A note about closed timelike curves in Gödel space-time

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A greatest lower bound for the total (integrated) energy of closed timelike curves in Gödel space-time is derived. (Here "energy" is determined relative to the velocity field of the major mass points of the universe.) The derivation is then used to reconstruct and extend a remark of Gödel's concerning total (integrated) acceleration requirements for "time travel" in his model universe.

I. INTRODUCTION

Gödel space-time,[1] of course, is not a live candidate for describing our universe. But it is an interesting geometric structure, and a source of insight into the possibilities allowed by relativity theory.

In this paper we present an elementary, but perhaps somewhat curious, proposition concerning the geometry of closed timelike curves in Gödel space-time (Proposition 2). It establishes a greatest lower bound for the total (integrated) energy of such curves (where "energy" is determined relative to the velocity field of the major mass points of the Universe). The proposition turns on the possibility of reducing questions about total energy (of closed timelike curves in Gödel space-time) to more tractable questions about area enclosure by curves in the hyperbolic plane (Proposition 1).

By way of application, we also invoke the proposition to reconstruct and extend a remark of Gödel's[2] concerning total (integrated) acceleration requirements for "time travel" in his model universe. It was this remark that first suggested our question about total energy. We close with a brief discussion of a conjecture on minimal total acceleration requirements.

II. PRELIMINARIES

In this section we recall several basic facts about Gödel space-time and introduce some notation.[3]

We take Gödel space-time to be the pair (M, g_{mn}) where M is $\mathbb{R}^4$ and g_{mn} is a Lorentz metric on M characterized by the condition that for some point (and hence, by homogeneity, any point) p in M, there is a global adapted (cylindrical) coordinate system t, r, φ, y on M in which $t(p) = r(p) = y(p) = 0$ and

$$\begin{aligned} g_{mn} = 4\mu^2[&(dt)_m(dt)_n - (dr)_m(dr)_n - (dy)_m(dy)_n \\ &+ (\text{sh}^4 r - \text{sh}^2 r)(d\varphi)_m(d\varphi)_n \\ &+ 2\sqrt{2}\,\text{sh}^2 r(d\varphi)_{(m}(dt)_{n)}]. \end{aligned}$$

(We use sh r and ch r, respectively, to abbreviate sinh r and cosh r.) Here $-\infty < t < \infty$, $-\infty < y < \infty$, $0 \leqslant r < \infty$, and $0 \leqslant \varphi < 2\pi$ with $\varphi = 0$ identified with $\varphi = 2\pi$. The metric g_{mn} is a solution to Einstein's equation

$$R_{mn} - \tfrac{1}{2} g_{mn} R = 8\pi\kappa[\rho\eta_m\eta_n - p(g_{mn} - \eta_m\eta_n)]$$

for a perfect fluid source with four-velocity $\eta^m = (\partial/\partial t)^m/2\mu$, mass density $\rho = 1/(16\pi\kappa\mu^2)$, and pressure $p = 1/(16\pi\kappa\mu^2)$.[4]

Here, η^m is a unit timelike Killing field, and defines a temporal orientation on (M, g_{mn}). The integral curves of the field, characterized by constant values for r, φ, and y, will be called *matter lines*. The $(\partial/\partial\varphi)^m$ is a rotational Killing field with squared norm $4\mu^2(\text{sh}^4 r - \text{sh}^2 r)$. Its (closed) integral curves, characterized by constant values for t, r, and y, will be called *Gödel circles*. Gödel circles with critical radius $r_c = \ln(1+\sqrt{2})$ are closed null curves (since sh $r_C = 1$). Those with radius $r > r_C$ are closed timelike curves. Here $(\partial/\partial y)^m$ is a covariantly constant field with squared norm $-4\mu^2$.

Let S be a $t = $ const, $y = $ const submanifold of M. Orthogonal projection of g_{mn} induces a (negative definite) metric

$$h_{mn} = g_{mn} - \left(\frac{1}{4}\mu^2\right)\left[\left(\frac{\partial}{\partial t}\right)_m\left(\frac{\partial}{\partial t}\right)_n - \left(\frac{\partial}{\partial y}\right)_m\left(\frac{\partial}{\partial y}\right)_n\right]$$

on S.[5] Now

$$\left(\frac{\partial}{\partial t}\right)_m = 4\mu^2[(dt)_m + \sqrt{2}\,\text{sh}^2 r(d\varphi)_m]$$

and

$$\left(\frac{\partial}{\partial y}\right)_m = -4\mu^2(dy)_m.$$

So

$$h_{mn} = -4\mu^2[(dr)_m(dr)_n + \tfrac{1}{4}\text{sh}^2 2r(d\varphi)_m(d\varphi)_n].$$

Once h_{mn} is presented in this form it is not difficult to verify that the pair $(S, -h_{mn})$ is a complete two-dimensional Riemannian manifold with constant curvature $-1/\mu^2$.[6]

In what follows we use the following notation. Given a timelike curve[7] γ in (M, g_{mn}), we take its four-velocity (i.e., unit tangent vector field) to be ξ^m, and set

$\alpha^m = \xi^n\nabla_n\xi^m$ (the acceleration of γ),

$a = (-\alpha^m\alpha_m)^{1/2}$ (the magnitude of γ's acceleration)

$E = \xi^m\eta_m$

(γ's energy with respect to the unit Killing field η^m).

We also use the parameter s for arc length (= elapsed proper time) along γ, and set

$$PT(\gamma) = \int_\gamma ds \quad \text{(total elapsed proper time of } \gamma\text{)},$$

$$TA(\gamma) = \int_\gamma a\, ds \quad \text{(total acceleration of } \gamma),$$

$$TE(\gamma) = \int_\gamma E\, ds \quad \text{(total energy of } \gamma).$$

Note that $E \geqslant 1$ (since ξ^m and η^m are both future directed, unit timelike vectors), and that $E = 1/(1 - v^2)^{1/2}$, where v is the speed of γ relative to matter lines. In terms of the coordinates above, E is given by

$$E = 2\mu\left[\left(\frac{dt}{ds}\right) + \sqrt{2}\, \text{sh}^2 r\left(\frac{d\varphi}{ds}\right)\right].$$

In the special case where γ is a Gödel circle of radius $r > r_C$, we have

$$\xi^m = \left(\frac{d\varphi}{ds}\right)\left(\frac{\partial}{\partial\varphi}\right)^m$$

where

$$\frac{d\varphi}{ds} = \frac{1}{2\mu(\text{sh}^4 r - \text{sh}^2 r)^{1/2}},$$

and hence,[8]

$$E = \sqrt{2}\, \text{sh}^2 r/(\text{sh}^4 r - \text{sh}^2 r)^{1/2},$$

$$a = (1/4\mu)\text{sh}\, 2r(2\, \text{sh}^2 r - 1)/(\text{sh}^4 r - \text{sh}^2 r),$$

$$PT(\gamma) = 4\pi\mu(\text{sh}^4 r - \text{sh}^2 r)^{1/2},$$

$$TA(\gamma) = \pi\, \text{sh}\, 2r(2\, \text{sh}^2 r - 1)/(\text{sh}^4 r - \text{sh}^2 r)^{1/2},$$

$$TE(\gamma) = 4\sqrt{2}\pi\mu\, \text{sh}^2 r.$$

III. ENERGY AND AREA

Clearly, $4\sqrt{2}\pi\mu$ is the (unrealized) greatest lower bound of $TE(\gamma)$ as γ ranges over Gödel circles of radius $r > r_C$. In this section we prove that it is actually the greatest lower bound as γ ranges over *all* closed timelike curves. The first step in the argument is to give $TE(\gamma)$ an intuitive geometric interpretation.

In what follows let γ be some closed timelike curve, let S be some $t = \text{const}$, $y = \text{const}$ submanifold of M, and let γ^* be the closed (at least piecewise smooth) curve that results from projecting γ into S. Notice first that since γ is closed, we have (using our coordinate expression for E)

$$TE(\gamma) = 2\sqrt{2}\mu \int_\gamma \text{sh}^2 r\, d\varphi.$$

The integrand on the right depends only on r and φ. So we may perform the integration over γ^* rather than γ. Thus

$$TE(\gamma) = 2\sqrt{2}\mu \int_{\gamma^*} \text{sh}^2 r\, d\varphi.$$

We can evaluate the right-hand integral using Stokes' theorem. Let S be assigned the orientation, say, determined by the field $(\partial/\partial\varphi)^m$. Assume for the moment that γ^* is a simple (i.e., non-self-intersecting) curve. Then it forms the boundary of an (oriented) region G in S, and we have

$$\int_{\gamma^*} \text{sh}^2 r\, d\varphi = \int_G d(\text{sh}^2 r\, d\varphi)$$

$$= \int_G \text{sh}\, 2r\, dr\, d\varphi = \frac{1}{2\mu^2}\int_G dA,$$

where dA is the area element $2\mu^2\, \text{sh}\, 2r\, dr\, d\varphi$ on S. Now notice that the formula, suitably interpreted, holds even in the case where γ is allowed to be self-intersecting. For in this case γ^* can be decomposed as a "sum" of simple closed curves, and we can associate with it a corresponding sum G of oriented regions bounded by these curves. To extend the formula we simply apply it to each element in the sum, and add.

In what follows, "area" should be understood in the extended sense of "signed, summed area." On that understanding we can formulate our conclusion as follows.

Proposition 1: Let γ be a closed timelike curve, and let G be the (oriented, summed) region obtained by projecting γ into any $t = \text{const}$, $y = \text{const}$ submanifold S. Then

$$TE(\gamma) = (\sqrt{2}/\mu)\cdot \text{the area of } G.$$

Now we determine a greatest lower bound for the right-hand side of the equation. We do so using an "isoperimetric inequality." Consider any complete two-dimensional Riemannian manifold of constant curvature k. Let L and A, respectively, be the length of, and area enclosed by, a (possibly self-intersecting) closed curve in the manifold. Then

$$L^2 \geqslant (4\pi - kA)A,$$

and equality holds iff the curve is a circle.[9] (It follows that of all closed curves of given length, area is strictly maximized by circles.) The case of interest to us is that in which $k = -1/\mu^2$.

Let γ, γ^*, and G be as above, let ξ^m be the four-velocity of γ, and let σ^m be the component of ξ^m orthogonal to $(\partial/\partial t)^m$ and $(\partial/\partial y)^m$. Then

$$-\sigma^m\sigma_m = E^2 - E_y{}^2 - 1,$$

where $E_y = \xi^m(\partial/\partial y)_m/2\mu$. So if L is the length of γ^* and A is the area of G, we have (by Proposition 1)

$$L = \int_\gamma (E^2 - E_y{}^2 - 1)^{1/2}\, ds < \int_\gamma E\, ds = (\sqrt{2}/\mu)A.$$

Combining our two inequalities (with $k = -1/\mu^2$) we arrive at our principal result.

Proposition 2: Let γ, γ^*, and G be as above. Let L be the length of γ^*, and let A be the area of G. Then

(a) $A > 4\pi\mu^2$ and $L > 4\sqrt{2}\pi\mu$.

Hence (by Proposition 1),

(b) $TE(\gamma) > 4\sqrt{2}\pi\mu$.

Given our previous remarks about Gödel circles, it follows that $4\sqrt{2}\pi\mu$ is the greatest lower bound of $TE(\gamma)$ as γ ranges over all closed timelike curves. It also follows that the two lower bounds in (a) are greatest. For this we need only observe that Gödel circles γ of radius $r > r_C$ have area and length

$$A = (\mu/\sqrt{2})TE(\gamma) = 4\pi\mu^2\, \text{sh}^2 r,$$

$$L = \int_\gamma (E^2 - 1)^{1/2}\, ds = (E^2 - 1)^{1/2}PT(\gamma) = 2\pi\mu\, \text{sh}\, 2r.$$

We can think of clause (a) as asserting, simply, that no closed timelike curve has an associated area, after projection, that is as small as the area of a disk of critical radius r_C (or a length, after projection, as small as the circumference of that disk).

IV. GÖDEL'S REMARK

In a paper devoted to a discussion of the philosophical significance of his discoveries in general relativity, Gödel cites a calculation of "fuel requirements" for travel along closed timelike curves in his universe:

"Basing the calculation on a mean density of matter equal to that observed in our world, and assuming one were able to transform matter completely into energy, the weight of the "fuel" of the rocket ship, in order to complete the voyage in t years (as measured by the traveler), would have to be of the order of magnitude of $10^{22}/t^2$ times the weight of the ship (if stopping, too, is effected by recoil). This estimate applies to $t \ll 10^{11}$ yr. Irrespective of the value of t, the velocity of the ship must be at least $1/\sqrt{2}$ of the velocity of light."[2]

It seems likely that Gödel was considering time travel along Gödel circles, and calculated the fuel required to accelerate from zero velocity to the velocities characteristic of those circles, and then back again.[10] (Here "velocity" is understood to mean "speed relative to matter lines.") That is why he can refer to *the* (unchanging) velocity of the ship.[11] Using Proposition 2, it will be possible for us to recover Gödel's numbers without assuming that the time traveler traverses Gödel circles (or sections thereof).

We make use of a lemma[12] that connects total acceleration to changes in energy value.

Lemma 3: Let γ be a timelike curve connecting points p and q. Then

$$TA(\gamma) > |\ln E(q) - \ln E(p)|.$$

[Here, of course, $E(q)$ is the value of E that γ assumes at q.]

Proof: Let $g'_{mn} = g_{mn} - \xi_m \xi_n$ be the (negative definite) metric that results from projecting g_{mn} orthogonal to ξ^m. Since η^m is a Killing field, we have

$$\frac{dE}{ds} = \xi^n \nabla_n E = \xi^n \xi^m \nabla_n \eta_m + \eta_n \alpha^n = \eta_n \alpha^n = g'_{mn} \eta^m \alpha^n.$$

Hence, by the Schwarz inequality (applied to $-g'_{mn}$),

$$\left|\frac{dE}{ds}\right| = |-g'_{mn} n^m \alpha^n|$$
$$\leqslant (-g'_{mn}\alpha^m\alpha^n)^{1/2}(-g'_{mn}\eta^m\eta^n)^{1/2}$$
$$= a(E^2 - 1)^{1/2} < aE.$$

So $a > |d(\ln E)/ds|$, and therefore

$$TA(\gamma) = \int_\gamma a\, ds > |\ln E(q) - \ln E(p)|. \qquad \blacksquare$$

The corollary we now state concerns closed timelike curves that have initial (and perhaps final) four-velocity η^m. They represent the trajectories of time travelers who start out (and perhaps end up) at rest relative to the major mass points of the Universe.

Corollary 4: Let γ be a closed timelike curve.

(a) If γ has initial four-velocity η^m, then

$TA(\gamma) > |\ln(4\sqrt{2}\pi\mu/PT(\gamma))|$.

(b) If γ has *both* initial and final four-velocity η^m, then

$TA(\gamma) > 2|\ln(4\sqrt{2}\pi\mu/PT(\gamma))|$.

Proof: Let p be the initial (= terminal) point of γ, and let q be a point on γ at which E achieves its *average* value. By Proposition 2,

$$E(q) \cdot PT(\gamma) = \int_\gamma E\, ds > 4\sqrt{2}\pi\mu.$$

Let $E(p^+)$ and $E(p^-)$ be the initial and terminal values of E at p. In case (a) we have $E(p^+) = 1$, and the assertion follows immediately if we apply the lemma to that stretch of γ running from p to q. In case (b) we have $E(p^-) = 1$ as well, and so we can apply the lemma, in addition, to the return stretch of γ running from q back to p. ■

Now we establish the connection between total acceleration and "fuel consumption." [12] Suppose γ represents the trajectory of a point particle "rocket ship." Let m be its mass, and J^n the energy momentum of its exhaust. Let us assume that the rocket is suitably isolated during its trip. (It is not refueled, nor hit by meteors.) Then the energy momentum of the rocket's exhaust must balance precisely the rate at which the rocket itself loses energy momentum, i.e.,

$$J^n = -\xi^p \nabla_p (m\xi^n) = -(\xi^n \xi^p \nabla_p m + m\alpha^n).$$

And the mass of the rocket must be nonincreasing (i.e., $\xi^p \nabla_p m \leqslant 0$) since the rocket is consuming fuel. Hence, since J^n is causal (i.e., $J^n J_n \geqslant 0$),

$$a \leqslant (-\xi^n \nabla_n m)/m = -d(\ln m)/ds.$$

Let m_p be the mass of the rocket's payload (the rocket with empty fuel tanks), and let m_f be the mass of the fuel with which it starts. Assuming that the rocket arrives with empty fuel tanks, we have (by integration)

$$(m_p + m_f)/m_p \geqslant e^{TA(\gamma)}.$$

Now let us insert some numbers. Recall that the parameter μ is correlated with cosmic mass density ρ by the relation $\rho = 1/(16\pi\kappa\mu^2)$. If we take for ρ the value 10^{-30} g/cm^3 (the estimated mass density of our universe), then $\mu \approx 10^{10}$ yr, and $4\sqrt{2}\pi\mu \approx 10^{11}$ yr. Hence, in our two cases (a) and (b), assuming $PT(\gamma) \ll 10^{11}$ yr,

case (a): $m_f/m_p \geqslant 10^{11}/PT(\gamma)$,

case (b): $m_f/m_p \geqslant 10^{22}/(PT(\gamma))^2$

[where $PT(\gamma)$ is given in years].

V. A CONJECTURE

Corollary 4 applies only to closed timelike curves γ that are initially tangent to matter lines. And even within this restricted class, it places no lower bound on $TA(\gamma)$. It leaves open the possibility that $TA(\gamma)$ can be made arbitrarily small if $PT(\gamma)$ is allowed to be arbitrarily large. (A sufficiently patient time traveler might not need much fuel for his rocket ship.)

It seems natural to ask what the greatest lower bound of $TA(\gamma)$ is as γ ranges over *all* closed timelike curves. Let GLB be this number.[13] In earlier work we showed that GLB > 0.[3] It now seems to us overwhelmingly likely that $\text{GLB} = 2\pi(9 + 6\sqrt{3})^{1/2} \approx 28$. (This would yield a fuel consumption ratio m_f/m_p larger than 10^{12}.)

One arrives at that particular number by considering Gödel circles. As noted in Sec. II., Gödel circles γ of radius $r > r_C$ have total acceleration

$$TA(\gamma) = \pi \operatorname{sh} 2r (2 \operatorname{sh}^2 r - 1)/(\operatorname{sh}^4 r - \operatorname{sh}^2 r)^{1/2}.$$

This expression assumes a minimal value of $2\pi(9 + 6\sqrt{3})^{1/2}$ when $\operatorname{sh}^2 r = (1 + \sqrt{3})/2$.

One might hope to prove the conjecture using ideas related to those in Sec. III, i.e., by reducing it to an assertion about closed curves in the hyperbolic plane, and then invoking the "isoperimetric inequality" (or something similar). But we have not been able to do so. The best we have done so far,[14] is to show that *Gödel circles of the required special radius are the only closed timelike curves that minimize total acceleration against local variation*. So if the value GLB is realized by any closed timelike curve, the conjecture must be true. It seems overwhelmingly likely that the value *is* realized (because of the nature of the sectional curvatures of the Gödel metric).

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[1]See, for example, K. Gödel, Rev. Mod. Phys. **21**, 447 (1949); W. Kundt, Z. Phys. **145**, 611 (1956); S. Chandrasekhar and J. P. Wright, Proc. Natl. Acad. Sci. USA **47**, 341 (1961); H. Stein, Philos. Sci. **37**, 589 (1970); J. Pfarr, Gen. Relativ. Gravit. **13**, 1073 (1981); and other references cited by Pfarr.

[2]K. Gödel, "A remark about the relationship between relativity theory and idealistic philosophy," in *Albert Einstein: Philosopher–Scientist*, edited by P. Schilpp (Open Court, La Salle, IL, 1949).

[3]To make the paper self-contained, we here (and in Sec. V) include some material previously presented in D. Malament, J. Math. Phys. **26**, 774 (1985).

[4]In Gödel's original paper he interpreted his model as a solution to Einstein's equation

$$R_{mn} - \tfrac{1}{2} g_{mn} R + \lambda g_{mn} = 8\pi\kappa\rho' \eta_m \eta_n$$

with cosmological constant $\lambda = 1/(2\mu^2)$, for a pressureless fluid source with mass density $\rho' = 1/(8\pi\kappa\mu^2)$.

[5]This statement is slightly delicate. The Killing field η^m is not hypersurface orthogonal. So strictly speaking, the metric h_{mn} does not live *on* S (or on any other submanifold of M). However, we can here invoke a canonical one-to-one (tensor operation preserving) correspondence between tensor fields on S and fields on M that are (i) orthogonal to $(\partial/\partial y)^m$ and η^m in all indices, and (ii) Lie derived by $(\partial/\partial y)^m$ and η^m. [See the Appendix in R. Geroch, J. Math Phys. **12**, 918 (1971).] That h_{mn} is Lie derived by $(\partial/\partial y)^m$ and η^m follows from the fact that η^m is a Killing field of constant length and $(\partial/\partial y)^m$ is covariantly constant.

[6]One way to see this is the following. Consider new coordinates on S defined by

$$x_1 = \mu \operatorname{ch} 2r, \quad x_2 = \mu \operatorname{sh} 2r \cos\varphi, \quad x_3 = \mu \operatorname{sh} 2r \sin\varphi.$$

Clearly, $x_1 > 0$ and $x_1{}^2 - x_2{}^2 - x_3{}^2 = \mu^2$ for all r and φ. Furthermore, in these coordinates the metric $-h_{mn}$ assumes the form

$$-h_{mn} = -(dx_1)_m (dx_1)_n + (dx_2)_m (dx_2)_n + (dx_3)_m (dx_3)_n.$$

Thus $(S, -h_{mn})$ is isometric to the upper half of a two-sheeted hyperboloid of radius μ in $\mathbf{R}^3$, with respect to the metric induced on the latter by a background flat metric of signature $(-,+,+)$. It is a standard result that this hyperboloid (under the induced metric) is a complete Riemannian manifold with constant curvature $-1/\mu^2$. [See, for example, B. O'Neill, *Semi-Riemannian Geometry* (Academic, New York, 1983).] *Correction:* In the paper cited in Ref. 3, we worked throughout with the value $\mu = \frac{1}{2}$, and mistakenly asserted a curvature of $-\frac{1}{4}$, rather than -4. This slip did not affect our argument.

[7]"Timelike curves" will be understood to be future directed and smooth (everywhere) unless they are closed, in which case smoothness will be allowed to fail at initial (= terminal) points.

[8]We derive, e.g., the expression for a. Let $f = 1/[2\mu(\operatorname{sh}^4 r - \operatorname{sh}^2 r)^{1/2}]$. Then $\xi^n = f(\partial/\partial\varphi)^n$. Clearly, $\xi^n \nabla_n f = 0$. Hence, since $(\partial/\partial\varphi)^n$ is a Killing field,

$$\begin{aligned} \alpha_n &= f^2 \left(\frac{\partial}{\partial\varphi}\right)^m \nabla_m \left(\frac{\partial}{\partial\varphi}\right)_n \\ &= -f^2 \left(\frac{\partial}{\partial\varphi}\right)^m \nabla_n \left(\frac{\partial}{\partial\varphi}\right)_m \\ &= -(f^2/2)\nabla_n [4\mu^2 (\operatorname{sh}^4 r - \operatorname{sh}^2 r)] \\ &= -2f^2\mu^2 \operatorname{sh} 2r (2\operatorname{sh}^2 r - 1)\nabla_n r. \end{aligned}$$

Our expression for a now follows from the fact that $(\nabla_n r) = (-1/4\mu^2)(\partial/\partial r)_n$.

[9]See, for example, C. Bandle, *Isoperimetric Inequalities and Applications* (Pitman, London, 1980), p. 35. The inequality is usually proved only for non-self-intersecting curves. But it can easily be extended to the more general case we are considering (if area is interpreted in the sense explained).

[10]This is the way J. Pfarr (see Ref. 1) reconstructs Gödel's remark.

[11]It is clear on this construal how Gödel arrives at the number $1/\sqrt{2}$. As noted in Sec. II., a Gödel circle with radius $r > r_c$ has energy $E = \sqrt{2} \operatorname{sh}^2 r/(\operatorname{sh}^4 r - \operatorname{sh}^2 r)^{1/2}$, and so velocity $v = \operatorname{ch} r/(\sqrt{2} \operatorname{sh} r) > 1/\sqrt{2}$.

[12]S. Chakrabarti, R. Geroch, and S. Liang, J. Math. Phys. **24**, 597 (1983).

[13]Note that GLB is a dimensionless constant. The scale dependencies of acceleration and elapsed proper time cancel each other.

[14]Joint work with R. Geroch and L. Lindblom (unpublished).