

**The Reduction of the State Vector and Limitations on Measurement
in the Quantum Mechanics of Closed Systems***

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“... persuaded of these principles, what havoc must we make?” – Hume

ABSTRACT

Measurement is a fundamental notion in the usual approximate quantum mechanics of measured subsystems. Probabilities are predicted for the outcomes of measurements. State vectors evolve unitarily in between measurements and by reduction of the state vector at measurements. Probabilities are computed by summing the squares of amplitudes over alternatives which could have been measured but weren't. Measurements are limited by uncertainty principles and by other restrictions arising from the principles of quantum mechanics. This essay examines the extent to which those features of the quantum mechanics of measured subsystems that are explicitly tied to measurement situations are incorporated or modified in the more general quantum mechanics of closed systems in which measurement is not a fundamental notion. There, probabilities are predicted for decohering sets of alternative time histories of the closed system, whether or not they represent a measurement situation. Reduction of the state vector is a necessary part of the description of such histories. Uncertainty principles limit the possible alternatives at one time from which histories may be constructed. Models of measurement situations are exhibited within the quantum mechanics of the closed system containing both measured subsystem and measuring apparatus. Limitations are derived on the existence of records for the outcomes of measurements when the initial density matrix of the closed system is highly impure.

* *To appear in the festschrift for Dieter Brill, edited by B.-L. Hu and T. Jacobson, Cambridge University Press, Cambridge, 1993*

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0. Preface

In 1959, then an undergraduate at Princeton in search of a senior thesis topic, I was introduced by John Wheeler to his young colleague, Dieter Brill. This was fortunate from my point of view, for Dieter proved to have the patience, time and talent not only to introduce me to the beauties of Einstein’s general relativity but also give me instruction and guidance in the practice of research. Our subject – the method of the self-consistent field in general relativity and its application to the gravitational geon – was also fortunate. Through it we helped lay the foundations for the short wavelength approximation for gravitational radiation (Brill and Hartle, 1964). In particular, building on ideas of Wheeler (1964), we introduced what Richard Isaacson (Isaacson 1968ab) was later kind enough to call the “Brill-Hartle” average for the effective stress-energy tensor of short wavelength radiation, and which was to prove such a powerful tool when made precise in his hands in his general theory of this approximation. It would be difficult to imagine a more marvelous introduction to research. I have not written a paper with Dieter Brill since, but each day I use the lessons learned from him so long ago. It is a pleasure to thank him with this small essay on the occasion of his 60th birthday.

I. Introduction

“Measurement” is central to the usual formulations of quantum mechanics. Probabilities are predicted for the outcomes of measurements carried out on some subsystems of the universe by others. In a Hamiltonian formulation of quantum mechanics, states of a subsystem evolve unitarily in between measurements and by reduction of the state vector at them. In a sum-over-histories formulation, amplitudes are squared and summed over alternatives “which could have been measured but weren’t” to calculate the probabilities of incomplete measurements. In these and other ways the notion of measurement plays a fundamental role in the usual formulations of quantum theory.

The quantum mechanics of a subsystem alone, of course, does not offer a quantum mechanical description of the workings of the measuring apparatus which acts upon it, but it does limit what can be measured. We cannot, for instance, carry out simultaneous ideal measurements of the position and momentum of a particle to arbitrary accuracies. Ideal measurements are defined to leave the subsystem in eigenstates of the measured quantities and there are no states of the subsystem for which position and momentum are specified to accuracies better than those allowed by the Heisenberg uncertainty principle. Analyses of the workings of measuring apparatus and subsystem as part of a single quantum system reveal further quantum mechanical limitations on ideal measurements, as in the work of Wigner (1952) and Araki and Yanase (1960).

Cosmology is one motivation for generalizing the quantum mechanics of measured subsystems to a quantum mechanics of closed systems in which measurement plays no fundamental role. Simply providing a more coherent and precise formulation of quantum mechanics, free from many of the usual interpretive difficulties, is motivation enough for many. Today, because of the efforts of many over the last thirty-five years, we have a quantum mechanics of closed systems.* In this formulation, it is the internal consistency of probability sum rules that determines the sets of alternatives of the closed system for which probabilities are predicted rather than any external notion of “measurement” (Griffiths, 1984; Omnès, 1988abc; Gell-Mann and Hartle, 1990). It is the absence of quantum mechanical interference between the individual members of a set of alternatives, or decoherence, that is a sufficient condition for the consistency of probability sum rules. It is the initial condition of the closed system that, together with its Hamiltonian, determines which sets of alternatives decohere and which do not. Alternatives describing a measurement situation decohere, but an alternative does not have to be part of a measurement situation in

* A pedagogical introduction to the quantum mechanics of closed systems can be found in the author’s other contribution to these volumes, (Hartle, 1993a), where references to some of the literature may be found.

order to decohere. Thus, for example, with an initial condition and Hamiltonian are such that they decohere, probabilities are predicted for alternative sizes of density fluctuations in the early universe or alternative positions of the moon whether or not they are ever measured.

The familiar quantum mechanics of measured subsystems is an approximation to this more general quantum mechanics of closed systems. It is an approximation that is appropriate when certain approximate features of measurement situations can be idealized as exact. These include the decoherence of alternative configurations of the apparatus in which the result of the measurement is registered, the correlation of these with the measured alternatives, the short duration of certain measurement interactions compared to characteristic dynamical time scales of the measured subsystems, the persistence of the records of measurements, etc. etc.* The question naturally arises as to the extent to which those features of the quantum mechanics of measured subsystems that were tied to measurement situations are incorporated, modified, or dispensed with in the more general quantum mechanics of closed systems. Are two laws of evolution still needed? Is there reduction of the state vector, and if so, when? What becomes of a rule like “square amplitudes and sum over probabilities that one could have measured but didn’t”? What becomes of the limitations on measurements in a more general theory where measurement can be described but does not play a fundamental role. This essay is devoted to some thoughts on these questions.

II. The Reduction of the State Vector

In the approximate quantum mechanics of measured subsystems the Schrödinger picture state of the subsystem is described by a time-dependent vector, $|\psi(t)\rangle$, in the subsys-

* For more discussion of ideal measurement models in the context of the quantum mechanics of closed systems see Section IV and Hartle (1991a)

tem's Hilbert space. In between measurements the state vector evolves unitarily:

$$i\hbar \frac{\partial |\psi(t)\rangle}{\partial t} = H |\psi(t)\rangle . \quad (\text{twoone})$$

If a measurement is carried out at time t_k , the probabilities for its outcomes are

$$p(\alpha_k) = \|s_{\alpha_k}^k |\psi(t_k)\rangle\|^2 . \quad (\text{two two})$$

Here, the $\{s_{\alpha_k}^k\}$ are an exhaustive set of orthogonal, Schrödinger picture, projection operators describing the possible outcomes. The index k denotes the *set* of outcomes at time t_k , for example, a set of ranges of momentum, or a set of ranges of position, etc. The index α_k denotes the particular alternative within the set – a *particular* range of momentum, a *particular* range of position, etc. If the measurement was an “ideal” one, that “disturbed the system as little as possible”, the state vector is reduced at t_k by the projection that describes the outcome of the measurement:

$$|\psi(t_k)\rangle \rightarrow \frac{s_{\alpha_k}^k |\psi(t_k)\rangle}{\|s_{\alpha_k}^k |\psi(t_k)\rangle\|} . \quad (\text{two three})$$

This is the “second law of evolution”, which together with the first (twoone), can be used to calculate the probabilities of sequences of ideal measurements.

The two laws of evolution can be given a more unified expression. For example, in the Heisenberg picture, the joint probability of a sequence of measured outcomes is given by the single expression*:

$$p(\alpha_n, \dots, \alpha_1) = \|s_{\alpha_n}^n(t_n) \cdots s_{\alpha_1}^1(t_1) |\psi\rangle\|^2 \quad (\text{two four})$$

where $|\psi\rangle$ is the Heisenberg state vector and

$$s_{\alpha_k}^k(t_k) = e^{iht_k/\hbar} s_{\alpha_k}^k e^{-iht_k/\hbar} \quad (\text{two five})$$

* The utility of the Heisenberg picture in giving a compact expression for the two laws of evolution has been noted by many authors, Groenewold (1952) and Wigner (1963), among the earliest. Similar unified expressions can be given in the sum-over-histories formulation of quantum mechanics (Caves 1986, 1987 and Stachel 1986)

are the Heisenberg picture projection operators with h the Hamiltonian of the subsystem. Nevertheless, even in such compact expressions one can distinguish unitary evolution from the action of projections at an “ideal” measurement.

One gains the impression from parts of the literature that some think the law of state vector reduction to be secondary in importance to the law of unitary evolution. Perhaps by understanding the quantum mechanics of large, “macroscopic” systems that include the measuring apparatus the second law of evolution can be derived from the first. Perhaps the law of the reduction of the state vector is unimportant for the calculation of realistic probabilities of physical interest. No ideas could be further from the truth in this author’s opinion. Certainly the second law of evolution is less precisely formulated than the law of unitary evolution because the notion of an “ideal” measurement is vague and many realistic measurements are not very ideal. However, as shown conclusively by Wigner (1963), the second law of evolution is not reducible to the first and it is essential for the calculation of probabilities of realistic, everyday interest as we shall now describe.

Scattering experiments can perhaps be said to involve but a single measurement of the final state once the system has been prepared in an initial state. Many everyday probabilities, however are for *time sequences* of measurements. For instance, in asserting that the moon moves on a certain classical orbit one is asserting that successions of suitably crude measurements of the moon’s position and momentum will be correlated in time by Newton’s deterministic law. Thus, measured classical behavior involves probabilities for time sequences like (two-four). Successive state vector reductions are essential for their prediction as well as many other questions of interest in quantum mechanics.

Since the state vector of a subsystem evolves unitarily except when that subsystem is measured by an external device, some have argued that one could dispense with the “second law of evolution” in the quantum mechanics of a closed system. All predictions would be derived from a state vector, $|\Psi(t)\rangle$, of the closed system that evolves in time only

according to the Schrödinger equation (Everett, 1957; DeWitt, 1970). However, a state vector is a function of one time and can, therefore, be used to predict only the probabilities of alternatives that are at one time according to the generalization of (twotwo)

$$p(\alpha_k) = \|P_{\alpha_k}^k |\Psi(t_k)\rangle\|^2 . \quad (\text{twosix})$$

Here, the $\{P_{\alpha_k}^k\}$ are an exhaustive set of orthogonal, Schrödinger picture, projection operators representing alternatives of the closed system at a moment of time. For instance, in a description of the system in terms of hydrodynamic variables they might represent alternative ranges of the energy density averaged over suitable volumes. In a description of a measurement situation, the P 's might represent alternative registrations of that variable by an apparatus.

The restriction to a unitary law of evolution and the action of projections at a single time as in (twosix) would rule out the calculation of probabilities for time histories of the closed system. Some have suggested that probabilities at the single marvelous moment of time “now” are enough for all realistic physical prediction and retrodiction.* In this view, for example, probabilities referring to past history are more realistically understood as the probabilities for correlations among present records. However, just to establish whether a physical system is a good record, one needs to examine the probability for the correlations between the present value of that record and the past event it has supposed to have recorded. That is a probability for correlation between alternatives at two times — the probability of a history. For this and other reasons, probabilities of histories are just as essential in the quantum mechanics of closed systems as they were in the quantum mechanics of measured subsystems.

There is a natural generalization of expressions like (twofour) to give a framework for predicting the joint probabilities of time sequences of alternatives in the quantum

* For a recent expression of this point of view, see Page and Wootters (1983).

mechanics of closed systems (Griffiths, 1984; Omnès, 1988abc; Gell-Mann and Hartle, 1990). The joint probability of a history of alternatives is

$$p(\alpha_n, \dots, \alpha_1) = \|P_{\alpha_n}^n(t_n) \cdots P_{\alpha_1}^1(t_1)|\Psi\rangle\|^2 \quad (\text{twoseven})$$

where the Heisenberg P 's evolve according to

$$P_{\alpha_k}^k(t_k) = e^{iHt_k/\hbar} P_{\alpha_k}^k(0) e^{-iHt_k/\hbar} . \quad (\text{twoeight})$$

and the times in (twoseven) are ordered with the earliest closed to $|\Psi\rangle$. Here, projection operators, state vectors, the Hamiltonian H , etc all refer to the Hilbert space of a closed system, containing both apparatus and measured subsystem if any. This is most generally the universe, in an approximation in which gross quantum fluctuations in the geometry of spacetime can be neglected.* The Heisenberg state vector $|\Psi\rangle$ represents the initial condition of the closed system, assumed here to be a pure state for simplicity.

The occurrence of the projections in (twoseven) can be described by saying that the state vector is “reduced” at each instant of time where an alternative is considered. However, the important point for the present discussion of state vector reduction is that the projections in (twoseven) are not, perforce, associated with a measurement by some external system. This is a quantum mechanics of a closed system! The P 's can represent any alternative at a moment of time. Measurement situations within a closed system of apparatus and measured subsystem can be described by appropriate P 's (see Section IV) but the P 's do not necessarily have to describe measurement situations. They might describe alternative positions of the moon whether or not it is being observed or alternative values of density fluctuations in the early universe where ordinary measurement situations of any kind are unlikely to have existed. Thus, the state vector can be said to be reduced in (twoseven) by the action of the projections and one might even say that there are “two laws

* For a generalized quantum mechanics of closed systems that includes quantum spacetime see Hartle (1993b) and the references therein.

of evolution” present, but those reductions and evolutions have nothing to do, in general, with measurement situations. In the author’s view, it is clearer not to use the language of “reduction” and “two laws of evolution”, but simply to regard (twoseven) as the law for the joint probability of a sequence of alternatives of a closed system. Projections occur therein because they are the way alternatives are represented in the quantum mechanics of closed systems.

There is a good reason why the probabilities (twofour) of a sequence of alternatives of a subsystem refer only to the results of *measured* alternatives. It would be inconsistent generally to calculate probabilities of histories that have not been measured because the sum rules of probability theory would not be satisfied as a consequence of quantum mechanical interference. In the two-slit experiment, for instance, the probability to arrive at a point on the screen is not the sum of the probabilities to go through the alternative slits and arrive at that point unless the alternative passages have been measured and the interference between them destroyed. Thus, probabilities are not predicted for all possible sets of histories of a subsystem but only those which have been “measured”.

Probabilities are not predicted for every set of alternative histories of a closed system either. But it is not an external notion of “measured” that discriminates those sets for which probabilities are predicted from those which are not. Rather, it is the internal consistency of the probability sum rules that distinguishes them (Griffiths, 1984). Probabilities are consistent for a set of histories, when, in a partition of the set of histories into an exhaustive set of exclusive classes, the probabilities of the individual classes are the sums of the probabilities of the histories they contain for all allowed partitions. A sufficient condition for the consistency of probabilities is the absence of interference between the individual histories in the set as measured by the overlap

$$\langle \Psi | P_{\alpha'_n}^n(t_n) \cdots P_{\alpha'_1}^1(t_1) \cdot P_{\alpha_1}^1(t_1) \cdots P_{\alpha_n}^n(t_n) | \Psi \rangle \propto \delta_{\alpha'_1 \alpha_1} \cdots \delta_{\alpha'_n \alpha_n}. \quad (\text{twonine})$$

Sets of histories that satisfy (twonine) are said to *decohere*.^{*} Decoherence implies the consistency of the probability sum rules. In the quantum mechanics of closed systems, probabilities are predicted for just those sets of alternative histories that decohere according to (twonine) as a consequence of the system’s Hamiltonian and initial quantum state $|\Psi\rangle$.

III. Uncertainty Principles

The state of a single particle cannot be simultaneously an eigenstate of position and momentum. It follows from their commutation relations that position and momentum cannot be specified to accuracies greater than those allowed by the Heisenberg uncertainty principle

$$\Delta x \Delta p \geq \frac{1}{2} \hbar . \quad (\text{threetwo})$$

Following the standard discussion, we infer from the mathematical inequality (threetwo) that it is not possible to simultaneously perform ideal measurements of position and momentum to accuracies better than that allowed by the uncertainty principle (threetwo). There can be no such ideal measurement because there is no projection operator, s , that could represent its outcome in (twothree).

The limitations on ideal measurements implied by the uncertainty principle (twothree) are usually argued to extend to non-ideal measurements as well. Examination of quantum mechanical models of specific measurement situations have for the most part verified the consistency of this extension although some have maintained otherwise.^{*} No such elaborate analysis is needed to demonstrate the impossibility of ideal measurements of position and momentum to accuracies better than those allowed by the uncertainty principle. That limitation follows from the quantum mechanics of the subsystem alone.

^{*} There are several possible decoherence conditions. This is *medium decoherence* in the terminology of Gell-Mann and Hartle (1990b).

^{*} For example, Margenau (1958) and Prugovečki (1967). We are not discussing here, nor do we discuss later, “unsharp” observables or “effects”. For those see *e.g.*, Busch (1987).

The mathematical derivation of the uncertainty relation (threetwo) is, of course, no less valid in the Hilbert space of a closed system than it is for that of a subsystem. In the quantum mechanics of a closed system, however, the absence of projection operators that specify x and p to accuracies better than (threetwo) is not to be interpreted as a limitation on external measurements of this closed system. By hypothesis there are none! Rather, in the quantum mechanics of closed systems, uncertainty relations like (threetwo) are limitations on how a closed system can be *described*. There are no histories in which position and momentum can be simultaneously specified to accuracies better than allowed by Heisenberg's principle.

Although there are no projection operators that specify position and momentum *simultaneously* to accuracies better than the limitations of the uncertainty principle, we can consider histories in which position is specified sharply at one time and momentum at another time. Let $\{\Delta_\alpha\}$ be an exhaustive set of exclusive position intervals, $\{\tilde{\Delta}_\beta\}$ be an exhaustive set of exclusive momentum intervals, and $\{P_\alpha(t')\}$ and $\{\tilde{P}_\beta(t)\}$ be the corresponding Heisenberg picture projection operators at times t' and t respectively. An individual history in which the momentum lies in the interval $\tilde{\Delta}_\beta$ at time t and the position in interval Δ_α at a *later* time t' would correspond to a branch of the initial state vector of the form:

$$P_\alpha(t')\tilde{P}_\beta(t)|\Psi\rangle . \tag{threethree}$$

As α and β range over all values, an exhaustive set of alternative histories of the closed system is generated. Probabilities are assigned to these histories when the set decoheres, that is, when the branches (threethree) are sufficiently orthogonal according to (twonine).

Nothing prevents us from considering the case when t' coincides with t . If the alternative histories decohere, one would predict the joint probability $p(\alpha, \beta)$ that the momentum is in the interval $\tilde{\Delta}_\beta$ at one time and *immediately* afterwards the position is in the interval Δ_α . That would give a different meaning to the probability of a simultaneous specification

of position and momentum.

Even if the intervals $\{\Delta_\alpha\}$ and $\{\tilde{\Delta}_\beta\}$ are infinitesimal, corresponding to a sharp specification of position and momentum there are some states $|\Psi\rangle$ for which these alternatives decohere. Eigenstates of momentum provide one example. However, for no state $|\Psi\rangle$ will the marginal probability distributions of position and momentum have variances that violate the uncertainty principle. That is because decoherence implies the probability sum rules so that

$$p(\alpha) \equiv \sum_{\beta} p(\alpha, \beta) = \|P_\alpha(t)|\Psi\rangle\|^2 \quad (\text{threefoura})$$

and

$$p(\beta) \equiv \sum_{\alpha} p(\alpha, \beta) = \|\tilde{P}_\beta(t)|\Psi\rangle\|^2 \quad (\text{threefourb})$$

where, in each case, the last equality follows from decoherence. However, the left-hand sides of (threefour) are just the usual probabilities for position and momentum computed from a single state. Their variances must satisfy the uncertainty principle.

One would come closer to the classical meaning of simultaneously specifying the position and momentum if histories of coincident position and momentum projections decohered independently of their order. That is, if the set of histories

$$\tilde{P}_\beta(t)P_\alpha(t)|\Psi\rangle \quad (\text{threefive})$$

were to decohere in addition to the set defined by (threethree) with $t' = t$. In that case it is straightforward to show that the joint probabilities $p(\alpha, \beta)$ are independent of the order of the projections as a consequence of decoherence. Whether states can be exhibited in which both (threethree) and (threefive) decohere is a more difficult question.

IV. Limitations on Ideal Measurements

While measurement is not fundamental to a formulation of the quantum mechanics of a closed system, measurement situations can be described within it. That is because

we can always consider a closed system consisting of measuring apparatus and measured subsystem or most generally and accurately the entire universe. Roughly speaking, a *measurement situation* is one in which a variable of the measured subsystem, perhaps not normally decohering, becomes correlated with high probability with a variable of the apparatus that decoheres because of *its* interactions with the rest of the universe. The variable of the apparatus is called a *record* of the measurement outcome. The decoherence of the alternative values of this record leads to the decoherence of the measured alternatives because of their correlation. Measurement situations can be described quantitatively in the quantum mechanics of closed systems by using the overlap (twonine) to determine when measured alternatives decohere and using the resulting probabilities to assess the degree of correlation between record and measured variable. By such means, any measurement situation, ideal or otherwise, may be accurately handled in the quantum mechanics of the closed system containing both measuring apparatus and measured subsystem.

Conventional discussions of measurement in quantum mechanics often focus on *ideal measurement models* in which certain approximate features of realistic measurement situations are idealized as exact.* In particular an ideal measurement is one that leaves a subsystem that is initially in an eigenstate of a measured quantity in that same eigenstate after the measurement. The subsystem is thus “disturbed as little as possible” by its interaction with the apparatus. Of course, not very many realistic measurements are ideal in this sense. Typically, after a measurement, subsystem and apparatus are not even in a product state for which it makes sense to talk about the “state of the subsystem”. Probably the reason for the focus on ideal measurement models is that they are models of the sorts of measurements for which the “reduction of the state vector” could accurately model the evolution of the measured subsystem interacting with the measuring apparatus. In particular, a reduction of the state of the apparatus will leave the subsystem in the

* Some classic references are von Neumann (1932), London and Bauer (1939), and Wigner (1963) or see almost any text on quantum mechanics.

correlated eigenstate of the measured variable.

Quantum mechanics severely restricts the possible ideal measurement situations. Wigner (1952) and Araki and Yanase (1960) showed that, even given arbitrary latitude in the choice of Hamiltonian describing the combined system of apparatus and measured subsystem, only quantities that commute with additive, conserved quantities can be ideally measured. This is a very restrictive conclusion. It rules out, for example, precise, ideal measurements of the position and momentum of a particle with a realistic Hamiltonian. (They do not commute with the additive, conserved angular momentum.) Araki and Yanase showed that, in a certain sense, ideal measurements were *approximately* possible for such quantities, but in a strict sense quantum mechanics prohibits them.

Impurity of the initial state of a closed system limits ideal measurements in another way. To derive this limitation it is necessary to discuss more precisely ideal measurement models in the quantum mechanics of closed systems.* We will use the more of the formulation of the quantum mechanics of closed systems than has been developed here. The reader can find the necessary background in the author's other contribution to these volumes, (Hartle 1993a).

We consider a closed system in which we can identify alternatives of a subsystem that are to be “measured”. Let $\{S_{\alpha_k}^k(t_k)\}$, $\alpha_k = 1, 2, 3, \dots$ be the Heisenberg projection operators corresponding to these alternatives at a set of times $\{t_k\}$, $k = 1, 2, 3, \dots$. In a more detailed ideal measurement model we might assume that the Hilbert space of the closed system is a tensor product of a Hilbert space \mathcal{H}^s defining the subsystem and a Hilbert space \mathcal{H}^r defining the rest of the universe outside the subsystem. In the Schrödinger picture, projection operators representing the alternatives of the subsystem would have the form $S_{\alpha_k}^k = s_{\alpha_k}^k \otimes I^r$ where the s 's act on \mathcal{H}^s alone. However, such specificity is not needed for the result that we shall derive.

* For more detail than can be offered here see Hartle (1991a), Section II.10.

Let us consider how a sequence of ideal measurements of alternatives of the subsystem $S_{\alpha_k}^k(t_k)$ at times $t_1 < \dots < t_n$ is described. A history of specific alternatives $(\alpha_1, \dots, \alpha_n) \equiv \alpha$ is represented by the corresponding chain of projections:

$$C_\alpha = S_{\alpha_n}^n(t_n) \cdots S_{\alpha_1}^1(t_1) . \quad (\text{fourone})$$

One defining feature of an ideal measurement situation is that there should exist at a time $T > t_n$ a *record* of the outcomes of the measurement that is exactly correlated with the measured alternatives of the subsystem. That is, there should be a set of orthogonal, *commuting*, projection operators $\{R_\beta(T)\}$ with $R_\beta(T) \equiv R_{\beta_n \dots \beta_1}(T)$ which are always exactly correlated with the measured alternatives C_α in histories that contain them both, as a consequence of the system's initial condition. The degree of correlation is defined by the decoherence functional $D(\beta', \alpha'; \beta, \alpha)$ which measures the interference between a history consisting of a sequence of measured alternatives $\alpha = (\alpha_1, \dots, \alpha_n)$ followed by a record $\beta = (\beta_1, \dots, \beta_n)$ at time T and a similar history with alternatives α' and β' . If the records are exactly correlated with the measured alternatives.

$$D(\beta', \alpha'; \beta, \alpha) = Tr[R_{\beta'}(T)C_{\alpha'}\rho C_\alpha^\dagger R_\beta(T)] \propto \delta_{\beta'\alpha'}\delta_{\beta\alpha} \quad (\text{fourtwo})$$

where ρ is the Heisenberg picture initial density matrix of the closed system of apparatus and measured subsystem and $\delta_{\beta\alpha}$ means $\delta_{\beta_1\alpha_1}\delta_{\beta_2\alpha_2}\cdots\delta_{\beta_n\alpha_n}$, etc.

The existence of exactly correlated records as described by (fourtwo) ensures the decoherence of the histories of the subsystem and permits the prediction of their probabilities. That is because the records are orthogonal and exhaustive:

$$R_\beta(T)R_{\beta'}(T) = \delta_{\beta\beta'}R_\beta(T), \quad \sum_\beta R_\beta(T) = I . \quad (\text{fourthree})$$

These properties together with the cyclic property of the trace are enough to show that

$$Tr[C_{\alpha'}\rho C_\alpha^\dagger] \propto \delta_{\alpha'\alpha} \quad (\text{fourfour})$$

follows from (fourtwo). This is the generalization of the decoherence condition (twonine) for an initial density matrix. The measurement correlation thus effects the decoherence of the measured alternatives.

Of course, much more is usually demanded of an ideal measurement situation than just decoherence of the measured alternatives. There is the idea that ideal measurements “disturb the measured subsystem as little as possible” and, in particular, that values of measured quantities are not disturbed. These are described in more detail in the context of the quantum mechanics of closed systems in Hartle (1991a). For our discussion, however, we need only the feature that ideal measurements assume *exactly* correlated records of measurement outcomes, for we shall now show that if the density matrix is highly impure such records cannot exist for non-trivial sets of measured histories.*

We begin by introducing bases of complete sets of states in which the density matrix ρ and the commuting set of projection operators $\{R_\beta(T)\}$ are diagonal, *viz.* :

$$\rho = \sum_r |r\rangle \pi_r \langle r| , \quad (\text{fourfive})$$

$$R_\beta(T) = \sum_n |\beta, n\rangle \langle \beta, n| . \quad (\text{foursix})$$

where π_r are the diagonal elements of ρ . When “diagonal” elements of the condition (fourtwo) (those with $\alpha = \alpha', \beta = \beta'$) are written out in terms of these bases they take the form

$$\sum_{r,n} \pi_r |\langle \beta, n | C_\alpha | r \rangle|^2 \propto \delta_{\alpha\beta} . \quad (\text{fourseven})$$

The left-hand side is a sum of positive numbers so that this implies

$$\langle r | C_\alpha | \beta, n \rangle = 0 , \text{ when } \alpha \neq \beta , \quad (\text{foureeight})$$

for all r for which $\pi_r \neq 0$.

* The argument we shall give is a straightforward extension of that used by M. Gell-Mann and the author to analyze the possibility of “strong decoherence” in Gell-Mann and Hartle (1993a). Thanks are due to M. Gell-Mann for permission to publish here what is essentially a joint result.

If the density matrix ρ is highly impure, so that $\pi_r \neq 0$ for a *complete* set of states $\{|r\rangle\}$, the relation (foureight) implies the operator condition

$$C_\alpha|\beta, n\rangle = 0, \quad \alpha \neq \beta. \quad (\text{fournine})$$

Therefore, C_α is non-zero only on the subspace defined by $R_\alpha(T)$ where $R_\alpha(T)$ is effectively unity. Thus we have

$$R_\beta(T)C_\alpha = \delta_{\beta\alpha}C_\alpha. \quad (\text{foureleven})$$

Summing this relation over C_α and utilizing the fact that $\sum_\alpha C_\alpha = I$, we find

$$C_\alpha = R_\alpha(T). \quad (\text{fourtwelve})$$

which says that the string of projections is itself a projection. This can happen only if the string consists of a single projection or if all the projections in the string commute with each other. To see the latter fact write (fourtwelve) in detail as

$$S_{\alpha_n}^n(t_n) \cdots S_{\alpha_1}^1(t_1) = R_{\alpha_n \cdots \alpha_1}(T). \quad (\text{fourthirteen})$$

Summation implies

$$S_{\alpha_k}^k(t_k) = \sum_{\alpha_j \neq \alpha_k} R_{\alpha_n \cdots \alpha_1}(T) \quad (\text{fourfourteen})$$

but since the R 's commute with each other the S 's must also. Even in the case that C_α consists of a single projection, (fourtwelve) shows that record and projection are identical. If C_α consists of projections that refer to a subsystem defined by a Hilbert space as described above then the records cannot be elsewhere in the universe. Thus, if the initial density matrix is highly impure, in the sense that it has non-zero probabilities for a complete set of states, there cannot be exactly correlated records of measurement outcomes. In particular there cannot be ideal measurements.

Of course, in realistic measurement situations we do not expect to find records that are *exactly* correlated with measured variables of a subsystem. Neither do we necessarily

expect *exact* decoherence of measured alternatives or many of the other idealizations of the ideal measurement situation as very experimentalist knows! It, therefore, becomes an interesting question to investigate quantitatively the connection between the $\{\pi_r\}$ of the density matrix and the degree to which approximate records defined by a relaxed (fourtwo) exist.

V. Interfering Alternatives

The starting point for Feynman’s sum-over-histories formulation of quantum mechanics is the prescription of the amplitude for an elementary (completely fine-grained) history of a measured subsystem as

$$\exp[iS(\text{history})/\hbar] \tag{fiveone}$$

where S is the action functional summarizing the subsystem’s dynamics. As an example, we may think of a non-relativistic particle moving in one dimension. In this case the elementary histories are the possible paths of the particle, $x(t)$, and the action is the usual

$$S[x(\tau)] = \int dt \left[\frac{1}{2}m \left(\frac{dx}{dt} \right)^2 - V(x) \right] . \tag{fivetwo}$$

We will use this example for all illustrative purposes in what follows.

A given experimental situation determines some parts of the subsystem’s path but leaves undetermined many other parts. For instance, consider a measurement that determines whether or not a particle is in a position interval Δ at time t . In that case the measurement leaves undetermined the positions at times other than t and the relative position within Δ at time t . Given an initial state at time t_0 represented by a wave function $\psi(x_0)$, we may compute the probabilities for the outcomes that are determined by the measurement as follows: We first divide the *undetermined* alternatives into “interfering” and “non-interfering” (or “exclusive”) alternatives according to the experimental situation. We sum amplitudes for histories weighted by the initial wave function over the interfering

alternatives, square that, and sum the square over the non-interfering alternatives. The result is the probability for the measured determination. For example, in the case of the measurement mentioned above that localized a particle to an interval Δ at time t , the probability of this outcome is:

$$p(\Delta) = \int_{\Delta} dx_f \left| \int_{x_f} \delta x e^{iS[x(\tau)]/\hbar} \psi(x_0) \right|^2. \quad (\text{fivethree})$$

The path integral is over all paths in the time interval $[t_0, t]$ that end in x_f , and includes an integral over the initial position x_0 . These are the “interfering alternatives”. The square of the amplitude is summed over the final position within Δ . These positions are the “non-interfering” alternatives.

What determines whether an undetermined alternative is interfering or not? Certainly it is not whether it is measured in the experimental situation. In the above example, positions at times other than t were not measured and they were “interfering”. But the experiment also did not measure the relative position within Δ and this was “non-interfering”. According to Feynman and Hibbs (1965):

“It is not hard, with a little experience, to tell what kind of alternatives is involved. For example, suppose that information about alternatives is available (*or could be made available without altering the result*) [author’s italics], but this information is not used. Nevertheless, in this case a sum of probabilities (in the ordinary sense) must be carried out over *exclusive* alternatives. These exclusive alternatives are those which *could* have been separately identified by the information.”

Thus, in the above example, the value of x at a time other than t is an interfering alternative because we could not have acquired information about it without disturbing the later probability that x is in Δ at t . By contrast, the precise value of x within Δ is a non-interfering or exclusive alternative because we *could* have measured it precisely and left

the probability for the particle to lie in Δ undisturbed. Indeed, one way to determine whether the particle is in Δ is simply to measure the position at t precisely.

The author has always found this distinction between types of alternatives confusing. He did not doubt Feynman's ability "to tell what kind of alternative is involved", but he was less sure of his own. This was especially the case since the distinction seemed to involve analyzing, not only the particular experiment in question, but also many others that *might* have been carried out. No precise rules for analyzing a given experimental situation seemed to be available. This situation is considerably clarified in the quantum mechanics of closed systems.

In the quantum mechanics of closed systems, we cannot have a fundamental distinction between "interfering" and "non-interfering" alternatives based on different types of measurement situations, because alternatives are not necessarily associated with measurement situations. Whether alternatives interfere with one another, or do not, depends on the boundary conditions and Hamiltonian that define the closed system. A quantitative measure for the degree of interference is provided by the decoherence functional. To illustrate this idea, let us consider the single particle model we have been discussing on the time interval $[t_0, t]$. The fine-grained histories are the particle paths on this interval. Sets of alternatives correspond to partitions of these paths into an exhaustive set of exclusive classes $\{c_\alpha\}$, $\alpha = 1, 2, \dots$. The classes are coarse-grained alternatives for the closed system. For example, one could partition the paths by which of an exhaustive set of position intervals they pass through at one time, which of a different set of position intervals they pass through at another time, etc. There are many more general possibilities (see, e.g. Hartle, 1991). The decoherence functional is a complex valued functional on pairs of coarse-grained alternatives defined in a sum-over-histories formulation of the quantum

mechanics of a closed system by:

$$D(\alpha', \alpha) = N \int_{c_{\alpha'}} \delta x' \int_{c_{\alpha}} \delta x \rho_f(x_f, x'_f) \exp \{i (S[x'(\tau)] - S[x(\tau)]) / \hbar\} \rho_i(x'_0, x_0). \quad (\text{fiveten})$$

The first sum is over paths $x'(t)$ in the class $c_{\alpha'}$ and includes a sum over their initial endpoints x'_0 and final endpoints x'_f . The sum over paths $x(t)$ is similar. The normalization factor is $N = 1/\text{Tr}(\rho_f \rho_i)$ where the ρ 's are the operators whose matrix elements appear in (fiveten). We have written the decoherence functional for a general, time-neutral, formulation of quantum mechanics* in which both an initial and a final condition enter symmetrically, represented by density matrices $\rho_i(x'_0, x_0)$ and $\rho_f(x'_f, x_f)$ respectively. The final condition which seems to best represent our universe and ensures causality is a final condition of indifference with respect to final state in which the final density operator is $\rho_f \propto I$.

The “off-diagonal” elements of the decoherence functional ($\alpha' \neq \alpha$) are a measure of the degree of interference between pairs of alternatives. When the interference is negligible between all pairs in an exhaustive set, the “diagonal” elements ($\alpha' = \alpha$) are the probabilities of the alternatives and obey the correct probability sum rules as a consequence of the absence of interference. The orthogonality of the branches in (twonine) is an operator transcription of this condition in the special case that $\rho_f \propto I$.

The important point for a discussion of “interfering” and “non-interfering” alternatives is that *all* alternatives are potentially interfering in the quantum mechanics of closed systems. For this reason amplitudes are summed over them in the construction of the decoherence functional (fiveten). Whether alternatives are interfering or not depends on the measure of interference provided by (fiveten), but in its construction all sets of alternatives are treated the same. When interference between each pair is negligible the probabilities

* See, e.g. Aharonov, Bergmann and Lebovitz (1964) in the quantum mechanics of measured subsystems, and Griffiths (1984) and Gell-Mann and Hartle (1993) in the quantum mechanics of closed systems.

for coarser-grained alternatives may be constructed either directly from (fiveten) by summing amplitudes, or by summing the probabilities for the finer-grained alternatives in the coarser-grained ones. The equivalence between the two is the content of decoherence.

Thus, there is no distinction between kinds of alternatives generally in the formalism, but distinctions may emerge between different kinds of alternatives because of particular properties of ρ_i and ρ_f . In particular, if $\rho_f \propto I$ any alternatives at the last time will decohere. Thus, indifference with respect to final states is, in a time-neutral formulation of quantum mechanics, the origin of the usual rule that final alternatives are “non-interfering” rather than an analysis of whether “one could have measured them but didn’t”.

VI. Conclusion

In the quantum mechanics of closed systems, projections act on states in the formula for the probabilities of histories, but those reductions are not necessarily associated with a measurement situation within the system and certainly not with one from without. Uncertainty principles limit what kinds of alternatives a set of projections can describe, but these limitations need not be of our ability to carry out a measurement. Interfering alternatives can be distinguished from non-interfering ones, not by analyzing what might have been measured, but by using the decoherence functional as a quantitative measure of interference. Probabilities can be consistently assigned only to non-interfering sets of alternative histories but decoherence as a consequence of a particular initial condition and Hamiltonian rather than measurement decides which sets these are.

The fundamental role played by measurement in formulating a quantum mechanics of subsystems is replaced by decoherence in the quantum mechanics of a closed system. In the opinion of the author, the result is not only greater generality so that the theory can be applied to cosmology, but also greater clarity. An important reason for this is the disassociation of the notion of alternative from an ideal measurement. As we saw from the

work of Wigner, Araki and Yanase, and the argument of Section IV, ideal measurements are almost impossible to realize exactly within quantum mechanics, and are therefore of limited value as approximations to realistic measurement situations. But in the usual quantum mechanics of measured subsystems, the second law of evolution is stated for ideal measurements, not realistic ones. To discuss the evolution under realistic alternatives it appears necessary to consider more and more of the universe beyond the subsystem of interest until one obtains a subsystem large enough such that measurements of *it* may be approximated as “ideal”. By contrast, the alternatives used in the quantum mechanics of closed systems are general enough to describe realistic measurement situations. The theory can provide quantitative estimates of their closeness to “ideal” and therefore to how closely the quantum mechanics of measured subsystems approximates the more general quantum mechanics of closed systems.

Thus, little havoc needs be made to achieve a quantum mechanics of closed systems. All that is needed is a more general formulation in which decoherence rather than measurement is fundamental, but in which most features of the approximate quantum mechanics of subsystems that were tied to measurement reëmerge in a more general and conceptually clearer light.

Acknowledgment

The work described in this essay is an outgrowth of the author’s program with M. Gell-Mann to explore and clarify quantum mechanics. It is a pleasure to thank him for many discussions that are reflected, in part, in the work presented here.

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