Notes on Bell's Theorem*

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1 Bell's Theorem

In this first section, we reconstruct several versions of Bell's theorem. We work with a standard set-up – to be described in class – involving a pair of photons in the singlet state, and consider the probability that they will both pass through polarizer sheets having specified orientations.

We take $pr_{QM}(A, B|a, b)$ to be the probability, given by QM, for a joint *outcome* of A on the left and B on the right, given polarizer *orientations* a on the left and b on the right. (Thus A and B take as values "*yes*" (the photon passes through the polarizer) and "*no*" (it does not pass through the polarizer), while a and b take as values lines in the plane orthogonal to the motion of the particles.) We take $pr_{QM}(A, \_|a, b)$ to be the probability of outcome A on the left (regardless of the outcome on the right) given polarizer settings a on the left and b on the right. Since the outcome on the right must be either "*yes*" or "*no*" (and cannot be both) we have, for example,

$$pr_{QM}(yes, \_|a, b) = pr_{QM}(yes, yes|a, b) + pr_{QM}(yes, no|a, b).$$

(Of course, $pr_{QM}(\_, B|a, b)$ is handled similarly.)

The predictions of QM are fully characterized by the following two conditions. (Here $\angle(a, b)$ is the (acute) angle between a and b.)

(QM1) For all a, b, $pr_{QM}(yes, yes|a, b) = \frac{1}{2}\cos^2\angle(a, b)$.

(QM2) ("*yes-no*" symmetry) For all a, b,

$$pr_{QM}(yes, no|a, b) = pr_{QM}(yes, yes|a, b^{\perp})$$
$$pr_{QM}(no, yes|a, b) = pr_{QM}(yes, yes|a^{\perp}, b)$$
$$pr_{QM}(no, no|a, b) = pr_{QM}(yes, yes|a^{\perp}, b^{\perp}).$$

(Here, $b^{\perp}$ is understood to be the line othogonal to b (in the plane orthogonal to the motion of the particles).) In light of the symmetry conditions in (QM2), we lose nothing in what follows if we restrict consideration to "*yes-yes*" outcomes.) It follows from (QM1) and (QM2) that the "single side" probabilities generated by QM satisfy the following condition.

(QM3) For all a, b, $pr_{QM}(yes, \_|a, b) = \dfrac{1}{2} = pr_{QM}(\_, yes|a, b)$.

The computation is straight-forward. Since $\angle(a, b^{\perp}) = \dfrac{\pi}{2} - \angle(a, b)$, we have

$$\begin{aligned} pr_{QM}(yes, \_|a, b) &= pr_{QM}(yes, yes|a, b) + pr_{QM}(yes, no|a, b) \\ &= pr_{QM}(yes, yes|a, b) + pr_{QM}(yes, yes|a, b^{\perp}) \\ &= \frac{1}{2}\cos^2\angle(a, b) + \frac{1}{2}\cos^2[\frac{\pi}{2} - \angle(a, b)] \\ &= \frac{1}{2}\cos^2\angle(a, b) + \frac{1}{2}\sin^2\angle(a, b) = \frac{1}{2}. \end{aligned}$$

(The other case, of course, is handled similarly.) (QM3) asserts that the (single side) probability that a photon will pass through a polarizer is $\frac{1}{2}$, whatever the orientation of the polarizer, even though the joint probability for passage through both polarizers is a function of $\angle(a,b)$. Thus, except for the very special case in which $\angle(a,b) = \frac{\pi}{4}$ (and hence $\frac{1}{2}\cos^2(\angle(a,b)) = \frac{1}{4}$),

$$pr_{QM}(yes, yes|a, b) \neq pr_{QM}(yes, \_|a, b) \cdot pr_{QM}(\_, yes|a, b),$$

i.e., the outcomes on the two sides are statistically correlated.

The "EPR" case, where $b = a^{\perp}$ (and so $\angle(a,b) = \frac{\pi}{2}$), is of special interest. We have the following conditions.

(QM4) For all c,

$$pr_{QM}(yes, yes|c, c^{\perp}) = 0 = pr_{QM}(no, no|c, c^{\perp})$$
$$pr_{QM}(yes, no|c, c^{\perp}) = \frac{1}{2} = pr_{QM}(no, yes|c, c^{\perp}).$$

(The two equalities in the first line follow immediately from (QM1) and (QM2). The two in the second line follow from the first two and (QM3). So, for example,

$$pr_{QM}(yes, no|c, c^{\perp}) = pr_{QM}(yes, \_|c, c^{\perp}) - pr_{QM}(yes, yes|c, c^{\perp}) = \frac{1}{2} - 0 = \frac{1}{2}.)$$

Notice that in the EPR case, the outcomes on the two sides exhibit perfect anti-correlation, i.e., the probability that they differ (either in the pattern "*yes-no*" or "*no-yes*") is 1.

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Now we consider possible "hidden-variable theories" that posit a space Λ of hidden states, and determine probabilities $pr_{HV}(A, B|a, b; \lambda)$ for each $\lambda \in \Lambda$. We show that if pr_{HV} satisfies certain constraints, then it is *not* possible to recover pr_{QM} from pr_{HV} – more precisely, it is not possible to represent pr_{QM} in the form

$$pr_{QM}(A, B|a, b) = \int_{\Lambda} pr_{HV}(A, B|a, b; \lambda)\rho(\lambda)d\lambda$$

where ρ is a probability density on Λ, i.e., an (integrable) function $\rho : \Lambda \to [0, 1]$ satisfying $\int_{\Lambda} \rho(\lambda)d\lambda = 1$. (We think of $\rho(\lambda)$ as the probability that the photon pair is in hidden state λ.)

Technical note: We have been deliberately vague in the preceding paragraph. We have made reference to "spaces" and to integration over those spaces without indicating exactly what mathematical structures and operations we have in mind. We have done so because it really makes no difference. The proofs that follow invoke only such elementary properties of integrals as are exhibited by all species. We could be thinking about Riemann integration over suitably chosen sets in $\mathbb{R}^n$, such as one studies in a calculus course. Or, for example, we could be thinking about integration over abstract measure spaces.

We will be interested in the following four conditions on hidden variable theories.

Quasi-determinism: For all A,B, a, b, λ, $pr_{HV}(A, B|a, b; \lambda) = 0/1$

Screening-off: For all A,B, a, b, λ, $pr_{HV}(A, B|a, b; \lambda) = pr_{HV}(A, \_|a, b; \lambda) \cdot pr_{HV}(\_, B|a, b; \lambda)$

Locality: For all A, B, a, a', b, b', λ,

$$pr_{HV}(A, \_|a, b; \lambda) = pr_{HV}(A, \_|a, b'; \lambda)$$
$$pr_{HV}(\_, B|a, b; \lambda) = pr_{HV}(\_, B|a', b; \lambda)$$

Anti-correlation: For all A, c, λ, $pr_{HV}(A, A|c, c^{\perp}; \lambda) = 0$.

The first is clear. It asserts that conditionalization on the hidden state λ pushes all HV-probabilities to 0 or 1. (Notice that our notion of hidden variable theory does not build-in this condition from the beginning, i.e., we allow for "non-deterministic hidden variable theories".) The second asserts that the observed correlation between *outcomes* (*yes* or *no*) on the two sides is "screened-off" by the underlying hidden variable state . The third asserts that (after conditionalization on the hidden state λ) the HV-probability for an outcome on one side is independent of the polarizer orientation (or *setting*) on the other side. (The distinction between "outcome-outcome correlations" and "setting-outcome correlations" is crucially important here.) The fourth condition asserts that HV-probabilties (after conditionalization on the hidden state λ) exhibit the same the anti-correlation pattern as QM-probabilities. (Recall condition (QM4).)

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We turn to our first two versions of Bell's theorem. (There will be three altogether.) The first rules out hidden variable theories that satisfy the locality and screening-off conditions. The second (really just a corollary of the first) rules out theories that satisfy the locality and quasi-determinism conditions.

Proposition 1.1. *Let $pr_{HV}(A, B|a, b; \lambda)$ satisfy the locality and screening-off conditions. Let ρ be a probability density on Λ, and let $pr_{HV}(A, B|a, b)$ be defined by*

$$pr_{HV}(A, B|a, b) = \int_{\Lambda} pr_{HV}(A, B|a, b; \lambda)\rho(\lambda)d\lambda.$$

Then it is not the case that $pr_{QM}(A, B|a, b) = pr_{HV}(A, B|a, b)$ for all A, B, a, b.

To prove the theorem, we show that if the stated hypotheses hold, then pr_{HV} must satisfy the following inequality (the "Clauser-Horne inequality"):

$$\begin{aligned} 0 \ \leq \ & pr_{HV}(yes, \_|a, \_) + pr_{HV}(\_, yes|\_, b) + pr_{HV}(yes, yes|a', b') \\ & -pr_{HV}(yes, yes|a, b') - pr_{HV}(yes, yes|a', b) - pr_{HV}(yes, yes|a, b) \ \leq \ 1. \end{aligned}$$

for all a, b, a', b'. (The expressions $pr_{HV}(yes, \_|a, \_)$ and $pr_{HV}(\_, yes|\_, b)$ make sense if the locality condition holds for then

$$pr_{HV}(yes, \_|a, b) = \int_{\Lambda} pr_{HV}(yes, \_|a, b; \lambda)\rho(\lambda)d\lambda$$

does not depend on b (and similarly $pr_{HV}(\_, yes|a,b)$ does not depend on a).) This will suffice, since QM probabilities do *not* satisfy the counterpart inequality for all a, b, a', b'. For example, if $\angle(a', b') = \frac{\pi}{2}$ and $\angle(a,b) = \angle(a,b') = \angle(a',b) = \frac{\pi}{6}$, then

$$\begin{aligned} pr_{QM}(yes, yes|a, b') &= pr_{QM}(yes, yes|a', b) \\ &= pr_{QM}(yes, yes|a, b) \;=\; \frac{1}{2}\cos^2(\frac{\pi}{6}) \;=\; \frac{3}{8} \\ pr_{QM}(yes, yes|a', b') &= \frac{1}{2}\cos^2(\frac{\pi}{2}) \;=\; 0 \\ pr_{QM}(yes, \_|a, \_) &= pr_{QM}(\_, yes|\_, b) \;=\; \frac{1}{2} \end{aligned}$$

(The expressions $pr_{QM}(yes, \_|a, \_)$ and $pr_{QM}(\_, yes|\_, b)$ make sense since, by (QM4), $pr_{QM}(yes, \_|a, b)$ does not depend on b, and $pr_{QM}(\_, yes|a, b)$ does not depend on a.) Hence the sum of six terms in the inequality is

$$\frac{1}{2} + \frac{1}{2} + 0 - 3(\frac{3}{8}) = -\frac{1}{8}$$

(which is not between 0 and 1).

Proof. First note that for all numbers x, x', y, y' in the interval $[0, 1]$,

$$0 \leq x + y + x'y' - xy' - x'y - xy \leq 1. \tag{1.1}$$

There are various ways to see this. One involves a simple consideration of three cases: (i) $x \leq x'$, (ii) $y \leq y'$, (iii) $x > x'$ and $y > y'$. In case (i), we have

$$0 \leq \underline{x(1-y) + y(1-x') + y'(x'-x)} \leq x + (1-x') + (x'-x) = 1.$$

But the underlined expression is equal to $x + y + x'y' - xy' - x'y - xy$. So (*) holds. Similarly, in case (ii), we have

$$0 \leq \underline{y(1-x) + x(1-y') + x'(y'-y)} \leq y + (1-y') + (y'-y) = 1.$$

Finally, in case (iii), we have

$$0 \leq \underline{(x-x')(y-y') + y(1-x) + x(1-y)} \leq xy + y(1-x) + (1-y) = 1.$$

Thus (1.1) holds in all three cases. Now assume pr_{HV} satisfies the locality and screening-off conditions. Then we can express $pr_{HV}(yes, yes|a, b; \lambda)$ in the form

$$pr_{HV}(yes, yes|a, b; \lambda) = pr_{HV}(yes, \_|a, \_; \lambda) \cdot pr_{HV}(\_, yes|\_, b; \lambda). \tag{1.2}$$

Given lines a, a', b, b', and hidden state λ, let

$$\begin{aligned} x &= pr_{HV}(yes, \_|a, \_; \lambda) \\ x' &= pr_{HV}(yes, \_|a', \_; \lambda) \\ y &= pr_{HV}(\_, yes|\_, b; \lambda) \\ y' &= pr_{HV}(\_, yes|\_, b'; \lambda). \end{aligned}$$

Then x, x', y, and y' are all in the interval $[0,1]$ and so, by (1.1) and (1.2),

$$\begin{aligned} 0 \;\leq\; & pr_{HV}(yes, \_|a, \_; \lambda) \;+\; pr_{HV}(\_, yes|\_, b; \lambda) \;+\; pr_{HV}(yes, yes|a', b'; \lambda) \\ & -pr_{HV}(yes, yes|a, b'; \lambda) \;-\; pr_{HV}(yes, yes|a', b; \lambda) \;-\; pr_{HV}(yes, yes|a, b; \lambda) \;\leq\; 1. \end{aligned}$$

We move from this inequality to the Clauser-Horne inequality with a simple integration. Let $X(a, a', b, b', \lambda)$ be the (middle) sum of six terms, and let ρ be a probability density on Λ, Then

$$0 \;=\; \int_\Lambda 0 \cdot \rho(\lambda) d\lambda \;\leq\; \int_\Lambda X(a, a', b, b', \lambda) \cdot \rho(\lambda) d\lambda \;\leq\; \int_\Lambda 1 \cdot \rho(\lambda) d\lambda \;=\; 1,$$

and

$$\begin{aligned} \int_\Lambda X(a, a', b, b', \lambda) \cdot \rho(\lambda) d\lambda \;\; = & \\ & pr_{HV}(yes, \_|a, \_) + pr_{HV}(\_, yes|\_, b) + pr_{HV}(yes, yes|a', b') \\ & -pr_{HV}(yes, yes|a, b') - pr_{HV}(yes, yes|a', b) - pr_{HV}(yes, yes|a, b). \end{aligned}$$

□

It is not hard to show that the quasi-determinism condition implies the screening off condition (and we leave this as an exercise).

Problem 1.1. *Show*

(a) quasi-determinism $\Longrightarrow$ *screening-off*

(b) screening-off & *locality* & *anti-correlation* $\Longrightarrow$ *quasi-determinism*

So we have the following immediate corollary. (Everything remains the same except that reference to the latter condition is replaced by reference to the former.)

Proposition 1.2. *Let* $pr_{HV}(A, B|a, b; \lambda)$ *satisfy the locality and quasi-determinism conditions. Let* ρ *be a probability density on* Λ, *and let* $pr_{HV}(A, B|a, b)$ *be defined by*

$$pr_{HV}(A, B|a, b) = \int_\Lambda pr_{HV}(A, B|a, b; \lambda) \rho(\lambda) d\lambda.$$

Then it is not the case that

$$pr_{QM}(A, B|a, b) = pr_{HV}(A, B|a, b)$$

for all A, B, a, b.

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Now we consider a third version of the theorem that is a bit different in character from the first two. It is of interest in its own right. (For one thing, it is perfectly precise in formulation. No reference is made to some not fully specified sense of integration.) And it will prepare the way for our discussion of Itamar Pitowsky's work on the geometric interpretation of Bell type inequalities.

For the moment, let us continue within the framework of section 1.2. Suppose we have a space Λ of hidden states for our two-photon system, a probability function $pr_{HV}(yes, yes|a, b; \lambda)$ over Λ, and a probability density over Λ, i.e., an (integrable) function $\rho : \Lambda \rightarrow [0, 1]$ satisfying $\int_\Lambda \rho(\lambda)d\lambda = 1$. Further suppose that $pr_{HV}(A, B|a, b)$ is defined (as above) by

$$pr_{HV}(A, B|a, b) = \int_\Lambda pr_{HV}(A, B|a, b; \lambda)\rho(\lambda)d\lambda.$$

Finally, suppose that $pr_{HV}(A, B|a, b; \lambda)$ satisfies both locality and quasi-determinism.

For all lines a and b, consider sets:

$$\begin{aligned} X_{ab} &= \{\lambda \in \Lambda : pr_{HV}(yes, yes|a, b; \lambda) = 1\} \\ L_a &= \{\lambda \in \Lambda : pr_{HV}(yes, \_|a, \_; \lambda) = 1\} \\ R_a &= \{\lambda \in \Lambda : pr_{HV}(\_, yes|\_, b; \lambda) = 1\} \end{aligned}$$

The latter two are well defined by locality. By locality and quasi-determinism (which implies screening off)

$$pr_{HV}(yes, yes|a, b; \lambda) = pr_{HV}(yes, \_|a, \_; \lambda) \cdot pr_{HV}(\_, yes|\_, b; \lambda).$$

So

$$pr_{HV}(yes, yes|a, b; \lambda) = 1 \iff pr_{HV}(yes, \_|a, \_; \lambda) = 1 \text{ and } pr_{HV}(\_, yes|\_, b; \lambda) = 1.$$

Hence,

$$\lambda \in X_{ab} \iff \lambda \in L_a \ \& \ \lambda \in R_b$$

or, equivalently,

$$X_{ab} = L_a \cap R_b$$

for all a and b.

Now consider the measure μ on Λ defined by setting

$$\mu(C) = \int_C \rho(\lambda)d\lambda.$$

(We understand C to be in the domain of μ if the integral is well defined.) It then follows by quasi-determinism that, for all a and b,

$$\begin{aligned} pr_{HV}(yes, yes|a, b) &= \int_\Lambda pr_{HV}(yes, yes|a, b; \lambda)\rho(\lambda)d\lambda \\ &= \int_{X_{ab}} 1 \cdot \rho(\lambda)d\lambda + \int_{(X - X_{ab})} 0 \cdot \rho(\lambda)d\lambda = \int_{X_{ab}} \rho(\lambda)d\lambda \\ &= \mu(X_{ab}) = \mu(L_a \cap R_b). \end{aligned}$$

(Notice that it is quasi-determinism that allows us to divide the first integral into two subintegrals – one over the set X_{ab} where $pr_{HV}(yes, yes|a, b; \lambda)$ is 1, and one over the complement set $(X - X_{ab})$ where $pr_{HV}(yes, yes|a, b; \lambda)$ is 0.) Similarly, it follows that

$$\begin{aligned} pr_{HV}(yes, \_|a, b) &= \int_{\Lambda} pr_{HV}(yes, \_|a, \_; \lambda)\rho(\lambda)d\lambda = \int_{L_a} \rho(\lambda)d\lambda = \mu(L_a) \\ pr_{HV}(\_, yes|a, b) &= \int_{\Lambda} pr_{HV}(\_, yes|\_, b; \lambda)\rho(\lambda)d\lambda = \int_{R_b} \rho(\lambda)d\lambda = \mu(R_b) \end{aligned}$$

for all a and b. Now suppose it were the case that $pr_{QM}(A, B|a, b) = pr_{HV}(A, B|a, b)$ for all a, b. Then it would follow that

$$\left.\begin{aligned} pr_{QM}(yes, yes|a, b) &= \mu(L_a \cap R_b) \\ pr_{QM}(yes, \_|a, \_) &= \mu(L_a) \\ pr_{Q}M(\_, yes|\_, b) &= \mu(R_b) \end{aligned}\right\} \quad (*)$$

for all a and b.

We now have another way to set up the second version of the theorem. We forget about our route to the three equations in $(*)$ and use them, in effect, to *characterize* a (local, deterministic) hidden variable theory. We think of μ as just some probability measure (or other) on Λ, and pay no attention to whether it arises from a probability density. For any subset C of X (in the domain of μ), we understand $\mu(C)$ as the probability that the exact underlying state of the sytem is, in fact, in C. In particular, we interpret $\mu(L_a \cap R_b)$ as the probability that the underlying state of the system happens to be in one in which it is determined that, with orientations a and b, both photons will pass through the polarizer sheets. The surprising result, of course, is that we cannot *have* a hidden variable theory in the sense just characterized. This will be our third version of Bell's theorem.

Before stating it, we need to recall the definition of a *probability space*. It is a structure (X, Σ, μ) where X is a non-empty set, Σ is a set of subsets of X satisfying three conditions

(F1) $X \in \Sigma$

(F2) For all subsets A of X, $A \in \Sigma \Rightarrow (X - A) \in \Sigma$

(F3) For all subsets A_1, A_2 of X, A_1, $A_2 \in \Sigma \Rightarrow (A_1 \cup A_2) \in \Sigma$;

and μ is a probability measure on Σ, i.e., a map $\mu : \Sigma \to [0, 1]$ such that

(M1) $\mu(X) = 1$

(M2) For all sets A_1, A_2 in Σ that are pairwise disjoint, $\mu(A_1 \cup A_2) = \mu(A_1) + \mu(A_2)$.

It follows immediately by induction, of course, that the conditions in (F3) and (M2) can be extended to arbitrary finite unions. Furthermore, it follows from (F2) and (F3) that Σ is closed under finite intersections (as well as unions) – since

$$(A_1 \cap A_2 \cap ... \cap A_n) = X - [(X - A_1) \cup (X - A_2) \cup ... \cup (X - A_n)].$$

Technical note: Standardly one replaces (F3) with the stronger requirement that Σ be closed under countable unions, and replaces (M2) with the stronger requirement of countable additivity. But the difference between the two formulations (finite versus countable) is irrelevant for our purposes.

Now we have all the pieces in place.

Proposition 1.3. *There does not exist a classical probability space* (X, Σ, μ) *and, for all directions* a, b, *sets* L_a *and* R_b *in* Σ, *such that the three equations in* $(*)$ *hold.*

(There are clear advantages to this formulation. The disadvantage is that it depends crucially on the assumption of quasi-determinism. There is no variant that uses only the screening-off condition (and so serves as counterpart to our first version of Bell's thorem).)

Proof. We already know that quantum mechanical probabilities violate the Clauser-Horne inequality. So it suffices to show that given a probability space (X, Σ, μ), and any four sets L_a, L'_a, R_b, R'_b in Σ, the corresponding inequality

$$0 \leq \mu(L_a) + \mu(R_b) + \mu(L_{a'} \cap R_{b'}) - \mu(L_a \cap R_{b'}) - \mu(L_{a'} \cap Rb) - \mu(L_a \cap R_b) \leq 1$$

is satisfied. We do so using a low brow computation. Every point λ in X is either in L_a or in its complement $L_a^- = X - L_a$. Similarly, it is either in $L_{a'}$ or in its complement $L_{a'}^-$; the same for R_b and $R_{b'}$. Thus we can partition X into sixteen disjoint sets. Typical members are

$$(L_a \cap L_{a'} \cap R_b^- \cap R_{b'})$$
$$(L_a^- \cap L_{a'}^- \cap R_b^- \cap R_{b'}).$$

The first is the set of all points λ in X that are in L_a, $L_{a'}$, and $R_{b'}$, but not in R_b. The second is the set of all such points that are in $R_{b'}$, but not in L_a, $L_{a'}$, or R_b.

To proceed, we just have to express the six sets that enter into the Clauser-Horne inequality in terms of the elements of the partitition, and then do some cancelling. For example, $(L_a \cap R_b)$ can be expressed as the disjoint union of the four sets

$$(L_a \cap L_{a'} \cap R_b \cap R_{b'})$$
$$(L_a \cap L_{a'} \cap R_b \cap R_{b'}^-)$$
$$(L_a \cap L_{a'}^- \cap R_b \cap R_{b'})$$
$$(L_a \cap L_{a'}^- \cap R_b \cap R_{b'}^-).$$

(Notice that L_a and R_b appear in all four (rather than their complements L_a^- and R_b^-).) So it follows from the basic additivity property of μ that

$$\begin{aligned}\mu(L_a \cap R_b) = \mu(L_a \cap L_{a'} \cap R_b \cap R_{b'})\\ +\mu(L_a \cap L_{a'} \cap R_b \cap R_{b'}^-)\\ +\mu(L_a \cap L_{a'}^- \cap R_b \cap R_{b'})\\ +\mu(L_a \cap L_{a'}^- \cap R_b \cap R_{b'}^-).\end{aligned}$$

We list the decompositions for the six sets below. For ease of reading, we use a simple labelling scheme for the 16 sets in the partition. So, for example, $\mu(1011)$ stands for the set $\mu(L_a \cap L_{a'}^- \cap R_b \cap R_{b'})$. (The marks on the extreme right (checks, circles, infinity signs) indicate a cancellation pattern that we will consider shortly.

$$\begin{aligned}\mu(L_a) = \mu(L_a \cap L_{a'} \cap R_b \cap R_{b'}) &\quad \mu(1111) \quad \checkmark\\ +\mu(L_a \cap L_{a'} \cap R_b \cap R_{b'}^-) &\quad \mu(1110) \quad \checkmark\checkmark\\ +\mu(L_a \cap L_{a'}^- \cap R_b \cap R_{b'}) &\quad \mu(1011) \quad \checkmark\checkmark\checkmark\\ +\mu(L_a \cap L_{a'}^- \cap R_b \cap R_{b'}^-) &\quad \mu(1010) \quad \checkmark\checkmark\checkmark\checkmark\\ +\mu(L_a \cap L_{a'} \cap R_b^- \cap R_{b'}) &\quad \mu(1101) \quad \circ\circ\\ +\mu(L_a \cap L_{a'} \cap R_b^- \cap R_{b'}^-) &\quad \mu(1100)\\ +\mu(L_a \cap L_{a'}^- \cap R_b^- \cap R_{b'}) &\quad \mu(1001) \quad \circ\circ\circ\circ\\ +\mu(L_a \cap L_{a'}^- \cap R_b^- \cap R_{b'}^-) &\quad \mu(1000)\end{aligned}$$

$$\begin{aligned}\mu(R_b) = \mu(L_a \cap L_{a'} \cap R_b \cap R_b') &\quad \mu(1111) \quad \infty\\ +\mu(L_a \cap L_{a'} \cap R_b \cap R_{b'}^-) &\quad \mu(1110) \quad \infty\infty\\ +\mu(L_a \cap L_{a'}^- \cap R_b \cap R_{b'}) &\quad \mu(1011) \quad \circ\,\circ\circ\\ +\mu(L_a \cap L_{a'}^- \cap R_b \cap R_{b'}^-) &\quad \mu(1010)\\ +\mu(L_a^- \cap L_{a'} \cap R_b \cap R_{b'}) &\quad \mu(0111)\\ +\mu(L_a^- \cap L_{a'} \cap R_b \cap R_{b'}^-) &\quad \mu(0110) \quad \infty\infty\infty\infty\\ +\mu(L_a^- \cap L_{a'}^- \cap R_b \cap R_{b'}) &\quad \mu(0011)\\ +\mu(L_a^- \cap L_{a'}^- \cap R_b \cap R_{b'}^-) &\quad \mu(0010)\end{aligned}$$

$$\begin{aligned}\mu(L_{a'} \cap R_{b'}) = \mu(L_a \cap L_{a'} \cap R_b \cap R_{b'}) &\quad \mu(1111) \quad \circ\\ +\mu(L_a \cap L_{a'} \cap R_b^- \cap R_{b'}) &\quad \mu(1101)\\ +\mu(L_a^- \cap L_{a'} \cap R_b \cap R_{b'}) &\quad \mu(0111) \quad \infty\infty\infty\\ +\mu(L_a^- \cap L_{a'} \cap R_b^- \cap R_{b'}) &\quad \mu(0101)\end{aligned}$$

$$
\begin{aligned}
\mu(L_a \cap R_b') = {} & \mu(L_a \cap L_{a'} \cap R_b \cap R_{b'}) \quad \mu(1111) \quad \circ \\
& +\mu(L_a \cap L_{a'} \cap R_b^- \cap R_{b'}) \quad \mu(1101) \quad \circ\circ \\
& +\mu(L_a \cap L_{a'}^- \cap R_b \cap R_{b'}) \quad \mu(1011) \quad \circ\circ\circ \\
& +\mu(L_a \cap L_{a'}^- \cap R_b^- \cap R_{b'}) \quad \mu(1001) \quad \circ\circ\circ\circ
\end{aligned}
$$

$$
\begin{aligned}
\mu(L_{a'} \cap R_b) = {} & \mu(L_a \cap L_{a'} \cap R_b \cap R_{b'}) \quad \mu(1111) \quad \infty \\
& +\mu(L_a \cap L_{a'} \cap R_b \cap R_{b'}^-) \quad \mu(1110) \quad \infty\infty \\
& +\mu(L_a^- \cap L_{a'} \cap R_b \cap R_{b'}) \quad \mu(0111) \quad \infty\infty\infty \\
& +\mu(L_a^- \cap L_{a'} \cap R_b \cap R_{b'}^-) \quad \mu(0110) \quad \infty\infty\infty\infty
\end{aligned}
$$

$$
\begin{aligned}
\mu(L_a \cap R_b) = {} & \mu(L_a \cap L_{a'} \cap R_b \cap R_{b'}) \quad \mu(1111) \quad \checkmark \\
& +\mu(L_a \cap L_{a'} \cap R_b \cap R_{b'}^-) \quad \mu(1110) \quad \checkmark\checkmark \\
& +\mu(L_a \cap L_{a'}^- \cap R_b \cap R_{b'}) \quad \mu(1011) \quad \checkmark\checkmark\checkmark \\
& +\mu(L_a \cap L_{a'}^- \cap R_b \cap R_{b'}^-) \quad \mu(1010) \quad \checkmark\checkmark\checkmark\checkmark
\end{aligned}
$$

To compute the Clauser-Horne expression

$$
\mu(L_a) + \mu(R_b) + \mu(L_{a'} \cap R_{b'}) - \mu(L_a \cap R_{b'}) - \mu(L_{a'} \cap R_b) - \mu(L_a \cap R_b)
$$

we add the (8+8+4) terms arising from

$$
\mu(L_a) + \mu(R_b) + \mu(L_{a'} \cap R_{b'})
$$

and then subtract the (4+4+4) terms arising from

$$
\mu(L_a \cap R_{b'}) + \mu(L_{a'} \cap R_b) + \mu(L_a \cap R_b).
$$

But as a simple inspection confirms, every term in the second group already appears in the first group, and eight terms in the first group are left over. (See the cancellation pattern indicated with check marks and related symbols.) When the dust clears, we have

$$
\begin{aligned}
& \mu(L_a) + \mu(R_b) + \mu(L_{a'} \cap R_{b'}) - \mu(L_a \cap R_{b'}) - \mu(L_{a'} \cap R_b) - \mu(L_a \cap R_b) \\
= & \mu(1100) + \mu(1000) + \mu(1010) + \mu(0111) + \mu(0011) + \mu(0010) + \mu(1101) + \mu(0101).
\end{aligned}
$$

The right side is a sum of terms, each of which is ≥ 0. So the sum is ≥ 0. And the sum is clearly ≤ 1 since the sum of all 16 terms of form $\mu(\_\_\_\_)$ is 1. So the expression on the left side of the equality is clearly bounded by 0 and 1, as claimed. $\square$

2 The Geometric Interpretation of Bell Type Inequalities

Let us put quantum mechanics aside for the moment and consider a quite general question that Pitowsky [4] poses and answers.

Let $n \geq 2$ be given, and let S be a non-empty subset of $\{\langle i, j \rangle : 1 \leq i < j \leq n\}$. Further, assume we are given $n + |S|$ numbers

$$\begin{aligned} & p_i \quad i = 1, ..., n \\ & p_{ij} \quad \langle i, j \rangle \in S \end{aligned}$$

(Here $|S|$ is the number of elements in S.) It is helpful to think of the numbers as determining an $(n+|S|)$-tuple $\langle p_1, ..., p_n, ..., p_{ij}, ... \rangle$ where, let us agree, the p_{ij} are ordered by their indices, and the latter are ordered lexicographically. We say that the $(n + |S|)$-tuple admits a *probability space representation* if there exists a probability space (X, Σ, μ), and (not necessarily distinct) sets $A_1, ..., A_n \in \Sigma$ such that, for all $i \in \{1, 2, ..., n\}$ and all $\langle i, j \rangle \in S$,

$$\begin{aligned} p_i &= \mu(A_i) \\ p_{ij} &= \mu(A_i \cap A_j). \end{aligned}$$

Question: Under what conditions does $\langle p_1, ..., p_n, ..., p_{ij}, ... \rangle$ admit a probability space representation?

Pitowsky gives a beautifully simple answer. Let $\{0, 1\}^n$ be the set of all n-tuples of 0's and 1's. Given any such n-tuple $\epsilon = \langle \epsilon_1, ..., \epsilon_n \rangle$, let p^ϵ be the $(n + |S|)$-tuple $\langle \epsilon_1, ..., \epsilon_n, ..., \epsilon_i\, \epsilon_j, ... \rangle$ where the product term $\epsilon_i \epsilon_j$ appears precisely if $\langle i, j \rangle \in S$. (For example, if $n = 3$, $S = \{\langle 1, 2 \rangle, \langle 2, 3 \rangle\}$, and $\epsilon = \langle 0, 1, 1 \rangle$,

$$p^\epsilon = \langle 0, 1, 1, 0, 1 \rangle.)$$

Now let $c(n, S)$ to be the closed, convex polytope in $\mathbb{R}^{(n+|S|)}$ whose vertices are the 2^n vectors of form p^ϵ, where $\epsilon \in \{0, 1\}^n$, i.e., the set of all vectors that can be expressed as convex sums of these 2^n vectors. (Recall that, quite generally, given vectors $v_1, , ..., v_m$ in a vector space (over $\mathbb{R}$), a *convex sum* of those vectors is a sum of the form $\lambda_1\, v_1 + ... + \lambda_m\, v_m$ where $\lambda_i \geq 0$ for all i, and $\lambda_1 + ... + \lambda_m = 1$.)

Example If $n = 2$ and $S = \{\langle 1, 2 \rangle\}$, $c(n, S)$ is the set of all vectors in $\mathbb{R}^3$ of the form

$$\begin{aligned} & \lambda(0,0)\,(0,0,0) \;+\; \lambda(1,0)\,(1,0,0) \;+\; \lambda(0,1)\,(0,1,0) \;+\; \lambda(1,1)\,(1,1,1) \\ & \qquad = (\ (\lambda(1,0) + \lambda(1,1)),\quad (\lambda(0,1) + \lambda(1,1)),\quad \lambda(1,1)\ ) \end{aligned}$$

where the four coefficients $\lambda(0,0)$, $\lambda(1,0)$, $\lambda(0,1)$, $\lambda(1,1)$ are non-negative and sum to 1. (See figure 2.1.)

Proposition 2.1. *For all n and S, $\langle p_1, ..., p_n, ..., p_{ij}, ... \rangle$ admits a probability space representation iff it belongs to $c(n, S)$.*

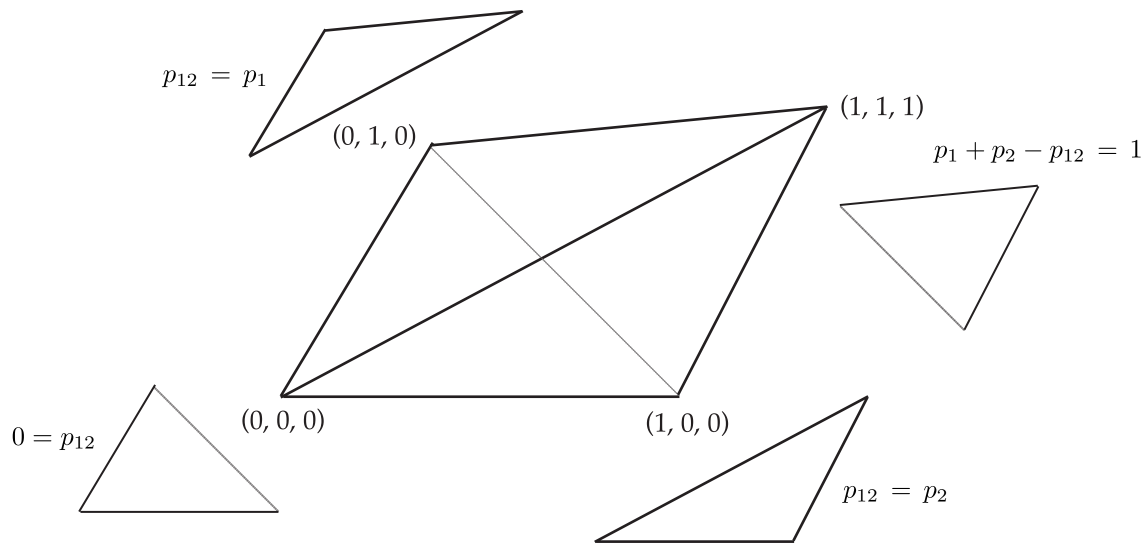

Figure 2.1: The $c(n, S)$ polytope in the case where $n = 2$ and $S = \{\langle 1, 2\rangle\}$. The planes of the bounding faces are identified.

Proof. Let n, S, and p be given. Assume first that p admits a classical representation, i.e., assume there is a probability space (X, Σ, μ) and sets $A_1, ..., A_n \in \Sigma$ such that, for all $i \le n$ and all $\langle i, j\rangle \in S$, $p_i = \mu(A_i)$ and $p_{ij} = \mu(A_i \cap A_j)$. Given a set $A \in \Sigma$, let $A^1 = A$ and $A^0 = X - A$. Further, given $\epsilon = \langle \epsilon_1, ..., \epsilon_n\rangle \in \{0, 1\}^n$, let

$$A(\epsilon) = A_1^{\epsilon_1} \cap A_2^{\epsilon_2} \cap ... \cap A_n^{\epsilon_n}.$$

The sets $A(\epsilon)$ form a partition of X as ϵ ranges over $\{0, 1\}^n$, i.e., $\epsilon \neq \epsilon' \Rightarrow A(\epsilon) \cap A(\epsilon') = \varnothing$ and $\cup\{A(\epsilon)\colon \epsilon \in \{0, 1\}^n\} = X$. Finally, let $\lambda(\epsilon) = \mu(A(\epsilon))$. Clearly, $\lambda(\epsilon) \ge 0$ for all ϵ in $\{0, 1\}^n$, and $\Sigma_{\epsilon \in I} \lambda(\epsilon) = 1$, where $I = \{0, 1\}^n$. For all $i \in \{1, 2, ..., n\}$ and all $\langle i, j\rangle \in S$,

$$A_i = \cup\{A(\epsilon)\colon \epsilon \in I \text{ and } \epsilon_i = 1\}$$
$$A_i \cap A_j = \cup\{A(\epsilon)\colon \epsilon \in I \text{ and } \epsilon_i = \epsilon_j = 1\}.$$

Hence,

$$p_i = \mu(A_i) = \Sigma_{\{\epsilon \in I : \epsilon_i = 1\}} \lambda(\epsilon) = \Sigma_{\epsilon \in I} \lambda(\epsilon)\epsilon_i = \Sigma_{\epsilon \in I} \lambda(\epsilon)(p^\epsilon)_i$$
$$p_{ij} = \mu(A_i \cap A_j) = \Sigma_{\{\epsilon \in I : \epsilon_i = \epsilon_j = 1\}} \lambda(\epsilon) = \Sigma_{\epsilon \in I} \lambda(\epsilon)\epsilon_i\epsilon_j = \Sigma_{\epsilon \in I} \lambda(\epsilon)(p^\epsilon)_{ij}.$$

Thus $p = \Sigma_{\epsilon \in I} \lambda(\epsilon)p^\epsilon$. So p belongs to $c(n, S)$.

Conversely, assume p belongs to $c(n, S)$. Then there exist numbers $\lambda(\epsilon) \ge 0$ such that $\Sigma_{\epsilon \in I} \lambda(\epsilon) = 1$ and $p = \Sigma_{\epsilon \in I} \lambda(\epsilon)p^\epsilon$. Let

$$X = I = \{0, 1\}^n$$
$$\Sigma = \mathcal{P}(X) \text{ (the power set of X)}$$

and for all A in Σ, let

$$\mu(A) = \Sigma_{\epsilon \in A} \lambda(\epsilon).$$

Clearly, (X, Σ, μ) is a probability space. Finally, for all $i \leq n$, let

$$A_i = \{\epsilon \colon \epsilon_i = 1\}.$$

Then

$$\begin{aligned}
\mu(A_i) &= \Sigma_{\epsilon \in A_i} \lambda(\epsilon) \;=\; \Sigma_{\{\epsilon \in I : \epsilon_i = 1\}} \lambda(\epsilon) \;=\; \Sigma_{\epsilon \in I} \lambda(\epsilon) \epsilon_i \;=\; \Sigma_{\epsilon \in I} \lambda(\epsilon) (p^\epsilon)_i \;=\; p_i, \\
\mu(A_i \cap A_j) &= \Sigma_{\epsilon \in A_i \cap A_j} \lambda(\epsilon) \;=\; \Sigma_{\{\epsilon \in I : \epsilon_i \,=\, \epsilon_j = 1\}} \lambda(\epsilon) \;=\; \Sigma_{\epsilon \in I} \lambda(\epsilon) \epsilon_i \epsilon_j \\
&= \Sigma_{\epsilon \in I} \lambda(\epsilon) (p^\epsilon)_{ij} \;=\; p_{ij}.
\end{aligned}$$

So p admits a classical representation. □

The polytope $c(n, S)$ can be characterized not only as the convex hull of its vertices, but also as the set of vectors bounded by its supporting hyperplanes, and thus as the set of vectors whose components satisfy a particular set of linear inequalities. It turns out that *it is precisely these hyperplane-describing inequalities, for simple choices of n and S, that we have come to know as "Bell-type inequalities"*.

Proposition 2.2. *(Examples)*

(a) Let $n = 2$ and let $S = \{\langle 1, 2\rangle\}$. A vector $\langle p_1, p_2, p_{12}\rangle$ belongs to $c(n, S)$ in this case iff

$$\begin{aligned}
0 \;\leq\; p_{12} \;\leq\; p_1 \;\leq\; 1 \\
0 \;\leq\; p_{12} \;\leq\; p_2 \;\leq\; 1 \\
p_1 + p_2 - p_{12} \;\leq\; 1.
\end{aligned}$$

(b) Let $n = 3$ and let $S = \{\langle 1, 2\rangle, \langle 1, 3\rangle, \langle 2, 3\rangle\}$. A vector $\langle p_1, p_2, p_3, p_{12}, p_{13}, p_{23}\rangle$ belongs to $c(n, S)$ in this case iff for all $\langle i, j\rangle \in S$,

$$\begin{aligned}
0 \;\leq\; p_{ij} \;\leq\; p_i \;\leq\; 1 \\
0 \;\leq\; p_{ij} \;\leq\; p_j \;\leq\; 1 \\
p_i + p_j - p_{ij} \;\leq\; 1
\end{aligned}$$

$$p_1 + p_2 + p_3 - p_{12} - p_{13} - p_{23} \;\leq\; 1$$

$$\begin{aligned}
0 \;&\leq\; p_1 - p_{12} - p_{13} + p_{23} \\
0 \;&\leq\; p_2 - p_{12} - p_{23} + p_{13} \\
0 \;&\leq\; p_3 - p_{13} - p_{23} + p_{12}.
\end{aligned}$$

(c) Let $n = 4$ and let $S = \{\langle 1,3\rangle, \langle 1,4\rangle, \langle 2,3\rangle, \langle 2,4\rangle\}$. A vector $\langle p_1, p_2, p_3, p_4, p_{13}, p_{14}, p_{23}, p_{24}\rangle$ belongs to $c(n, S)$ in this case iff for all $\langle i, j\rangle \in S$,

$$\begin{aligned} 0 \ \leq \ p_{ij} \ &\leq \ p_i \ \leq \ 1 \\ 0 \ \leq \ p_{ij} \ &\leq \ p_j \ \leq \ 1 \\ p_i \ + \ p_j \ - \ p_{ij} \ &\leq \ 1 \end{aligned}$$

$$\begin{aligned} -1 \ &\leq \ p_{13} \ + \ p_{14} \ + \ p_{24} \ - p_{23} \ - \ p_1 \ - \ p_4 \ \leq \ 0 \\ -1 \ &\leq \ p_{23} \ + \ p_{24} \ + \ p_{14} \ - \ p_{13} \ - \ p_2 \ - \ p_4 \ \leq \ 0 \\ -1 \ &\leq \ p_{14} \ + \ p_{13} \ + \ p_{23} \ - \ p_{24} \ - \ p_1 \ - \ p_3 \ \leq \ 0 \\ -1 \ &\leq \ p_{24} \ + \ p_{23} \ + \ p_{13} \ - \ p_{14} \ - \ p_2 \ - \ p_3 \ \leq \ 0. \end{aligned}$$

Of course, if one combines propositions 2.1 and 2.2, one can bypass reference to correlation polytopes, and assert directly that, in the particular cases considered, vectors in $\mathbb{R}^{n+|S|}$ admit a classical representation if and only if they satisfy the corresponding set of inequalities. The combined versions, for cases (b) and (c), were proved by Fine ([2], [3]). His argument, however, did not involve geometrical ideas, and did not readily lend itself to generalization. Pitowsky's does. Proposition 2.1 provides an algorithm for finding the set of "generalized Bell inequalities" corresponding to any choice of n and S. One can find it, at least in principle, by systematically formulating all conditions of form "vector p falls to one (specified) side of hyperplane H", as H ranges over all bounding hyperplanes of $c(n, S)$.

Proof. (a) Let $n = 2$ and let $S = \{\langle 1, 2\rangle\}$. The vertices of $c(n, S)$ are: $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, and $(1, 1, 1)$. All four satisfy the inequalities:

$$0 \ \leq \ p_{12} \ \leq \ p_1 \ \leq \ 1 \qquad 0 \ \leq \ p_{12} \ \leq \ p_2 \ \leq \ 1 \qquad p_1 \ + \ p_2 \ - \ p_{12} \ \leq \ 1.$$

(This is easy to check.) Furthermore, if vectors p and p' in $\mathbb{R}^{n+|S|}$ satisfy the inequalities, so does every convex combination $p'' \ = \ \lambda\, p \ + \ (1 - \lambda)\, p'$. For example, p'' satisfies the third inequality because

$$\begin{aligned} p''_1 \ + \ p''_2 \ - \ p''_{12} \ &= [\lambda\, p + (1-\lambda)\, p']_1 + [\lambda\, p + (1-\lambda)\, p']_2 - [\lambda\, p + (1-\lambda)\, p']_{12} \\ &= \lambda\, [p_1 + p_2 - p_{12}] + (1-\lambda)\, [p'_1 + p'_2 - p'_{12}] \ \leq \ \lambda + (1-\lambda) \ = \ 1. \end{aligned}$$

Thus every vector in $c(n, S)$, i.e., every convex combination of the four vertices, satisfies the three inequalities.

Conversely, assume that the vector p in $\mathbb{R}^{n+|S|}$ satisfies them. Then we can express p as a convex combination of $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, and $(1, 1, 1)$:

$$\begin{aligned} p \ &= (p_1, p_2, p_{12}) \\ &= (1 - p_1 - p_2 + p_{12})(0,0,0) + (p_1 - p_{12})(1,0,0) + (p_2 - p_{12})(0,1,0) + p_{12}(1,1,1). \end{aligned}$$

That is, p belongs to $c(n, S)$.

(b) Let $n = 3$ and let $S = \{\langle 1,2\rangle, \langle 1,3\rangle, \langle 2,3\rangle\}$. To prove that every vector in $c(n, S)$ satisfies the indicated inequalities it suffices to check that every vertex

$$p^{\epsilon} \;=\; (\epsilon_1,\, \epsilon_2,\, \epsilon_3,\, \epsilon_1\epsilon_2,\, \epsilon_1\epsilon_3,\, \epsilon_2\epsilon_3) \qquad \epsilon \;=\; \langle \epsilon_1,\, \epsilon_2,\, \epsilon_3\rangle \in \{0,1\}^3$$

does so. (Again, this is easy.) For the converse, assume that a vector p in $\mathbb{R}^{n+|S|}$ satisfies the inequalities. Then there exists a number α such that

$$\alpha \;\leq\; min\{p_{12},\, p_{13},\, p_{23},\, 1 - (p_1 + p_2 + p_3 - p_{12} - p_{13} - p_{23})\}$$

and

$$max\{0, (-p_1 + p_{12} + p_{13}),\, (-p_2 + p_{12} + p_{23}),\, (-p_3 + p_{13} + p_{23})\} \;\leq\; \alpha.$$

(The inequalities guarantee that every number in the second list is less than or equal to every number in the first list.) To every ϵ in $\{0,1\}^3$ we assign a number $\lambda(\epsilon) \geq 0$ as follows:

$$\begin{aligned}
\lambda(0,0,0) &= 1 - (p_1 + p_2 + p_3 - p_{12} - p_{13} - p_{23}) - \alpha \\
\lambda(1,0,0) &= \alpha + (p_1 - p_{12} - p_{13}) \\
\lambda(0,1,0) &= \alpha + (p_2 - p_{12} - p_{23}) \\
\lambda(0,0,1) &= \alpha + (p_3 - p_{13} - p_{23}) \\
\lambda(1,1,0) &= p_{12} - \alpha \\
\lambda(1,0,1) &= p_{13} - \alpha \\
\lambda(0,1,1) &= p_{23} - \alpha \\
\lambda(1,1,1) &= \alpha.
\end{aligned}$$

Clearly the sum of these eight numbers is 1. Furthermore, $p = \Sigma_{\epsilon \in \{0,1\}^3} \lambda(\epsilon) p^{\epsilon}$. For example,

$$\lambda(1,0,0) + \lambda(1,1,0) + \lambda(1,0,1) + \lambda(1,1,1) = [\alpha + (p_1 - p_{12} - p_{13})] + (p_{12} - \alpha) + (p_{13} - \alpha) + \alpha = p_1$$

and

$$\lambda(1,1,0) + \lambda(1,1,1) = (p_{12} - \alpha) + \alpha = p_{12}$$

as required. (The other cases are handled similarly.)

(c) Let $n = 4$ and let $S = \{\langle 1,3\rangle, \langle 1,4\rangle, \langle 2,3\rangle, \langle 2,4\rangle\}$. To prove that every vector in $c(n, S)$ satisfies the indicated inequalities it suffices to check that every vertex

$$p^{\epsilon} \;=\; (\epsilon_1,\, \epsilon_2,\, \epsilon_3,\, \epsilon_4,\, \epsilon_1\epsilon_3,\, \epsilon_1\epsilon_4,\, \epsilon_2\epsilon_3,\, \epsilon_2\epsilon_4) \qquad \epsilon \;=\; \langle \epsilon_1,\, \epsilon_2,\, \epsilon_3, \epsilon_4\rangle \in \{0,1\}^4$$

does so. For the converse, assume that a vector p in $\mathbb{R}^{n+|S|}$ satisfies the inequalities. Then there exists a number β such that

$$\beta \;\leq\; min\{p_1,\, p_2,\, (p_1 - p_{13} + p_{23}),\, (p_2 - p_{23} + p_{13}),\, (p_1 - p_{14} + p_{24}),\, (p_2 - p_{24} + p_{14})\}$$

and

$$max\{0, (p_1 + p_2 - 1), (p_{13} + p_{23} - p_3), (p_{14} + p_{24} - p_4),$$
$$(p_1 + p_2 + p_3 - p_{13} - p_{23} - 1), (p_1 + p_2 + p_4 - p_{14} - p_{24} - 1)\} \ \leq \ \beta.$$

(Again, the inequalities guarantee that every number in the second list is less than or equal to every number in the first list.) Now let $S' = \{\langle 1,3\rangle, \langle 2,3\rangle, \langle 1,3\rangle\}$, and consider $p' = (p'_1, p'_2, p'_3, p'_{12}, p'_{13}, p'_{23})$ in $\mathbb{R}^{3+|S'|}$ defined by

$$\begin{array}{lllllllll} p'_1 & = & p_1 & p'_2 & = & p_2 & p'_3 & = & p_3 \\ p'_{12} & = & \beta & p'_{13} & = & p_{13} & p'_{23} & = & p_{23}. \end{array}$$

One can easily check that p' satisfies all the inequalities cited in part (b). For example,

$$p'_1 \ + \ p'_2 \ + \ p'_3 \ - \ p'_{12} \ - \ p'_{13} \ - \ p'_{23} \ \leq \ 1$$

holds since the left side expression equals $(p_1 + p_2 + p_3 - \beta - p_{13} - p_{23})$ and $(p_1 + p_2 + p_3 - p_{13} - p_{23} - 1) \leq \beta$. Hence, p' belongs to $c(n, S')$, i.e., p' can be expressed as a convex sum of form

$$p' \ = \ \Sigma_{\epsilon \in \{0,1\}^3} \, \lambda'(\epsilon) \, p^{\epsilon}.$$

Similarly, the vector $p'' = (p''_1, p''_2, p''_3, p''_{12}, p''_{13}, p''_{23})$ in $\mathbb{R}^{3+|S'|}$ defined by

$$\begin{array}{lllllllll} p''_1 & = & p_1 & p''_2 & = & p_2 & p''_3 & = & p_4 \\ p''_{12} & = & \beta & p''_{13} & = & p_{14} & p''_{23} & = & p_{24} \end{array}$$

can be expressed as the convex sum

$$p'' \ = \ \Sigma_{\epsilon \in \{0,1\}^3} \, \lambda''(\epsilon) \, p^{\epsilon}.$$

Now for $\epsilon = (\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) \in \{0,1\}^4$ we set

$$\lambda(\epsilon) \ = \ \lambda(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) \ = \ \frac{\lambda'(\epsilon_1, \epsilon_2, \epsilon_3) \, \lambda''(\epsilon_1, \epsilon_2, \epsilon_4)}{\lambda'(\epsilon_1, \epsilon_2, 0) + \lambda'(\epsilon_1, \epsilon_2, 1)}$$

if the denominator is not zero. If it is zero, we set $\lambda(\epsilon) = \lambda(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) = 0$.

[Note that we could just as well have taken $\lambda''(\epsilon_1, \epsilon_2, 0) + \lambda''(\epsilon_1, \epsilon_2, 1)$ for the denominator because the two expressions are equal for all ϵ_1 and ϵ_2. This follows from the fact that

(i) $\lambda'(1,1,0) + \lambda'(1,1,1) \ = \ p'_{12} \ = \ p''_{12} \ = \ \lambda''(1,1,0) + \lambda''(1,1,1)$

(ii) $\lambda'(1,0,0) + \lambda'(1,0,1) + \lambda'(1,1,0) + \lambda'(1,1,1) \ =$
$p'_1 \ = \ p''_1 \ = \ \lambda''(1,0,0) + \lambda''(1,0,1) + \lambda''(1,1,0) + \lambda''(1,1,1)$

(iii) $\lambda'(0,1,0) + \lambda'(0,1,1) + \lambda'(1,1,0) + \lambda'(1,1,1) \ =$
$p'_2 \ = \ p''_2 \ = \ \lambda''(0,1,0) + \lambda''(0,1,1) + \lambda''(1,1,0) + \lambda''(1,1,1).$

There are four cases to consider. If $\epsilon_1 = 1$ and $\epsilon_2 = 1$, the desired equation is (i). If $\epsilon_1 = 1$ and $\epsilon_2 = 0$, we derive it by subtracting (i) from (ii). Similarly, if $\epsilon_1 = 0$ and $\epsilon_2 = 1$, we derive it by subtracting (i) from (iii). Finally, if $\epsilon_1 = 0$ and $\epsilon_2 = 0$, the desired equation follows from the three previous ones and the fact that

$$\Sigma_{\epsilon \in \{0,1\}^3} \lambda'(\epsilon) \ = \ 1 \ = \ \Sigma_{\epsilon \in \{0,1\}^3} \lambda''(\epsilon).]$$

It is clear that $\lambda(\epsilon) \geq 0$ for all $\epsilon \in \{0,1\}^4$. We also have $\Sigma_{\epsilon \in \{0,1\}^4} \lambda(\epsilon) = 1$. To see this note first that

$$\Sigma_{\epsilon \in \{0,1\}^4} \lambda(\epsilon) \ = \ \frac{\Sigma_{\epsilon_1 \epsilon_2} \Sigma_{\epsilon_3 \epsilon_4} [\lambda'(\epsilon_1, \epsilon_2, \epsilon_3)\, \lambda''(\epsilon_1, \epsilon_2, \epsilon_4)]}{\lambda'(\epsilon_1, \epsilon_2, 0) + \lambda'(\epsilon_1, \epsilon_2, 1)}.$$

But for all ϵ_1, ϵ_2,

$$\Sigma_{\epsilon_3 \epsilon_4} [\lambda'(\epsilon_1, \epsilon_2, \epsilon_3)\, \lambda''(\epsilon_1, \epsilon_2, \epsilon_4)] \ = \ [\lambda'(\epsilon_1, \epsilon_2, 0) + \lambda'(\epsilon_1, \epsilon_2, 1)]\, [\lambda''(\epsilon_1, \epsilon_2, 0) + \lambda''(\epsilon_1, \epsilon_2, 1)].$$

So

$$\Sigma_{\epsilon \in \{0,1\}^4} \lambda(\epsilon) \ = \ \Sigma_{\epsilon_1 \epsilon_2} [\lambda''(\epsilon_1, \epsilon_2, 0) + \lambda''(\epsilon_1, \epsilon_2, 1)] \ = \ \Sigma_{\ \epsilon \in \{0,1\}^3} \lambda''(\epsilon) \ = \ 1.$$

We claim, finally, that $p = \Sigma_{\epsilon \in \{0,1\}^4} \lambda(\epsilon)\, p^{\epsilon}$. We check just one representative component: p_{14}. We need to show that

$$p_{14} \ = \ \lambda(1,0,0,1) + \lambda(1,0,1,1) + \lambda(1,1,0,1) + \lambda(1,1,1,1).$$

Now if $\lambda'(1,0,0) + \lambda'(1,0,1) \neq 0$, then

$$\lambda(1,0,0,1) + \lambda(1,0,1,1) \ = \ \frac{\lambda'(1,0,0)\lambda''(1,0,1) + \lambda'(1,0,1)\lambda''(1,0,1)}{\lambda'(1,0,0) + \lambda'(1,0,1)} \ = \ \lambda''(1,0,1).$$

On the other hand, if $\lambda'(1,0,0) + \lambda'(1,0,1) = 0$, then $\lambda(1,0,0,1) + \lambda(1,0,1,1) = 0$. But in this case we also have $\lambda''(1,0,0) + \lambda''(1,0,1) = 0$, and hence $\lambda''(1,0,1) = 0$. So, in either case,

$$\lambda(1,0,0,1) + \lambda(1,0,1,1) \ = \ \lambda''(1,0,1).$$

Similarly,

$$\lambda(1,1,0,1) + \lambda(1,1,1,1) \ = \ \lambda''(1,1,1).$$

Therefore,

$$\lambda(1,0,0,1) + \lambda(1,0,1,1) + \lambda(1,1,0,1) + \lambda(1,1,1,1) \ = \ \lambda''(1,0,1) + \lambda''(1,1,1) \ = \ p''_{13} \ = \ p_{14}$$

as required. $\square$

3 One Attempt to Get Around Bell's Theorem

Here we consider one recent, somewhat non-standard, response to Bell's theorem by László E. Szabó. He argues that, the theorem notwithstanding, quantum mechanics *is* compatible with both "local determinism" and the classical character of probability. (See Szabó ([5], [6]), and Bana and Durt [1].)

Recall the set up in the first section. Given our pair of photons in the singlet state, we know that there exist orientations of the polarizers a, a', b, b' such that the associated probabilities

$$\begin{aligned}
p_a &= pr_{QM}(yes, _|a, _) \\
p_b &= pr_{QM}(_, yes|_, b) \\
p_{ab} &= pr_{QM}(yes, yes|a, b) \\
p_{ab'} &= pr_{QM}(yes, yes|a, b') \\
p_{a'b} &= pr_{QM}(yes, yes|a', b) \\
p_{a'b'} &= pr_{QM}(yes, yes|a', b')
\end{aligned}$$

have the values

$$p_a = p_b = \frac{1}{2} \qquad p_{ab} = p_{ab'} = p_{a'b} = \frac{3}{8} \qquad p_{a'b'} = 0.$$

These violate the Clauser-Horne inequality

$$0 \leq p_a + p_b - p_{ab} - p_{ab'} - p_{a'b} + p_{a'b'} \leq 1.$$

Hence (by theorem 1.3.1), we know these "probabilities" do not admit a probability space representation, i.e., there does *not* exist a probability space (X, Σ, μ) and sets $L_a^+, L_{a'}^+, R_b^+, R_{b'}^+ \in \Sigma$ such that

$$\left.\begin{aligned}
p_a &= \mu(L_a^+) \\
p_b &= \mu(R_b^+) \\
p_{ab} &= \mu(L_a^+ \cap R_b^+) \\
p_{ab'} &= \mu(L_a^+ \cap R_{b'}^+) \\
p_{a'b} &= \mu(L_{a'}^+ \cap R_b^+) \\
p_{a'b'} &= \mu(L_{a'}^+ \cap R_{b'}^+).
\end{aligned}\right\} \quad (**)$$

One straight-forward interpretation of this result is that "quantum probability" violates the constraints of classical probability (as codified by Kolmogorov). The starting point of Szabó's response is the observation that the "quantum probabilities" in question here are conditional in character. p_a, for example, is supposed to be the conditional probability that the left photon will pass through the polarizer *given* that the latter is oriented in direction a. What if we try to take into consideration just what the probability is that the polarizer *is* oriented in that direction? Or if we are casting the discussion in terms of determinism, what if we consider possible hidden variables that determine polarizer settings in addition to everything else (rather than treat the settings as independent variables under our control).

Szabó's proposal, in effect, is to consider a second, weaker sense in which one might try to give the numbers $p_a, p_b, p_{ab}, ..., p_{a'b'}$ a "probability space representation". Here we explicitly recognize the composite character of the events under consideration. Rather than looking for just four sets $L_a^+, L_{a'}^+, R_b^+, R_{b'}^+$ in Σ, we look for six sets $L_a, L_{a'}, R_b, R_{b'}, L^+, R^+$ in Σ. Intuitively, we think of L_a as the set of hidden states in which it is determined that the left polarizer will have orientation a (and similarly for $L_{a'}, R_b, R_{b'}$). We think of L^+ as the set of hidden states in which it is determined that the photon will pass through the left polarizer (and similarly for R^+). The conditions we require now are not (∗∗) above, but rather the following:

$$\left.\begin{array}{lcl} p_a & = & \dfrac{\mu(L_a \cap L^+)}{\mu(L_a)} \\[2ex] p_b & = & \dfrac{\mu(R_b \cap R^+)}{\mu(R_b)} \\[2ex] p_{ab} & = & \dfrac{\mu(L_a \cap L^+ \cap R_b \cap R^+)}{\mu(L_a \cap R_b)} \\[2ex] p_{ab'} & = & \dfrac{\mu(L_a \cap L^+ \cap R_{b'} \cap R^+)}{\mu(L_a \cap R_{b'})} \\[2ex] p_{a'b} & = & \dfrac{\mu(L_{a'} \cap L^+ \cap R_b \cap R^+)}{\mu(L_{a'} \cap R_b)} \\[2ex] p_{a'b'} & = & \dfrac{\mu(L_{a'} \cap L^+ \cap R_{b'} \cap R^+)}{\mu(L_{a'} \cap R_{b'})}. \end{array}\right\} \quad (*\,*\,*)$$

Actually, we need more than just these conditions. We are now, implicitly, relativizing our probability space representations (or, equivalently, our deterministic hidden variable theories) to particular experiments. In any one experiment, the polarizer orientations, right and left, occur with particular frequencies. These frequencies must also be recovered. (Maybe on one occasion, for example, the four possibilities (a, b), (a, b'), (a', b), (a', b') are observed with equal frequency – each, say, occurring 250 times in a run of 1000.) Nothing has been said so far about such frequencies because it has been assumed that they made no difference. Now we imagine that we have a particular experimental run in mind, and have observed experimental probabilities (or frequencies) for the different polarizer settings:

$$\begin{aligned} l_a &= \text{observed probability for orientation } a \text{ on the left} \\ l_{a'} &= \text{observed probability for orientation } a' \text{ on the left} \\ r_b &= \text{observed probability for orientation } b \text{ on the right} \\ r_{b'} &= \text{observed probability for orientation } b' \text{ on the right} \end{aligned}$$

What we must add to $(*\,*\,*)$ is the following set of conditions:

$$\left.\begin{array}{lcl} l_a & = & \mu(L_a) \\ l_{a'} & = & \mu(L_{a'}) \\ r_b & = & \mu(R_b) \\ r_{b'} & = & \mu(R_{b'}) \\ l_a r_b & = & \mu(L_a \cap R_b) \\ l_a r_{b'} & = & \mu(L_a \cap R_{b'}) \\ l_{a'} r_b & = & \mu(L_{a'} \cap R_b) \\ l_{a'} r_{b'} & = & \mu(L_{a'} \cap R_{b'}). \end{array}\right\} \quad (****)$$

Putting all this together, the question under consideration is whether, given a particular run of the two photon experiment, we can find a probability space (X, Σ, μ) and sets L_a, $L_{a'}$, R_b, $R_{b'}$, L^+, R^+ in Σ such that $(***)$ and $(****)$ hold. The answer is certainly 'yes'. Let's first verify that this is so, and then return to consider the significance of this fact.

	L_a		$L_{a'}$	
R_b	$L_a \cap L^+ \cap R_b \cap R^+$ $\frac{3}{8}(l_a r_b)$	$L_a \cap L^+ \cap R_b \cap R^-$ $\frac{1}{8}(l_a r_b)$	$L_{a'} \cap L^+ \cap R_b \cap R^+$ $\frac{3}{8}(l_{a'} r_b)$	$L_{a'} \cap L^+ \cap R_b \cap R^-$ $\frac{1}{8}(l_{a'} r_b)$
	$L_a \cap L^- \cap R_b \cap R^+$ $\frac{1}{8}(l_a r_b)$	$L_a \cap L^- \cap R_b \cap R^-$ $\frac{3}{8}(l_a r_b)$	$L_{a'} \cap L^- \cap R_b \cap R^+$ $\frac{1}{8}(l_{a'} r_b)$	$L_{a'} \cap L^- \cap R_b \cap R^-$ $\frac{3}{8}(l_{a'} r_b)$
$R_{b'}$	$L_a \cap L^+ \cap R_{b'} \cap R^+$ $\frac{3}{8}(l_a r_{b'})$	$L_a \cap L^+ \cap R_{b'} \cap R^-$ $\frac{1}{8}(l_a r_{b'})$	$L_{a'} \cap L^+ \cap R_{b'} \cap R^+$ 0	$L_{a'} \cap L^+ \cap R_{b'} \cap R^-$ $\frac{1}{2}(l_{a'} r_{b'})$
	$L_a \cap L^- \cap R_{b'} \cap R^+$ $\frac{1}{8}(l_a r_{b'})$	$L_a \cap L^- \cap R_{b'} \cap R^-$ $\frac{3}{8}(l_a r_{b'})$	$L_{a'} \cap L^- \cap R_{b'} \cap R^+$ $\frac{1}{2}(l_{a'} r_{b'})$	$L_{a'} \cap L^- \cap R_{b'} \cap R^-$ 0

Table 1: The displayed probabilities satisfy all conditions in $(***)$ and $(****)$.

It will be easiest to exhibit the requisite example with a diagram (see Table 1). (The elements of the background set X make no difference. They might as well be points in a region of the Euclidean plane.) In the diagram we label 16 distinct boxes, each the intersection of four sets, e.g., $(L_a \cap L^+ \cap R_b \cap R^+)$. (Notation: L^- and R^- are understood to be the complement sets $X - L^+$ and $X - R^+$.) The six sets L_a, $L_{a'}$, R_b, $R_{b'}$, L^+, R^+ individually, of course, can be reconstructed as appropriate unions of (eight of these) boxes. So, for example, L_a is the union of the boxes:

$$\begin{array}{rl} (L_a \cap L^+ \cap R_b \cap R^+) & (L_a \cap L^+ \cap R_b \cap R^-) \\ (L_a \cap L^- \cap R_b \cap R^+) & (L_a \cap L^- \cap R_b \cap R^-) \\ (L_a \cap L^+ \cap R_{b'} \cap R^+) & (L_a \cap L^+ \cap R_{b'} \cap R^-) \\ (L_a \cap L^- \cap R_{b'} \cap R^+) & (L_a \cap L^- \cap R_{b'} \cap R^-). \end{array}$$

In each box there is a displayed a number that should be understood as the probability assigned by the measure μ to that box. So, for example,

$$\mu(L_a \cap L^+ \cap R_b \cap R^+) = \frac{3}{8}(l_a r_b).$$

Assignments to disjoint unions of these boxes are determined by addition. Thus

$$\begin{aligned}\mu(L_a \cap R_b) &= \mu(L_a \cap L^+ \cap R_b \cap R^+) + \mu(L_a \cap L^+ \cap R_b \cap R^-) + \\ &\quad \mu(L_a \cap L^- \cap R_b \cap R^+) + \mu(L_a \cap L^- \cap R_b \cap R^-) \\ &= \frac{3}{8}(l_a r_b) + \frac{1}{8}(l_a r_b) + \frac{1}{8}(l_a r_b) + \frac{3}{8}(l_a r_b) \\ &= l_a r_b.\end{aligned}$$

It is straightforward to verify that all the conditions in $(***)$ and $(****)$ are satisfied. For example,

$$\frac{\mu(L_a \cap L^+)}{\mu(L_a)} = \frac{(\frac{3}{8}+\frac{1}{8})(l_a r_b) + (\frac{3}{8}+\frac{1}{8})(l_a r_{b'})}{l_a r_b + l_a r_{b'}} = \frac{1}{2} = p_a$$

and

$$\frac{\mu(L_a \cap L^+ \cap R_b \cap R^+)}{\mu(L_a \cap R_b)} = \frac{\frac{3}{8}(l_a r_b)}{l_a r_b} = \frac{3}{8} = p_{ab}.$$

The example we have just considered – with quantum mechanical probabilities arising from a pair of photons in the singlet state – is very specific, of course. The question naturally arises whether a similar treatment is available for all probabilities arising in quantum mechanics. The question is not yet precise, and we will not take the time to make it so. But this can be done (see Szabó [6] and Bana and Durt [1]) and the answer is 'yes'. Roughly speaking, the claim is this.

All probabilities involving experimental trials can be considered conditional in character. They can be understood to be of form $p(O|I)$, the probability that if an experiment characterized by initial conditions I is performed, the outcome will be O. Sometimes, as in our example, we consider, side by side, probabilities whose associated initial conditions are incompatible with one another. (We cannot simultaneously test the probability that the left photon will pass through the polarizer when it has orientation a, and also test the probability that it will pass through when polarizer has orientation a'.) It is in these cases that it sometimes becomes impossible to give the numbers in question (i.e., the numbers $p(O|I)$) a probability space representation in the initial sense.

The burden of the theorem under consideration is to make precise and prove the claim that in *all* cases – not just those involving quantum mechanics – one can have a modified probability space representation in which the numbers emerge as conditional probabilities of form $\mu(O \cap I)/\mu(I)$. In this sense, at least, Szabó argues, observed empirical data can *never* be in conflict with the principles of classical probability or with determinism.

It should be appreciated just how weak this sense is. Let's stay with the two photon example. It is a prediction of quantum mechanics that the probability for joint passage through the two polarizers is given by

$$pr_{QM}(yes, yes|a, b) = \frac{1}{2}\cos^2\angle(a, b).$$

(Recall assertion (QM1) on page 2.) This formula is confirmed by numerous experiments of the most diverse sort. It seems to express a fact about the the two photon system (in the single state) *itself*, about its disposition to behave *whenever* it is subjected to a test of the appropriate sort. One would like to have a hidden variable theory that reconstructs these probabilties *once and for all*, without reference to particular experimental tests. Instead, one gets from Szabó, in effect, a different hidden variable theory for each test or, to be more precise, a different theory for each set of non-negative real numbers l_a, $l_{a'}$, r_b, $r_{b'}$ summing to 1.

I hope to have further discussion of the significance of Szabo's work in class.

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