\alpha^2. \tag{V}$$

Now suppose α is constant and nonzero. Then $\xi \equiv 0$, from (V), and $DA^\mu/d\tau \equiv 0$, from (III). But this means $\alpha \equiv 0$ since $U_\mu DA^\mu/d\tau = \alpha^2$. Thus hyperbolic motion with $\alpha \neq 0$ is not included in this definition. If α is *not* constant, it follows from (II) and (V) that

$$m\alpha = k(\text{constant}), \tag{VI}$$

which shows that the *total* proper three force is constant and *not* the proper three force per unit rest mass, as in hyperbolic motion. In addition, the proper mass must vary in a prescribed manner: from (II), (IV), and (VI) it follows that $mm'' - m'^2 = -k^2$, whence $m = (k/B) \sinh(C - B\tau)$, B and C being arbitrary constants. Thus, from (VI),

$$\alpha = B \operatorname{csch}(C - B\tau), \tag{VII}$$

and this becomes infinite as $\tau \to C/B$. The purely geometric equations for the path result if in (III) we substitute for ξ and ξ' first from (IV) and (V) and then from (VII).