"Time Travel" in the Gödel Universe

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In the late 1940s Gödel (1949a) made a fundamental contribution to relativity theory by finding an important new solution to Einstein's equation. It represents a possible universe with quite remarkable properties. For one thing, the entire material content of the Gödel universe is in a state of uniform, rigid rotation. For another, free test particles in it exhibit a kind of boomerang effect. Most striking of all, the Gödel universe allows for the possibility of "time travel" in a certain sense. It is this third feature I want to discuss today.

Of course one can mean various things by "time travel". In traditional science fiction stories a classical background time structure is presupposed, and the time traveler is described as jumping from the present to the past (or to the future). In a spacetime diagram his worldline would appear as discontinuous.

In the Gödel universe our notions of "past", "present", and "future" break down to the point where such stories do not really make sense. I have something quite different in mind. I take "time travel" to be nothing more, and nothing less, than the act of starting at a particular point in spacetime, taking an otherwise conventional trip, and somehow returning to (or close to) that very point. No discontinuous motion is involved. Neither is superluminal velocity. In geometric terms, my time traveler is simply one whose worldline is closed (or almost closed).

I spent quite a bit of time last year thinking about "time travel" in the Gödel universe. I got hooked on a certain technical problem that had been posed by my colleague Bob Geroch. In a sense it has to do with how difficult it would be, as a practical matter, to execute time travel if one lived in the Gödel universe. It turns out that one can get a clean hold on this question in the language of spacetime geometry.

In the second half of my talk today I want to discuss the problem I worked on. I'll state one partial result, and one conjecture. But I'll be primarily concerned simply to explain and motivate the problem.

Before that I'll present a bit of background exposition. I hope this, at least, will be of quite general interest. It is my impression that a fair number of people have heard about the Gödel universe, and are curious about it, even though they have no particular expertise in relativity

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theory. This is in part simply because of their interest in Gödel himself. His was, after all, one of the truly powerful minds of our century; and relativity theory is the one topic on which he published outside of logic and the foundations of mathematics.

In any case, in the first half of my talk I'll try to explain just what the Gödel universe looks like, and how the possibility of time travel arises. I think I can do so without presupposing too much more than a certain minimal acquaintance with relativity theory.

1. Preliminaries

There are just a few facts about spacetime geometry that I am going to need. Let me take a moment to recall them.

It is helpful to think of relativity theory as determining a class of possible models for the large-scale spatiotemporal structure of the universe, either in its entirety, or in some restricted region of interest such as our solar system. Each of its models is an ordered pair (M, g_{ab}). Here M is a four-dimensional manifold which represents the totality of all point-event locations; and g_{ab} is a geometric object which represents the metric structure of spacetime. The latter, technically, is a semi-Riemannian metric of Lorentz signature on M. We can think of it, simply, as a function which assigns lengths to vectors at points of M, but which, characteristically, assigns negative and zero lengths to vectors as well as positive ones. It thus partitions the vectors at any point into three classes, and determines a cone structure. A vector is said to be *timelike*, *null*, or *spacelike* according to whether its length is positive, zero, or negative. Timelike vectors appear as those falling inside the cone. Null ones fall on the boundary.

Of course these so-called "null cones" have immediate physical significance. It is absolutely fundamental to relativity theory that there is an upper bound to the speeds with which particles can travel (as measured by any observer). If we think of vectors at a point as velocity vectors, then the cone can be interpreted as demarcating that upper bound. Particles with non-zero mass must have timelike velocity vectors; those with zero mass (e.g., photons) must have null ones.

Every null cone has two lobes. We say that a spacetime model is *temporally orientable* if, ranging over the entire spacetime manifold, we can label them as "past" and "future" lobes in a way which involves no discontinuity. One can come up with pathological models which are not temporally orientable. But the Gödel model is. That will be clear from the spacetime diagram we shall consider in a few moments. Though, in a sense, notions of past and future break down in the Gödel model, they do not do so locally.

Let us restrict attention, then, to spacetime models which are temporally orientable, and assume that some particular orientation has been specified. That will allow us to make reference to past and future lobes without further ado.

The principal geometric notion that I shall be using is that of a *future directed timelike curve*. It is simply a curve on the spacetime manifold, suitably smooth, whose tangent vector at every point is timelike and points to the future. Intuitively, it "threads" the cones of the points through which it passes. Such a curve represents the possible life history of a massive particle. Shortly, when we are looking at spacetime diagrams, I'll want to make claims of the form: "It would be possible for a massive particle (or rocket ship) to travel from point p to q." To support them it will be sufficient for me to exhibit, in each case, a future directed timelike curve running from p to q.

In saying that it would be possible for a massive particle to make the trip I mean only that it could do so without ever surpassing (or reaching) the maximal possible speed. Dynamical considerations play no role.

Given any future directed timelike curve, the spacetime metric associates with it various magnitudes, in particular, length and acceleration. Of course these too have immediate physical significance. Length is usually called elapsed *proper time*. It is another basic principle of relativity theory that natural clocks record its passage, at least approximately. Acceleration, i.e., invariant four-vector acceleration, enters into the second law of motion. Timelike curves with vanishing acceleration are timelike *geodesics*.

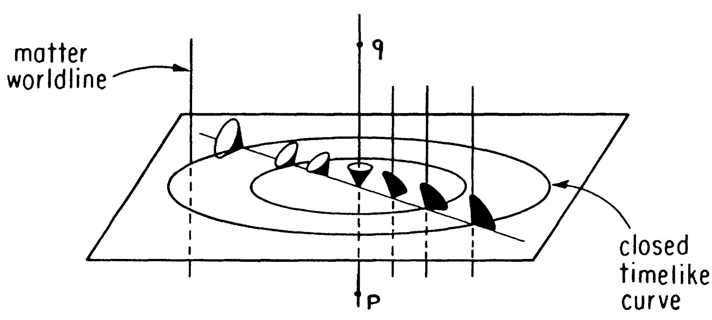
2. What the Gödel Universe Looks Like

With these remarks out of the way, let me turn to the topic of time travel. I'll say that a spacetime model allows for the possibility of time travel if it admits future directed timelike curves which are closed (or almost closed).

It is not difficult to find models having this property. For example, we can start with simple Minkowski spacetime and, using a standard quotient space construction, roll it up into a kind of cylindrical model. It is clear that in this rolled-up version of Minkowski spacetime there exist closed future directed timelike curves. Indeed, given any two points in the model there is a future directed timelike curve which runs from the first to the second. Intuitively, if the points are far apart, then the curve connecting them will simply wrap around the cylinder many times.

All the simplest examples of spacetime models allowing for time travel are rather like this one, and do not seem very interesting. All their closed timelike curves can be removed by dis-identifying points. More precisely, they disappear when one passes to a covering space. The Gödel universe is remarkable because its closed timelike curves cannot be removed in this way. It allows for the possibility of time travel, but not because its underlying spacetime manifold is topologically complex. Indeed, the manifold is just $\mathbf{R}^4$. It could not be any simpler.

Let us consider, now, a spacetime diagram of Gödel's model, and see how it is possible to have time travel without complex topology. In the diagram one spatial dimension has been suppressed, but no features of interest are lost as a result.



Gödel Spacetime (with one dimension suppressed)

The vertical lines are the worldlines of the major mass points of the universe--stars or galaxies. These are the objects which play a role in determining large-scale spacetime structure. (Of course in the context of cosmology the perturbing effects of small objects like tables and chairs and rocket ships are ignored.) I'll call these worldlines *matter lines.*

The diagram is intended to exhibit at least some of the symmetries of the Gödel universe. It suggests, as it should, rotational symmetry about the central matter line, and vertical translational symmetry as well. To know what the null cones look like at all spacetime points, we need only consider one representative radial direction. On the central matter line the cones are upright and oriented at 45^0 to the indicated horizontal plane. As one moves outward, the cones open up and tip over in a counter-clockwise direction. At a critical radius they become tangent to the plane. Beyond that radius they straddle it.

It is clear from our diagram that the Gödel model is temporally orientable. Indeed, only "future" lobes of null cones have been drawn. At every point there is a well-defined local sense of past and future.

It is also clear that, as had better be the case, all matter lines come out as timelike. To have things otherwise would be to attribute impossibly large speeds to the major mass points of the universe.[1]

Consider now the outer circle in the diagram. Notice that it threads the null cones of the points through which it passes. If we think of it as oriented in a counter-clockwise direction, then it qualifies as a closed future directed timelike curve. It is our first example of one. By itself it establishes the possibility of time travel in the Gödel universe. I'll refer to all such timelike circles concentric to the central matter line as *Gödel circles.*

Let us next consider a somewhat more striking example. The matter lines themselves are not closed. They exhibit, individually, a well-defined temporal order. With respect to that order, for example, point p on the central matter line is earlier than q. I claim that there is a future directed timelike curve running from q to p. Hence it is possible for a time traveler to start at p, coast along the matter line as far as q, and still make it back to p. The first half of his trip (from p to q) will be quite restful because the matter lines themselves are geodesics. All the acceleration will come in the second half.

One way to get from q to p is the following. The time traveler can first spiral his way "upward" and "outward" until he passes the critical radius. Then he can wind his way "downward" on a helix of narrow pitch (having the central matter line as its axis) until he suitably overshoots p. Then, finally, he can spiral "upward" and "inward" until he reaches p itself. It should be clear that he can, in this way, trace out a future directed timelike curve.

We could consider other interesting individual examples. But rather than do so, let me immediately up the ante and make the strongest possible claim. Given <u>any</u> two points in the Gödel universe there is a future directed timelike curve running from the first to the second. In this regard the situation is exactly the same as in rolled-up Minkowski spacetime. (I suggest that you can convince yourselves of this by trying a few examples. In every case some simple variant of the navigational strategy exhibited in the previous example will do the trick.) It follows that in the Gödel universe a time traveler can start at <u>any</u> spacetime point, return to it eventually, and stop off at any other spacetime point along the way!

There is still more information that we can extract from our diagram of the Gödel universe.

Let us consider the question of how long it would take a time traveler to return to his starting point as measured by his pocket stop watch. The diagram shows clearly that the round trip can be performed with arbitrarily small elapsed proper time. Consider the two circles drawn in the diagram. The inner one, with critical radius, is a closed null curve with length 0. The outer (Gödel) circle has positive length. Now suppose we shrink the radius of the latter. Continuity considerations alone guarantee that its length will converge to 0 as the critical inner radius is approached.

This special example illustrates a principle of navigational strategy that is quite general. Time travelers can always save on elapsed time and push its value to 0 by traveling at speeds which are asymptotic to the maximal possible speed.

I should mention that our diagram is misleading in one respect. It suggests that the central matter line is in some way special. In fact, the Gödel model is entirely homogeneous. Given any two points in it, one can find a global isometry which maps matter lines to matter lines, and takes the first point to the second. In a sense, the diagram depicts the Gödel universe from the point of view of one major mass point. But that one star or galaxy is no better than any other.

One encounters something similar in one's first exposure to special relativity when spacetime diagrams are used to explain, e.g., the relativity of simultaneity. In those diagrams one considers the worldlines of two non-accelerating observers. Generally one of the worldlines is in a vertical position, and the other is oblique to it. Of course that asymmetry of position does not reflect any physical asymmetry. It is not as if the first observer is the one who is really at rest.

3. Minimal Acceleration Requirements for Time Travel in the Gödel Universe [2]

So far we have ignored all consideration of dynamics. The focus has been on velocity rather than acceleration. Let me now turn to the latter, and with it the second part of my talk.

The first question to ask is whether there exist any closed timelike geodesics in the Gödel model. If so, it would be possible to execute time travel in a state of free fall. It turns out that the answer is 'no'. The possibility cannot be excluded *apriori*. Closed timelike geodesics do appear in some spacetime models, even those whose underlying manifold in $\mathbf{R}^4$. But they happen not to appear in the Gödel model. That much is easy to prove.

Much more difficult is the next question. In order to execute time travel at least some acceleration is required, at least sometime during the trip . So one would like to know whether there is some minimal amount that is required, or whether one can make do with arbitrarily small amounts. The question can be made precise as follows.

Consider an arbitrary future directed timelike curve γ (not necessarily closed) in an arbitrary spacetime model (not necessarily Gödel's). We can associate with γ a *total (integrated) acceleration* defined by:

$$TA(\gamma) = \int_\gamma a \, ds.$$

Here a is the magnitude of γ's acceleration at any point, and s is elapsed proper time along γ. Notice that $TA(\gamma) = 0$ if and only if γ is a geodesic.

I should make a remark about scale invariance. When one speaks of the metric of spacetime a certain ambiguity is involved. The metric is really only determined up to a scale factor or choice of units for spacetime length. This ambiguity infects certain defined magnitudes, but not others. Thus if I say that the elapsed proper time of a timelike curve is k, it is appropriate for you to ask whether I am using, for example, seconds or years as my unit of spacetime length. But no such question is appropriate if I say that the total acceleration of the curve is k. I could be using either. Total acceleration is a scale invariant magnitude. It is defined as a product of acceleration and elapsed proper time whose respective scale dependencies cancel each other. So we can speak of magnitudes of total acceleration without further qualification.

Our question can now be formulated as follows.

Problem 1. Does there exist a $k>0$ such that $TA(\gamma)\geq k$ for all closed timelike curves γ in the Gödel model?

The problem was first posed by Bob Geroch several years ago. It is actualy closely related to a classical problem in differential geometry. Let me take a moment to indicate the connection.

Let us temporarily leave relativity theory behind, and think about smooth curves in ordinary three-dimensional Euclidean space. Associated with every such curve and every point on it is a scalar magnitude of curvature. It is a measure of how much the curve there deviates from being a straight line. One intuitive characterization is the following. Given the curve and the point on it, there is a circle which best approximates the curve at that point. We take the curvature there to be just the reciprocal of the radius of that circle. In the limiting case where the curve is straight the best approximating circle has infinite radius. So the curvature comes out as 0, just as it should.

Our notion of scalar acceleration for timelike curves is the exact counterpart in relativistic spacetime geometry to this notion of scalar curvature for ordinary curves in three-dimensional Euclidean geometry. Indeed, the two are naturally subsumed under a common definition. So our notion of total acceleration is just the counterpart to the classical notion of total curvature. The latter results from integrating curvature over length.

Three dimensional Euclidean geometry	Four-dimensional spacetime geometry
c (curvature)	a (scalar acceleration)
$TC(\gamma) = \int_{\gamma} c\ dl$	$TA(\gamma) = \int_{\gamma} a\ ds$
Thm. $TC(\gamma)\geq 2\pi$ for all closed curves.	?

Consider, as an example, any circle in three-dimensional Euclidean space with radius r. Its curvature is everywhere $1/r$; and its length is $2\pi r$. So its total curvature is 2π. It is a classic result (due to Fenchel) that this value of 2π minimizes total curvature over the entire class of closed curves. You can convince yourselves that this is plausible by considering various examples. But all the proofs I know require a bit of work.

Our problem poses the question whether there is some counterpart lower bound for closed timelike curves in the Gödel model. But ours is more difficult in two respects. We are dealing with a Lorentzian metric rather than one that is positive definite; and our metric exhibits curvature rather than being flat. I have already mentioned that there exist some spacetime models admitting closed timelike geodesics--even ones whose underlying manifold is just $\mathbf{R}^4$. This alone indicates a radical asymmetry between the Euclidean and relativistic cases.

Well, so much for motivation. Let me tell you want I have learned. When I first started thinking about the problem I assumed that the answer was negative. I assumed that a would-be economical time traveler could make do with arbitrarily small quantities of total acceleration by properly choosing his navigational strategy. (For example, he might try using large bursts of acceleration for ultra-short periods of proper time, rather than sustaining acceleration over the entire trip. And he might try wandering over large regions of the spacetime manifold, rather than staying close to home.) But this turns out not to be the case. I can put a lower bound on the quantity of total acceleration that is required.

<u>Theorem</u> $TA(\gamma) \geq \ln(2 + \sqrt{5})$ for all closed timelike curves γ in the Gödel model.

(A proof is given in Malament (1985). $\ln(2 + \sqrt{5})$ comes out to approximately 1.44.) Unfortunately, my argument, which exploits certain weak inequalities, generates a lower bound that is far less than optimal. Even though Problem 1 is settled, it remains open, so far as I know, just how large the maximal lower bound is.

> <u>Problem 2</u>. What is the greatest lower bound of $TA(\gamma)$ as γ ranges over all closed timelike curves in the Gödel model?

All I can do on this problem is pose a conjecture. Consider again the simple Gödel circles we discussed earlier. They do <u>not</u> all have the same total acceleration. Rather, their total acceleration is a function of their radius. (This is yet another difference from the Euclidean case.) As this radius goes to infinity, or as it decreases to the critical minimal radius, total acceleration goes to infinity. At a certain intermediate radius a minimal value of $2\pi(9 + 6\sqrt{3})^{1/2}$ is achieved. (This comes out to approximately 27.67.) It seems likely to me that this value is not only minimal over the class of Gödel circles, but over the entire class of closed timelike curves in the Gödel model.

If this turns out to be the case, it will be quite interesting because, in terms of practical realization, 27 is a preposterously large value for total acceleration. Let me explain.

Let's imagine that a rocket ship traverses an arbitrary future directed timelike curve γ (in an arbitrary relativistic universe). The total acceleration of that curve, very roughly, is a measure of how much fuel will have to be expended in the course of the trip. The correlation can be made precise in the form of an interesting inequality.

Let's assume that the rocket ship is appropriately isolated during the trip. (For example, it is not hit by any meteors.) Then the composite system of rocket ship and exhaust will have to be one in which energy-momentum is conserved. Let's also take the the high road with respect to all questions of rocket engine efficiency, and assume that the rocket ship is powered by an ideally efficient reactor that extracts absolutely all internal energy from its fuel. ("$E = mc^2$" and all that.) Still, the following inequality will have to be satisfied.

$$\frac{m_r}{m_r + m_f} \leq e^{-TA(\gamma)}$$

Here m_r is the mass of the rocket (with empty fuel tanks), and m_f is the mass of the fuel expended during the flight. If the left-hand quantity seems like an odd one, just remember that we cannot expect a formula involving fuel expenditure alone. How much fuel is needed to produce the rocket's acceleration depends on the rocket's mass. More fuel is needed for a heavy rocket than for a light one.

Suppose, now, that we insert the value 27.67 for $TA(\gamma)$. Then the right-hand side of the inequality comes out to approximately 2×10^{-12}. Hence, if our rocket ship is to achieve a total acceleration of 27.67, for every two grams of payload the rocket will have to carry almost 10^{12} grams of fuel. And this is on the assumption that the rocket is powered by an ideally efficient atomic reactor! If it carries an engine anything like the ones we know how to build, the fuel required will be many, many orders of magnitude still higher.

All this should give pause to would-be time travelers in the Gödel universe. If my conjecture holds, or is anywhere close, then, though there is a sense in which time travel is possible in that universe, it is not a sense which has anything to do with practical possiblity.[3]

4. A Final Remark

I would like to close by making a remark on a rather different topic.

It is clear, I think, that Gödel's model is a fascinating geometric structure. But it is sometimes claimed that it is no more than a mathematical curiosity and has no "physical significance". This claim can mean, simply, that the Gödel model is not compatible with all currently available astrophysical data. That is certainly true. We have good evidence that our universe is expanding; Gödel's is not. But sometimes the claim is intended in a much stronger sense--namely, that Gödel's model can be rejected *a priori* insofar as it allows for the possibility of time travel. It should not even be thought of as representing a possible spatiotemporal structure for our universe. The view is that time travel, even our variety which involves no discontinuity of motion, is simply absurd and leads to logical contradiction.

You know how the argument goes. If time travel were possible, one could go backwards in time and undue the past. One could bring it about that both conditions P and not-P obtain at some point in spacetime. For example, I could go back and kill my earlier infant self making it impossible for that earlier self ever to grow up to be me.

I simply want to remark that arguments of this type have never seemed convincing to me. I do not believe that they alone suffice to rule out the Gödel model. And here let's forget about possible practical difficulties attendant to time travel. They are not what I have in mind.

The problem with these arguments is that they simply do no establish what they are supposed to. To be sure, if I could go back and kill my infant self, some sort of contradiction would arise. But the only conclusion to draw from this is that if I tried to go back and kill my infant self then, for some reason, I would fail. Perhaps I would trip at the last minute. The usual arguments do not establish that time travel is impossible, but only that if it were possible, certain actions could not be performed, certain mechanical devices could not be constructed, and so forth.

Actually I do not see that the usual arguments for the absurdity of time travel are any more persuasive than rather similar arguments that might be given for the absurdity of ordinary simple determinism at the level of human agency. If determinism does hold at this level, presumably, it is now determined whether I will raise my arm in 60 seconds. That strikes some people as very strange because we seem to be at liberty either to raise our arms at a given moment or not. But surely there is nothing approaching genuine absurdity here. Those who believe in the compatibility of free will and determinism will not see any problem in the first place. Those who do not believe in compatibility can say so much the worse for free will. Neither party is forced to give up determinism on pain of contradiction.

Again, we do have good evidence that the Gödel model does not describe our universe. I am only proposing here that it cannot be ruled out *a priori*. Neither can any other cosmological model just because it allows for the possibility for time travel.

Notes

[1]Curiously, the diagram of the Gödel model given in Hawking and Ellis (1973) is mistaken about just this point. It depicts the matter lines as spacelike! (The diagram is also mistaken in its depiction of the model's timelike geodesics.)

[2]This section is a partial summary of results presented in Malament (1985).

[3]In (1949b) Gödel, referring to the possibility of time travel in his model universe, asserts that: "...the velocities which would be necessary in order to complete the voyage *in a reasonable length of time* are far beyond everything than can be expected ever to become a practical possibility." (italics mine.) Here Gödel has in mind the sort of practical difficulties just described. (Clearly, no difficulty attends to being at high velocity (relative to the major mass points of the universe), only to attaining it. It is acceleration that is at issue.) Notice, however, that his claim is qualified. He does not assert that insuperable practical difficulties would rule out time travel altogether, only those instances where the trip is to be achieved in "a reasonable length of [elapsed proper] time". If my conjecture is correct, Gödel's qualification is unnecessary.

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