An Introduction to Kaluza-Klein Theory

A. Garrett Lisi

2nd March 2004

Department of Physics, University of California San Diego, La Jolla, CA 92093-0319

gar@lisi.com

1 Introduction

It is the aim of Kaluza-Klein theory to explain the existence and behavior of gauge fields solely as the dynamics of gravitation. Integral to the Kaluza-Klein approach is the assumption that the spacetime manifold in which we live is not four dimensional, but rather is a higher dimensional manifold in which the higher dimensions have curled up, or compactified, to a degree that dynamics in the compactified dimensions may be understood simply as the wrapping of the four dimensional spacetime around the compactified dimensions.

2 Manifold and Vielbein

Assume that the universe is an orientable, n-dimensional, pseudo-Riemannian manifold. The geometry of the manifold is completely described in any coordinate patch by a set of orthogonal unit vectors, the vielbein, frame, or tetrad,

$$\hat{e}_\alpha = (e_\alpha)^i \vec{\partial}_i \tag{1}$$

that satisfy

$$\hat{e}_\alpha \cdot \hat{e}_\beta = (e_\alpha)^i (e_\beta)^j g_{ij} = \eta_{\alpha\beta} \tag{2}$$

in which $\vec{\partial}_i$ is a coordinate basis vector, roman indices are coordinate indices, g is the metric, η is the Minkowski metric, and greek indices are labels raised and lowered by η. The vielbein naturally implies a set of basis 1-forms, the fielbein

$$\hat{\sigma}^\alpha = \vec{d}x^i \, (\sigma_i)^\alpha \tag{3}$$

dual to the vielbein vectors

$$(e_\alpha)^i(\sigma_i)^\beta = \delta_\alpha^\beta \tag{4}$$

or, more compactly in matrix notation, $e\sigma = I$, where the components of e are $(e_\alpha)^i$. On a pseudo-Riemannian manifold the 1-forms may be identified with vectors via the metric, $\vec{\partial}_i = g_{ij}\,\vec{d}x^j = g_{ij}\,\vec{\partial}^j$ and hence $\hat{\sigma}^\alpha = \eta^{\alpha\beta}\,\hat{e}_\beta$, and often go under the name of covariant vectors. Any vector may be represented in terms of the coordinate or vielbein basis vectors

$$\vec{v} = v^i\vec{\partial}_i = v^\alpha\hat{e}_\alpha = (v^i(\sigma_i)^\alpha)\hat{e}_\alpha = v^i\,g_{ij}\,\vec{d}x^j = v_\alpha\hat{\sigma}^\alpha \tag{5}$$

The fielbein can be considered a factorization of the metric, since

$$g_{ij} = (\sigma_i)^\alpha\,\eta_{\alpha\beta}\,(\sigma_j)^\beta \tag{6}$$

or $g = \sigma\eta\sigma^T$. However, the frame of orthogonal unit vectors tangent to a manifold seems a more satisfying intuitive description than the equivalent metric, and I interpret it as being more fundamental. Also note that the vielbein describes an orientation on the manifold, information absent from the metric. The metric is, however, a more compact description of the geometry, having $\frac{n(n+1)}{2}$ degrees of freedom compared to the vielbein's n^2. The metric is invariant under local orthonormal (Lorentz) transformations of the fielbein,

$$\hat{\sigma}^\alpha \mapsto L^\alpha{}_\beta\hat{\sigma}^\beta \tag{7}$$

with $L^\alpha{}_\beta\,\eta_{\alpha\mu}\,L^\mu{}_\nu = \eta_{\beta\nu}$ (or $L^T\eta L = \eta$), which leads to the natural unique decomposition of the fielbein matrix,

$$(\sigma_i)^\alpha = L^{+\alpha}{}_\beta(\gamma_i)^\beta \tag{8}$$

in which γ is Upper Triangular and L^+ is a proper Lorentz transformation, $\det L^+ > 0$. This decomposition capitalizes on the metric invariance (7) to factor the fielbein into a gravitational part, the UT fielbein, $\hat{\gamma}^\beta$, which has $\frac{n(n+1)}{2}$ degrees of freedom and gives $g = \sigma\eta\sigma^T = \gamma\eta\gamma^T$, an UDL decomposition, and the rotational part, L^+, which has $\frac{n(n-1)}{2}$ degrees of freedom. Note that this decomposition is not coordinate independent, but may always be performed anew after a coordinate transformation.

3 F

The exterior derivative of the basis 1-forms gives a set of 2-forms,

$$\begin{aligned} F^\alpha &= d\sigma^\alpha && (9) \\ &= \sum_{i<j}\left(\partial_i\,(\sigma_j)^\alpha - \partial_j(\sigma_i)^\alpha\right)dx^i\wedge dx^j && (10) \\ &= \sum_{\beta<\gamma} F_{\beta\gamma}{}^\alpha\sigma^\beta\wedge\sigma^\gamma && (11) \end{aligned}$$

where the field strength ${F_{\beta\gamma}}^\alpha = (e_\beta)^i (e_\gamma)^j (F_{ij})^\alpha$ and $(F_{ij})^\alpha = \partial_i(\sigma_j)^\alpha - \partial_j(\sigma_i)^\alpha$.

The field strength first appears in the equation for a geodesic. For a parameterized curve, $u^i(\tau)$, extremizing the square of the path length,

$$\int d\tau \dot{u}^i \dot{u}^j g_{ij} \tag{12}$$

gives the unit geodesic equation,

$$0 = \dot{v}^k + \Gamma^k_{ij} v^i v^j \tag{13}$$

where $v^i = \dot{u}^i = \frac{d}{d\tau} u^i$ and Γ is the torsionless connection. Or, in the vielbein basis,

$$0 = \dot{v}^\alpha + {F^\alpha}_{(\beta\gamma)} v^\beta v^\gamma \tag{14}$$

where $v^\alpha = v^i(\sigma_i)^\alpha$, parenthesis around indices indicate symetricisation, and the greek indices are raised and lowered with η.

4 Curvature and Gravitational Action

The covariant derivative applied to tensors is

$$D_i v^j = \partial_i v^j + \Gamma^j_{ik} v^k \tag{15}$$

The curvature scalar can be calculated from the vielbein as

$$R = (e^\alpha)^j (D_i D_j - D_j D_i)(e_\alpha)^i \tag{16}$$

Since the Einstein-Hilbert action is

$$S = \int dx^n \sqrt{|g|} R \tag{17}$$

we may rewrite R as

$$R = D_i[(e^\alpha)^j D_j(e_\alpha)^i] - D_j[(e^\alpha)^j D_i(e_\alpha)^i] + [D_i(e_\alpha)^i][D_j(e^\alpha)^j] - [D_i(e_\alpha)^j][D_j(e^\alpha)^i] \tag{18}$$

and drop the divergence terms, assuming no boundary contribution, to give us our equivalent action

$$S = \int dx^n |\sigma| L \tag{19}$$

in which we may now calculate L directly from the field strength as

$$L = -\frac{1}{4} F_{\alpha\beta\gamma} F^{\alpha\beta\gamma} - \frac{1}{2} F_{\alpha\beta\gamma} F^{\alpha\gamma\beta} + {F_{\alpha\beta}}^\beta {F^\alpha}_\gamma{}^\gamma \tag{20}$$

Note that this gravitational Lagrangian contains no second derivatives. Note also that the use of a vielbein and field strength, F, as well as the dropping of divergence terms in the action, is not a requirement of Kaluza-Klein theory, but gives a simpler exposition.

5 Kaluza-Klein theory

We adopt the Kaluza-Klein hypothesis that we live in a universe in which one or more spacelike dimensions have compactified, leaving an effective universe of four dimensional spacetime, S, and a collection of gauge fields. This may be imagined as our spacetime submanifold, S, embedded in a tightly wrapped, higher dimensional universe possessing symmetries (Killing vectors) in the wrapped directions (Figure **??**). For one compactified direction, $\vec{\partial}_4$, we begin by assuming a five dimensional vielbein of the form

$$(e_\alpha)^i = \begin{pmatrix} (e_{S\alpha})^i & A_\alpha \\ 0 & \frac{1}{\rho} \end{pmatrix}$$

where $(e_{S\alpha})^i$ are the vielbein components for the spacetime indices $0 \le \alpha, i \le 3$ and the vielbein is assumed independent of the compactified fourth spatial coordinate. ρ may be imagined as the radius of compactification and A will be our $U(1)$ gauge field. The corresponding basis 1-forms are

$$(\sigma_i)^\alpha = \begin{pmatrix} (\sigma_{Si})^\alpha & -\rho A_i \\ 0 & \rho \end{pmatrix} \tag{21}$$

Giving the field strength components

$$F_{\beta\gamma}{}^{\alpha<4} = F_{S\beta\gamma}{}^{\alpha} \tag{22}$$

$$F_{\beta 4}{}^{4} = \frac{1}{\rho}\partial_\beta \rho \tag{23}$$

$$F_{\beta\gamma}{}^{4} = -\rho(e_{S\beta})^i (e_{S\gamma})^j \left(\partial_i A_j - \partial_j A_i\right) = -\rho F_{\beta\gamma} \tag{24}$$

where we have used the abbreviation $\partial_\beta = (e_\beta)^i \partial_i$. Integrating over the compactified dimension gives the effective action

$$S = \int dx_S |\sigma_S| \rho (L_S - \frac{1}{4}\rho^2 F_{\alpha\beta} F^{\alpha\beta}) \tag{25}$$

The ρ multiplying L_S may be scaled away by multiplying the original vielbein by $\rho^{\frac{1}{3}}$ (a Weyl scaling) and dropping two resulting divergence terms, giving the effective action

$$S = \int dx_S |\sigma_S| (L_S - \frac{1}{4}\rho^2 F_{\alpha\beta} F^{\alpha\beta} + \frac{4}{3}\rho^{-2}(\partial_i \rho)(\partial^i \rho)) \tag{26}$$

Assuming constant ρ produces the action for gravity and our one gauge field. We note also that the equation for a geodesic through a Kaluza-Klein universe with flat spacetime,

$$0 = \dot{v}^\alpha + \rho v^4 F^\alpha{}_\beta v^\beta \tag{27}$$

is the Lorentz force law for a charged particle moving in an electromagnetic field. The U(1) field strength, $F^4 = d\sigma^4$, describes the curvature of S around the compactified dimension, a curvature that cannot be untwisted by coordinate transformation. A coordinate transformation of the compactified coordinate, $x^4 \mapsto x^4 - \lambda(x)$, results in the gauge transformation, $A_i \mapsto A_i + \partial_i \lambda$, via the corresponding Lorentz transformation, and leaves F^4 unchanged.

Although it is standard practice to assume that all fields are independent of the compactified coordinate, we may choose to keep this dependency and expand the fields in a Fourier series in the compact dimension. If this is done we obtain the U(1) field as the zero mode as well as an infinite collection of interacting higher modes. These higher mode gauge fields may be calculated to have a large mass due to the small compactification scale and are thus usually discarded.

6 SU(2)

We have obtained an effective U(1) gauge field by considering a universe with a compactified dimension admitting U(1) symmetry. Now we wish to obtain non-abelian SU(2) gauge fields. The double cover of S^2 is a maximally symmetric solution of Einstein's equations that has SU(2) symmetry. We begin by confirming this SU(2) symmetry explicitly by obtaining the Killing vectors, then use the Killing vectors to construct our SU(2) gauge theory via Kaluza-Klein theory.

We use polar coordinates $0 < \theta < \pi$ and $0 < \phi < 4\pi$, and use the vielbein on the sphere,

$$(e_\alpha)^i = \begin{pmatrix} \frac{1}{r} & 0 \\ 0 & \frac{1}{r\sin\theta} \end{pmatrix} \tag{28}$$

which implies the basis 1-forms and metric,

$$(\sigma_i)^\alpha = \begin{pmatrix} r & 0 \\ 0 & r\sin\theta \end{pmatrix}, g_{ij} = \begin{pmatrix} r^2 & 0 \\ 0 & r^2 \sin^2\theta \end{pmatrix} \tag{29}$$

The symmetries of a space are the coordinate transformations that leave the metric unchanged. The infinitesimal coordinate transformations can be written $x^i \mapsto x^i + \varepsilon^a {\xi_a}^i$, where $\vec{\xi_a}$ are the set of Killing vector fields, each corresponding to a metric symmetry, and ε^a are the parameters of the transformation. Each $\vec{\xi}$ represents a flow that leaves the space unchanged, and hence each must be a solution to Killing's equation,

$$0 = [L_{\vec{\xi}} g]_{ij} = \xi^k \partial_k g_{ij} + g_{kj} \partial_i \xi^k + g_{ik} \partial_j \xi^k \tag{30}$$

For our S^2 metric, this equation gives

$$0 = 2r^2 \partial_\theta \xi^1 \tag{31}$$
$$0 = r^2 \sin\theta^2 \partial_\theta \xi^2 + r^2 \partial_\phi \xi^1 \tag{32}$$
$$0 = 2r^2 \sin\theta \cos\theta \xi^1 + 2r^2 \sin\theta^2 \partial_\phi \xi^2 \tag{33}$$

which admit three Killing vectors as solutions,

$${\xi_x}^i = \begin{pmatrix} -\sin\phi \\ -\cos\phi\cot\theta \end{pmatrix}, {\xi_y}^i = \begin{pmatrix} \cos\phi \\ -\sin\phi\cot\theta \end{pmatrix}, {\xi_z}^i = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{34}$$

corresponding to rotations about the x, y and z axes. These vectors satisfy the commutation relations corresponding to the SU(2) Lie algebra,

$$[\vec{\xi}_a, \vec{\xi}_b] = L_{\vec{\xi}_a} \vec{\xi}_b = -\epsilon_{abc} \vec{\xi}_c \tag{35}$$

Or, in components,

$$[\vec{\xi}_a, \vec{\xi}_b]^i = \xi_a{}^k \partial_k \xi_b{}^i - \xi_b{}^k \partial_k \xi_a{}^i = -\epsilon_{abc} \xi_c{}^i \tag{36}$$

The vectors also satisfy the normalization relation,

$$\int_0^\pi \int_0^{4\pi} d\theta d\phi r^2 \sin\theta < \vec{\xi}_a, \vec{\xi}_b > = 8\pi r^2 \frac{2}{3} r^2 \delta_{ab} \tag{37}$$

Now we construct the Kaluza-Klein space for SU(2) as we did for U(1) by starting with the vielbein,

$$(e_\alpha)^i = \begin{pmatrix} (e_{S\alpha})^i & \xi_a^1 A_\alpha^a & \xi_a^2 A_\alpha^a \\ 0 & \frac{1}{r} & 0 \\ 0 & 0 & \frac{1}{r\sin\theta} \end{pmatrix} \tag{38}$$

where A^a will be our three SU(2) gauge fields corresponding to the three Killing vectors of the compactified space. The corresponding basis 1-forms are

$$(\sigma_i)^\alpha = \begin{pmatrix} (\sigma_{Si})^\alpha & -r\xi_a^1 A_i^a & -r\sin\theta \xi_a^2 A_i^a \\ 0 & r & 0 \\ 0 & 0 & r\sin\theta \end{pmatrix} \tag{39}$$

Giving the field strength components

$$F_{\beta\gamma}{}^{\alpha<4} = F_{S\beta\gamma}{}^\alpha \tag{40}$$

$$F_{45}{}^5 = \frac{1}{r}\cot\theta \tag{41}$$

$$F_{\beta 4}{}^4 = F_{\beta 5}{}^5 = \frac{1}{r}\partial_\beta r \tag{42}$$

$$F_{\beta 5}{}^4 = -F_{\beta 4}{}^5 = \sec\theta \xi_a^2 A_\beta^a \tag{43}$$

$$F_{\beta\gamma}{}^4 = -r\xi_c^1 F_{\beta\gamma}^c \tag{44}$$

$$F_{\beta\gamma}{}^5 = -r\sin\theta \xi_c^2 F_{\beta\gamma}^c \tag{45}$$

where we have used the Killing vector commutation relations to obtain the SU(2) field strength,

$$F_{\beta\gamma}^c = (e_{S\beta})^i (e_{S\gamma})^j \left(\partial_i A_j^c - \partial_j A_i^c - \epsilon_{abc} A_i^a A_j^b\right) \tag{46}$$

We sum the field strength components to obtain the action,

$$S = \int dx_S |\sigma_S| \int_0^\pi \int_0^{4\pi} d\theta d\phi r^2 \sin\theta (L_S - \frac{1}{4} < \vec{\xi}_a, \vec{\xi}_b > F_{\beta\gamma}^a F^{b\beta\gamma} + \frac{2}{r^2}(\partial_i r)(\partial^i r)) \tag{47}$$

and complete the integral over the compactified space, using the Killing vector normalization relation, to obtain the effective action,

$$S = \int dx_S |\sigma_S| 8\pi r^2 (L_S - \frac{1}{4}\frac{2}{3} r^2 F^c_{\beta\gamma} F^{c\beta\gamma} + \frac{2}{r^2}(\partial_i r)(\partial^i r)) \qquad (48)$$

which, after another Weyl scaling, we identify as the non-abelian SU(2) action.

A similar calculation for the manifold CP^2 produces the gauge fields of SU(3).