

# A TESSELLATION FOR ALGEBRAIC SURFACES IN $\mathbb{C}\mathbf{P}^3$

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In this paper we present a systematic and explicit algorithm for tessellating the algebraic surfaces (real 4-manifolds)  $F_n$  in  $\mathbb{C}\mathbf{P}^3$  defined by the equation

$$(1) \quad z_0^n + z_1^n + z_2^n + z_3^n = 0$$

in the standard homogeneous coordinates  $[z_0, z_1, z_2, z_3]$ , where  $n$  is any positive integer. Note that  $F_4$  in particular is a  $K3$  surface (see, e.g., [Griffiths & Harris]).

The tessellation we present contains a minimal number of vertices: they are the  $n$ -th roots of unity in the six standard projective lines  $\mathbb{C}\mathbf{P}^1$  in  $\mathbb{C}\mathbf{P}^3$ , and are the obvious vertices to start a construction of a natural tessellation for  $F_n$ . Our tessellation is invariant under the action of the obvious isomorphism group of  $F_n$  induced by permutations and phase multiplications of the coordinates, and the action is transitive on the set of 4-cells. The tessellation is built upon a similar triangulation for the corresponding algebraic curves in  $\mathbb{C}\mathbf{P}^2$ , and we believe it can be generalized to the corresponding algebraic hypersurfaces in  $\mathbb{C}\mathbf{P}^N$  for  $N > 3$ .

The tessellation is algorithmically programmable: For any given positive integer  $n$ , one first lists all the vertices; then all the edges, faces, 3-cells, and 4-cells can be produced symbolically from the list of vertices. One can then, for example, also formulate the simplicial complex boundary map matrices and compute the homology, etc., if one wishes.

Explicit representations of geometric objects such as manifolds are essential for any attempt to create visual images that help expose their features. While there exist many powerful mathematical methods that allow the calculation of the geometric and topological invariants of manifolds, human perception requires the construction of visual images. Thus, it can be useful to develop explicit descriptions of interesting families of manifolds that can be used in practice to create visual representations and pictures. Such explicit representations can also in principle be used to clarify the calculation and understanding of abstract invariants of the manifolds. Among the classes of geometric objects that have a long history of interest are the algebraic varieties defined by homogeneous polynomials in complex projective spaces. One such family, the algebraic curves in  $\mathbb{C}\mathbf{P}^2$  (see, e.g., [Hanson, 1994]), has recently

served the purpose of providing explicit images of cross-sections of Calabi-Yau spaces, and has been used to represent the hidden dimensions of string theory [Greene, 1999], for which very few other methods of producing images are available. While one might have guessed that the methods used for  $\mathbb{CP}^2$  could be extended trivially to  $\mathbb{CP}^3$  and higher dimensional projective spaces, the problem turns out to be fairly complex.

Let us now be more precise. We will show the following:

**Theorem.** *For any given positive integer  $n$ , there is a tessellation on  $F_n$  with  $6n^3$  4-cells. Each 4-cell is bounded by four pentahedrons. Each pentahedron is a pyramid with one quadrilateral face and four triangular faces. The tessellation is invariant under the action of the group  $\Gamma_n$ , where  $\Gamma_n$  consists of isomorphisms of  $F_n$  induced from permutations and phase multiplications of the homogeneous coordinates of  $\mathbb{CP}^3$ . The group  $\Gamma_n$  acts transitively on the set of 4-cells of the tessellation.*

Altogether, the tessellation has  $6n$  vertices,  $12n^2$  edges,  $8n^2 + 7n^3$  2-cells ( $3n^3$  quadrilaterals and  $8n^2 + 4n^3$  triangles),  $12n^3$  3-cells (pyramids) and  $6n^3$  4-cells. It is known that the Euler characteristic of any smooth algebraic surface of degree  $n$  in  $\mathbb{CP}^3$  is  $6n - 4n^2 + n^3$  (see, e.g., [Griffiths & Harris]). One handily verifies from our tessellation for  $F_n$  that this is equal to  $6n - 12n^2 + (8n^2 + 7n^3) - 12n^3 + 6n^3$ , i.e., the alternating sum of the numbers of vertices, edges, 2-cells, 3-cells, and 4-cells.

Notice that the restriction to  $F_n$  of the natural projection  $\mathbb{CP}^3 \setminus \{[0, 0, 0, 1]\} \rightarrow \mathbb{CP}^2$ , given by  $[z_0, z_1, z_2, z_3] \mapsto [z_0, z_1, z_2]$ , is a regular  $n$ -fold branched covering

$$(2) \quad \sigma : F_n \rightarrow \mathbb{CP}^2$$

which is branched over the algebraic curve in  $\mathbb{CP}^2$  defined by the equation

$$(3) \quad z_0^n + z_1^n + z_2^n = 0 .$$

The tessellation of  $F_n$  we present is a lift from  $\sigma$  of a tessellation of  $\mathbb{CP}^2$ , which is an extension of a tessellation (triangulation) of the algebraic curve (3). This approach greatly reduces the difficulty caused by the topological complexity of  $F_n$ , as the geometry and topology of  $\mathbb{CP}^2$  are much easier to handle and visualize. We also implicitly assume that  $\mathbb{CP}^2$  is equipped with the standard Fubini-Study Riemannian metric. In particular, every projective line  $\mathbb{CP}^1$  in  $\mathbb{CP}^2$  is totally geodesic, and, with the induced metric, is a round 2-sphere; the real projective planes are also totally geodesic and have induced metric of constant curvature.

1. TESSELLATION OF THE ALGEBRAIC CURVE

Denote by  $S_n$  the algebraic curve in  $\mathbb{CP}^2$  defined by (3). In this section, we will tessellate (i.e., triangulate)  $S_n$  in a specific way so that we can extend the tessellation to the  $\mathbb{CP}^2$  in the next section. The tessellation is in fact a lifting of a natural tessellation on  $\mathbb{CP}^1$  for the given  $n$ .

The projection  $\mathbb{CP}^2 \setminus \{[0, 0, 1]\} \rightarrow \mathbb{CP}^1$ , given by  $[z_0, z_1, z_2] \mapsto [z_0, z_1]$  induces a regular  $n$ -fold branched covering from  $S_n$  to the  $\mathbb{CP}^1$  branched at  $n$  points,

$$(4) \quad p_k := [1, e^{i(\pi+2k\pi)/n}], \quad k = 0, \dots, n - 1 .$$

We first formulate a tessellation for the  $\mathbb{CP}^1$ , which has  $n + 2$  vertices,  $3n$  edges and  $2n$  triangles:

Let

$$(5) \quad p_0 := [0, 1], \quad p_1 := [1, 0]$$

and join them by the following  $n$  paths,

$$(6) \quad e_k(t) := [\cos t, \sin t e^{i2k\pi/n}], \quad 0 \leq t \leq \frac{\pi}{2}, \quad k = 0, \dots, n - 1 .$$

Then the  $2n$  triangles of the tessellation are

$$(7) \quad f_k^k, \quad f_{k+1}^k, \quad k = 0, \dots, n - 1 \pmod{n},$$

where each of  $f_k^k, f_{k+1}^k$  is the triangle with vertices  $p_0, p_1, p_k$ , the edge  $e_k$ , or  $e_{k+1}$ , respectively, and the other two edges given by the minimizing geodesics joining  $p_k$  and  $p_0, p_1$  (see Figure 1).

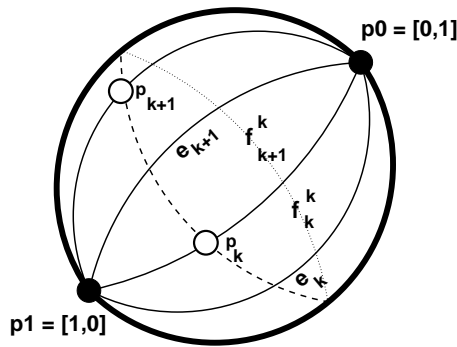


FIGURE 1

Lifting this triangulation through the branched covering, we then get a triangulation for  $S_n$ . There are  $3n$  vertices,

$$p_{0k} := [0, 1, e^{i(\pi+2k\pi)/n}], \quad p_{1k} := [e^{i(\pi+2k\pi)/n}, 0, 1], \quad p_{2k} := [1, e^{i(\pi+2k\pi)/n}, 0],$$

for  $k = 0, \dots, n-1 \pmod{n}$ , and  $2n^2$  triangles. It is not hard to see that these triangles, as lifts of  $f_k^k, f_{k+1}^k$  and expressed in terms of their vertices, are  $\triangle p_0 p_{j-k} p_{1-j-1} p_{2k}$ ,  $\triangle p_0 p_{j-(k+1)} p_{1-j-1} p_{2k}$ , respectively. We denote them by the following:

$$(8) \quad b_{j-k, -j-1, k}, \quad b_{j-(k+1), -j-1, k}, \quad j, k = 0, \dots, n-1 \pmod{n}.$$

To be more clear, we verify the indices in (8) by showing the edges of these triangles explicitly.

The three edges of  $b_{j-(k+1), -j-1, k}$ , in the order  $p_{2k} - p_{0_{j-(k+1)}} - p_{1-j-1} - p_{2k}$ , can be described as follows: notice that the first two coordinates give the edges of  $f_{k+1}^k$ , in the order  $p_k - p_0^{e_{k+1}} - p_1 - p_k$ , and the factor  $e^{i2j\pi/n}$  on the third coordinate specifies a certain branch to which  $f_{k+1}^k$  is lifted.

$$(9) \quad \begin{aligned} & [\cos t, \sin t e^{i(2k+1)\pi/n}, e^{i2j\pi/n} (-\cos^n t + \sin^n t)^{1/n}], \quad \pi/4 \leq t \leq \pi/2; \\ & [\sin t, \cos t e^{i2(k+1)\pi/n}, e^{i2j\pi/n} (-\sin^n t - \cos^n t)^{1/n}], \quad 0 \leq t \leq \pi/2; \\ & [\cos t, \sin t e^{i(2k+1)\pi/n}, e^{i2j\pi/n} (-\cos^n t + \sin^n t)^{1/n}], \quad 0 \leq t \leq \pi/4. \end{aligned}$$

Similarly, the three edges of  $b_{j-k, -j-1, k}$ , in the order  $p_{2k} - p_{1-j-1} - p_{0_{j-k}} - p_{2k}$ , as lifts of those of  $f_k^k$ , in the order  $p_k - p_1^{e_k} - p_0 - p_k$ , are

$$(10) \quad \begin{aligned} & [\sin t, \cos t e^{i(2k+1)\pi/n}, e^{i2j\pi/n} (-\sin^n t + \cos^n t)^{1/n}], \quad \pi/4 \leq t \leq \pi/2; \\ & [\cos t, \sin t e^{i2k\pi/n}, e^{i2j\pi/n} (-\cos^n t - \sin^n t)^{1/n}], \quad 0 \leq t \leq \pi/2; \\ & [\sin t, \cos t e^{i(2k+1)\pi/n}, e^{i2(j+1)\pi/n} (-\sin^n t + \cos^n t)^{1/n}], \quad 0 \leq t \leq \pi/4. \end{aligned}$$

Notice that there is a branch shift on the lift of  $p_0 - p_k$  (see Figure 2).

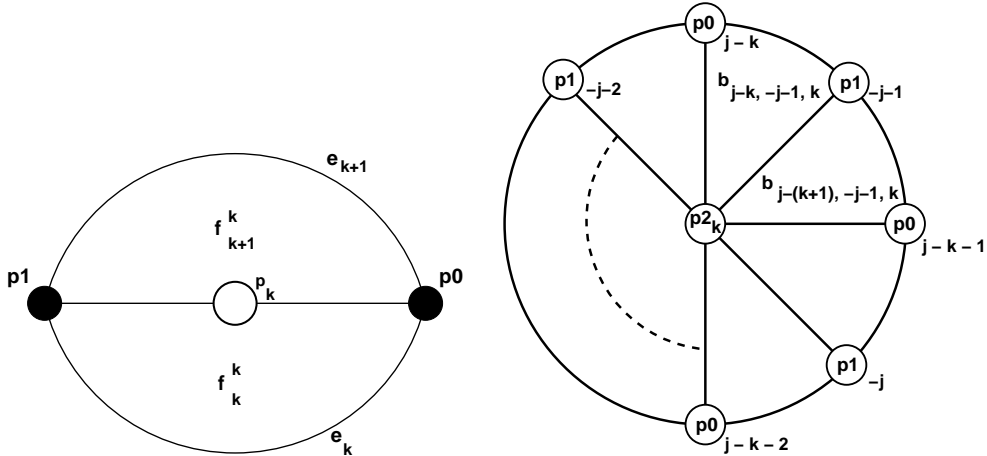


FIGURE 2

From (9) and (10) one then gets the vertices for the corresponding triangles easily. We note that any one of the indices for  $b$  in (8) is determined by the

other two according to the rule that the sum of the three indices is equal to  $-1$  or  $-2$ , respectively.

The tessellation is invariant under the isomorphisms of  $S_n$  induced from permutations and phase multiplications of the homogeneous coordinates of  $\mathbb{CP}^2$ ; here a phase multiplication means multiplying any of the coordinates by a number of the form  $e^{i2k\pi/n}$ . To see this, first notice that the tessellation on  $\mathbb{CP}^1$  is obviously invariant under the corresponding isomorphisms: the vertices are invariant and the edges are all geodesics while the isomorphisms are isometries. The tessellation is also obviously invariant under the phase multiplication of  $z_2$  because the latter is just a deck transformation of the branched covering. Therefore it suffices only to verify the invariance under interchanging the coordinates  $z_1$  and  $z_2$ .

After interchanging  $z_1$  and  $z_2$ , the three paths in (10) become

$$\begin{aligned} & [\sin t, e^{i2j\pi/n}(-\sin^n t + \cos^n t)^{1/n}, \cos t e^{i(2k+1)\pi/n}], \quad \pi/4 \leq t \leq \pi/2; \\ & [\cos t, e^{i2j\pi/n}(-\cos^n t - \sin^n t)^{1/n}, \sin t e^{i2k\pi/n}], \quad 0 \leq t \leq \pi/2; \\ & [\sin t, e^{i2(j+1)\pi/n}(-\sin^n t + \cos^n t)^{1/n}, \cos t e^{i(2k+1)\pi/n}], \quad 0 \leq t \leq \pi/4. \end{aligned}$$

They are the same as

$$\begin{aligned} & [\sin t, e^{i(2j+1)\pi/n}(\sin^n t - \cos^n t)^{1/n}, \cos t e^{i(2k+1)\pi/n}], \quad \pi/4 \leq t \leq \pi/2; \\ & [\cos t, e^{i(2j+1)\pi/n}(\cos^n t + \sin^n t)^{1/n}, \sin t e^{i2k\pi/n}], \quad 0 \leq t \leq \pi/2; \\ & [\sin t, e^{i2(j+1)\pi/n}(-\sin^n t + \cos^n t)^{1/n}, \cos t e^{i(2k+1)\pi/n}], \quad 0 \leq t \leq \pi/4; \end{aligned}$$

or

$$\begin{aligned} & [\cos t, e^{i(2j+1)\pi/n}(\cos^n t + \sin^n t)^{1/n}, e^{i2k\pi/n}(\sin^n t)^{1/n}], \quad 0 \leq t \leq \pi/2; \\ & [\sin t, e^{i2(j+1)\pi/n}(-\sin^n t + \cos^n t)^{1/n}, e^{i2k\pi/n}(-\cos^n t)^{1/n}], \quad 0 \leq t \leq \pi/4. \\ & [\sin t, e^{i(2j+1)\pi/n}(\sin^n t - \cos^n t)^{1/n}, e^{i2k\pi/n}(-\cos^n t)^{1/n}], \quad \pi/4 \leq t \leq \pi/2; \end{aligned}$$

which are the edges of  $b_{k-(j+1), -k-1, j}$ , a lift of  $f_{j+1}^j$ . Similarly, interchanging  $z_1$  and  $z_2$  transforms  $b_{j-(k+1), -j-1, k}$  to  $b_{k-j, -k-1, j}$ .

It is easy to see that the transformation under these isomorphisms is transitive on triangles. As the number of the isomorphisms is  $6n^2$ , the order of isotropy of each triangle is 3, consisting of the cyclic edge permutations. Therefore the transformation is also transitive on the edges, and obviously on the vertices as well.

We finally point out that the case  $n = 1$  is somewhat peculiar: the two triangles share the same three edges. Therefore extra care in labeling, e.g, specifying the orientation, is needed.

2. EXTENDED TESSELLATION ON  $\mathbb{CP}^2$ 

In this section, we extend the tessellation of  $S_n$  described in §1 to a tessellation of the  $\mathbb{CP}^2$ . Then, by lifting, that will automatically produce a tessellation of  $F_n$ .

Denote the projective line  $z_j = 0$  by  $L_j$ , for  $j = 0, 1, 2$ , and let

$$(11) \quad p01 := [0, 0, 1], \quad p12 := [1, 0, 0], \quad p20 := [0, 1, 0].$$

We start by specifying the other 2-cells for the tessellation.

Note that on  $L_0$ , the points  $p01, p20$  and the intersections with the  $S_n$ , namely  $p0_k$ ,  $k = 0, \dots, n-1$ , form the exact same configuration as (5) and (4) on  $\mathbb{CP}^1$  described in §1. We then add the corresponding  $2n$  triangles (7); similarly for the lines  $L_1$  and  $L_2$ . Therefore altogether there are  $6n$  new triangles, which we label as follows:

$$(12) \quad fj_k^k, \quad fj_{k+1}^k, \quad j = 0, 1, 2 \quad \text{and} \quad k = 0, \dots, n-1 \pmod{n}.$$

Label the edges corresponding to those in (6) by  $ej_k$ . Notice that, for example, as a path,  $e1_k(t) = [\sin t e^{i2k\pi/n}, 0, \cos t]$ .

In the next group, each triangle is formed by minimizing geodesics joining one of the vertices  $p01, p12, p20$  to the edge on  $S_n$ , e.g.,  $p2_j - p0_k$  in the case of  $p20$ . We denote these  $3n^2$  triangles as follows:

$$(13) \quad h01_{jk}, \quad h12_{jk}, \quad h20_{jk}, \quad j, k = 0, \dots, n-1 \pmod{n}.$$

We remark that all the triangles in (13) are totally geodesic; one sees, e.g., from (9) that they are pieces of real projective planes. In fact, all the new 2-cells we add will be totally geodesic.

There is one more group of  $n^2$  triangles that all have the same three vertices  $p01, p12, p20$ . For clarity, we write down the following explicit parameterizations for them:

$$g_{jk}(s, t) = [\cos s, \sin s \cos t e^{i2j\pi/n}, \sin s \sin t e^{i2k\pi/n}], \quad 0 \leq s, t \leq \pi/2,$$

The three edges of  $g_{jk}$  are  $e0_{k-j}, e1_{-k}, e2_j$ . For convenience, we will denote  $g_{jk}$  by

$$(14) \quad g_{k-j, -k, j}, \quad j, k = 0, \dots, n-1 \pmod{n},$$

noticing again that any one of the indices of  $g$  is determined by the other two according to the rule that the sum of the three indices is equal to 0.

The next set of 2-cells is a set of  $3n^2$  quadrilaterals. They are in one-to-one correspondence with the edges in  $S_n$ ; each edge is one side of exactly one quadrilateral. For example, the edge  $p0_j - p1_k$  is a side of the quadrilateral

having  $e_{2-j-k-1}$  as the opposite side of  $p_{0_j}-p_{1_k}$ ; recall that  $e_{2-j-k-1}$  is in  $L_2$  between the two vertices  $p_{2-j-k-2}$  and  $p_{2-j-k-1}$ , which are, respectively, the vertices of the two triangles in  $S_n$  having  $p_{0_j}-p_{1_k}$  as a common side. See Figure 3.

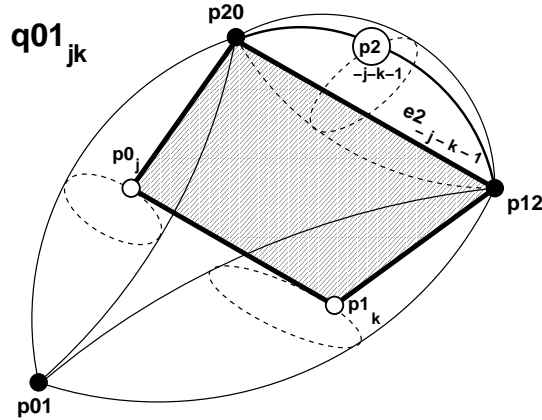


FIGURE 3

The quadrilateral is formed by minimizing geodesics joining the points on  $e_{2-j-k-1}$  to the distance-proportional points on  $p_{0_j}-p_{1_k}$ . In particular, the two edges in  $L_0, L_1$  joining  $p_{20}, p_{12}$  and  $p_{0_j}, p_{1_k}$ , respectively, are the other two sides of the quadrilateral. We denote this quadrilateral by  $q_{01_{jk}}$  and the set of quadrilaterals is

$$(15) \quad q_{01_{jk}}, \quad q_{12_{jk}}, \quad q_{20_{jk}}, \quad j, k = 0, \dots, n-1 \pmod{n} .$$

This concludes our construction of the 2-cells. The only new vertices added are then those in (11) and the only new edges are those in the  $L_j$ 's.

We now proceed to describe the 3-cells. It should be pointed out that, up to now, the cells constructed can be easily verified to be embedded in  $\mathbb{CP}^2$ , and there is no intersection among them in the interior of any cell. As the dimension of the cell becomes higher, this becomes less clear *a priori*. We will show later that the cells do form a tessellation for the  $\mathbb{CP}^2$ .

The 3-cells are divided into two groups. Each of them is in two-to-one correspondence with the set of edges in  $S_n$ , or the set of quadrilaterals. In fact, every quadrilateral is a face of exactly two 3-cells in each group.

In the first group, the two 3-cells corresponding to, say, the edge  $p_{0_j}-p_{1_k}$  are formed by interpolating between distance-proportional points on  $b_{j,k,-j-k-1}, b_{j,k,-j-k-2}$  and  $f_{2-j-k-1}^{-j-k-1}, f_{2-j-k-1}^{-j-k-2}$ , respectively, by minimizing geodesics (see Figure 4).

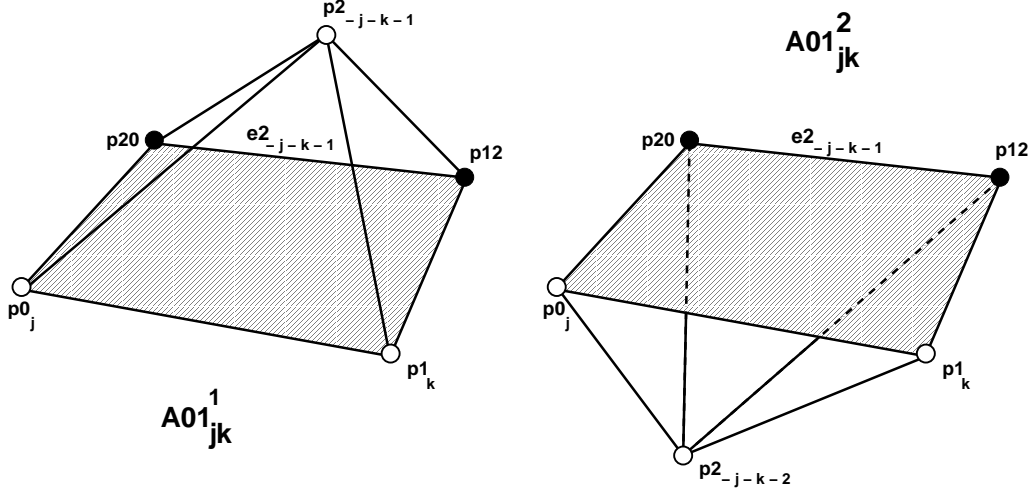


FIGURE 4

Clearly, the 3-cell is a pyramid. Besides the quadrilateral face  $q01_{jk}$ , the other four faces are the triangles

$$\{ b_{j,k,-j-k-1}, h12_{k,-j-k-1}, h20_{-j-k-1,j}, f2_{-j-k-1}^{-j-k-1} \},$$

or

$$\{ b_{j,k,-j-k-2}, h12_{k,-j-k-2}, h20_{-j-k-2,j}, f2_{-j-k-2}^{-j-k-1} \},$$

respectively. Denote these pyramids by  $A01_{jk}^1$ ,  $A01_{jk}^2$ , respectively. We can now list all the  $6n^2$  3-cells in the first group:

$$(16) \quad \begin{aligned} & A01_{jk}^1, A01_{jk}^2, \\ & A12_{jk}^1, A12_{jk}^2, \quad j, k = 0, \dots, n-1 \pmod{n}. \\ & A20_{jk}^1, A20_{jk}^2. \end{aligned}$$

In the second group of 3-cells, the two corresponding to, say again,  $p0_j-p1_k$  are formed by minimizing geodesic interpolation between  $h01_{jk}$  and  $g_{j,k+1,-j-k-1}$ ,  $g_{j+1,k,-j-k-1}$ , respectively (see Figure 5).

Clearly, each 3-cell is also a pyramid. Besides the quadrilateral face  $q01_{jk}$ , the other four faces are the triangles

$$\{ g_{j,k+1,-j-k-1}, f0_j^j, f1_{k+1}^k, h01_{jk} \},$$

or

$$\{ g_{j+1,k,-j-k-1}, f0_{j+1}^j, f1_k^k, h01_{jk} \},$$

respectively. Notice that, unlike the first group, whose two pyramids share only the quadrilateral face, these two pyramids share both the quadrilateral



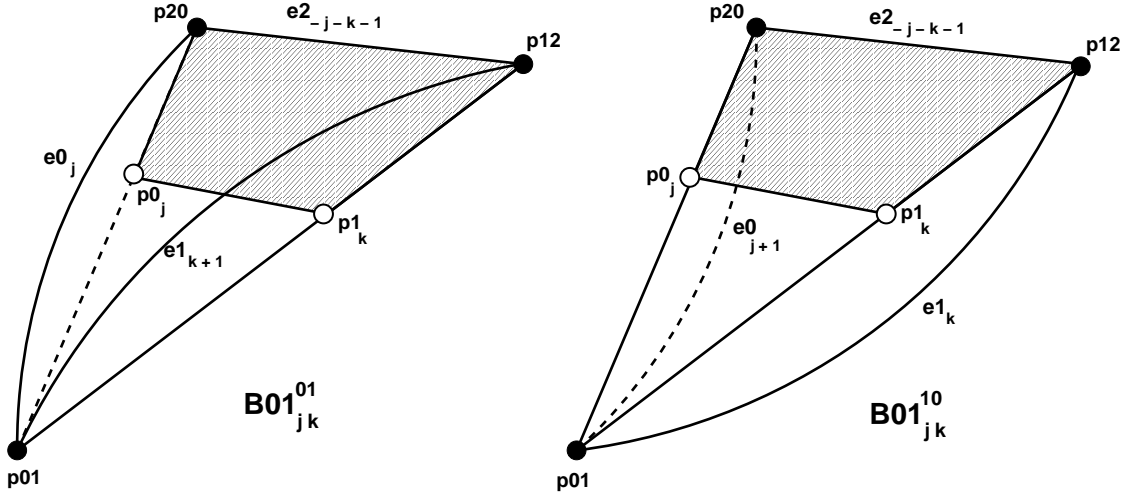


FIGURE 5

face  $q01_{jk}$  and the triangular face  $h01_{jk}$ . Denote these pyramids by  $B01_{jk}^{01}$ ,  $B01_{jk}^{10}$ , respectively. The list of all the  $6n^2$  3-cells in the second group then is:

$$(17) \quad \begin{aligned} & B01_{jk}^{01}, \quad B01_{jk}^{10}, \\ & B12_{jk}^{01}, \quad B12_{jk}^{10}, \quad j, k = 0, \dots, n-1 \pmod{n}. \\ & B20_{jk}^{01}, \quad B20_{jk}^{10}. \end{aligned}$$

We are now ready to tessellate the  $\mathbb{C}P^2$  by 4-cells. Each 4-cell is bounded by four pyramids, two from each of the groups (16) and (17); in fact, two from one determine the two from the other. Since every 3-cell should be the face of exactly two 4-cells, it follows that there are in all  $6n^2$  4-cells. We illustrate one of them as follows.

Start with  $A01_{jk}^1$  in (16). The other pyramid from (16) is either  $A12_{k,-j-k-1}^1$  or  $A20_{-j-k-1,j}^1$ , as these are the only other two pyramids in (16) sharing the triangular face  $b_{j,k,-j-k-1}$  with  $A01_{jk}^1$ . If, say, we pick  $A12_{k,-j-k-1}^1$ , then it is easy to see that the two pyramids from (17) must be  $B01_{jk}^{01}$  and  $B12_{k,-j-k-1}^{10}$ , in order to have the quadrilateral faces  $q01_{jk}$  and  $q12_{k,-j-k-1}$  shared, and for the two to have the triangular face from (14) in common. Therefore this 4-cell is bounded by the following four pyramids:

$$(18) \quad \{ A01_{jk}^1, \quad A12_{k,-j-k-1}^1, \quad B01_{jk}^{01}, \quad B12_{k,-j-k-1}^{10} \}$$

As illustrated in Figure 6, the pyramids in (18) indeed form a tessellation for a 3-sphere, at least combinatorially.

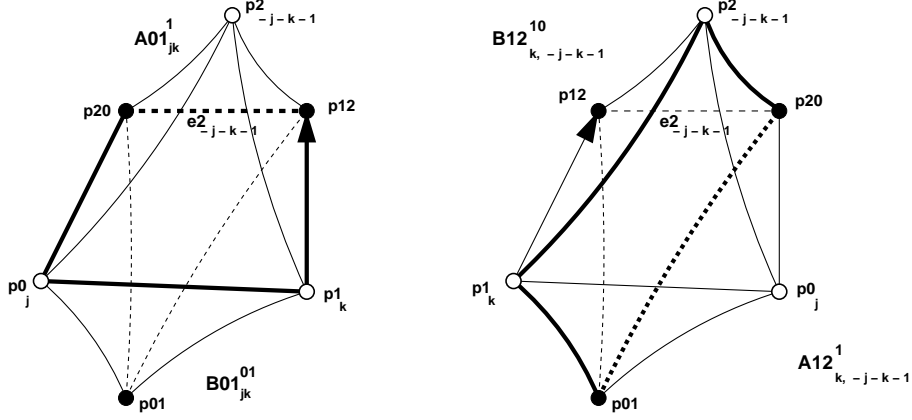


FIGURE 6

From the above, it is easy now to list all the 4-cells in terms of their boundary pyramids:

$$\begin{aligned}
 & \{A01_{jk}^1, A12_{k,-j-k-1}^1, B01_{jk}^{01}, B12_{k,-j-k-1}^{10}\}, \\
 & \{A01_{jk}^2, A12_{k,-j-k-2}^2, B01_{jk}^{10}, B12_{k,-j-k-2}^{01}\}, \\
 (19) \quad & \{A12_{jk}^1, A20_{k,-j-k-1}^1, B12_{jk}^{01}, B20_{k,-j-k-1}^{10}\}, \\
 & \{A12_{jk}^2, A20_{k,-j-k-2}^2, B12_{jk}^{10}, B20_{k,-j-k-2}^{01}\}, \\
 & \{A20_{jk}^1, A01_{k,-j-k-1}^1, B20_{jk}^{01}, B01_{k,-j-k-1}^{10}\}, \\
 & \{A20_{jk}^2, A01_{k,-j-k-2}^2, B20_{jk}^{10}, B01_{k,-j-k-2}^{01}\}.
 \end{aligned}$$

for  $j, k = 0, \dots, n-1 \pmod{n}$ . Combinatorially, these  $6n^2$  4-cells together form a simplicial 4-manifold. Combining this with the numbers of vertices, edges, 2-cells, and 3-cells we have obtained before, we find its Euler characteristic number to be

$$(3n + 3) - (3n^2 + 9n) + (2n^2 + 6n + 3n^2 + n^2 + 3n^2) - 12n^2 + 6n^2 = 3,$$

which is the Euler characteristic of  $\mathbb{C}\mathbf{P}^2$ . However, as we pointed out earlier, to show this is really a tessellation of the  $\mathbb{C}\mathbf{P}^2$ , one needs to verify that all the 4-cells are embedded and that there is no intersection among them at any of their interior points. We now confirm this.

For any fixed point  $p = [0, z_1, z_2] \in L_0$  let  $L_{0,p}$  be the projective line joining  $p12$  and  $p$ . Then  $\mathbb{C}\mathbf{P}^2 = \bigcup_{p \in L_0} L_{0,p}$ ; the union is disjoint except that all the  $L_{0,p}$ 's intersect at the single point  $p12$ . It is easy to verify, (i) if  $p \notin S_n$ , then  $L_{0,p}$  intersects  $S_n$  at exactly  $n$  different points in a similar position to those in (4) on  $\mathbb{C}\mathbf{P}^1$ , and (ii) if  $p \in S_n$  then  $p$  is the only intersection of  $L_{0,p}$  and  $S_n$ .

For  $p \notin S_n$ , we triangulate  $L_{0,p}$  similarly to  $\mathbb{CP}^1$ , using the points  $p12$ ,  $p$  (corresponding to the vertices in (5)), and the  $n$  intersections with  $S_n$  (see Figure 7).

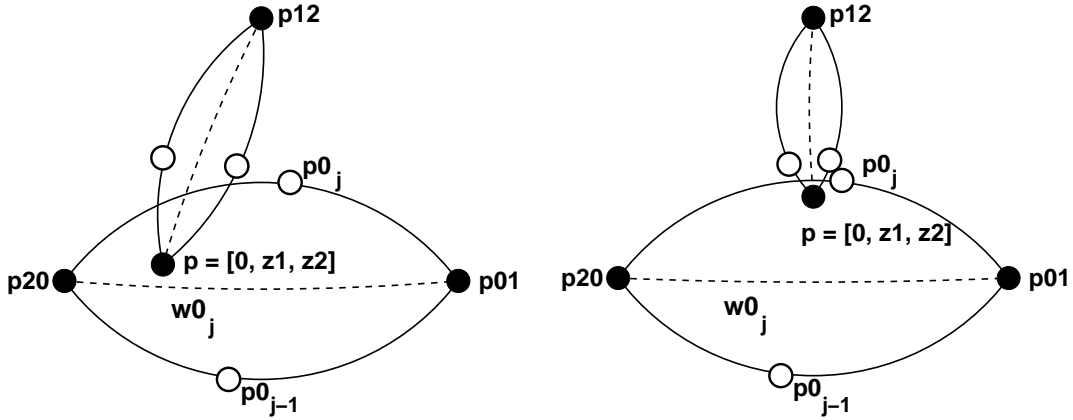


FIGURE 7

Notice that for  $p \in S_n$ , although we do not have the triangulation, there are  $n$  well defined paths from  $p12$  to  $p$  that are obtained as limits of the paths on  $L_{0,q}$ , for  $q$  near  $p \in S_n$  on  $L_0$ , corresponding to the edges joining  $p0$  in (5) and the  $p_k$ 's in (4). Also notice, in particular, that  $L_{0,p01} = L_1$  and  $L_{0,p20} = L_2$ .

Let

$$wl_k = fl_k^{k-1} \cup fl_k^k, \quad l = 0, 1, 2; \quad k = 0, \dots, n-1 \pmod{n}.$$

For given  $j$  and  $k$ , as  $p$  varies on  $w0_j$ , it is easy to see that we get a continuous family of regions  $w1_{k,p}$  in  $L_{0,p}$  with  $w1_{k,p01} = w1_k$ . The union of this family of regions then clearly forms an embedded 4-cell in the  $\mathbb{CP}^2$ , which we will denote by  $W_{j,k,-j-k}$ . It is also clear that  $\mathbb{CP}^2 = \bigcup_{j,k} W_{j,k,-j-k}$  and there is no intersection between different  $W_{j,k,-j-k}$ 's at any of their interior points.

For clarity and later convenience, we write down the following explicit parametrization for  $W_{j,k,-j-k}$ :

$$(20) \quad [\cos s e^{i\beta}, \cos r \sin s, \sin r \sin s e^{i\alpha}]$$

with  $0 \leq r, s \leq \pi/2$ ,  $(2j-1)\pi/n \leq \alpha \leq (2j+1)\pi/n$  and, if we denote  $\arg(-\cos^n r - \sin^n r e^{in\alpha})$  by  $a(r, \alpha)$  with  $0 \leq a(r, \alpha) \leq 2\pi$ , then

$$\frac{a(r, \alpha) + 2(j+k-1)\pi}{n} \leq \beta \leq \frac{a(r, \alpha) + 2(j+k)\pi}{n}.$$

We see in particular that  $w1_{k,p20} = w2_{-j-k}$ . See Figure 8.

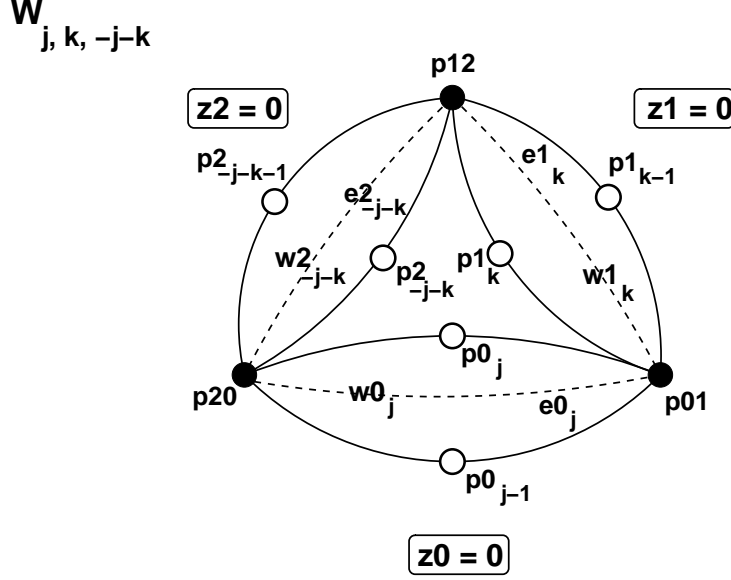


FIGURE 8

From (20), it follows that, in a way similar to the above,  $W_{j,k,-j-k}$  can also be described as a union of  $w_{2_{-j-k},p}$  over  $p \in w_{1_k}$ , or a union of  $w_{0_j,p}$  over  $p \in w_{2_{-j-k}}$ . Therefore the boundary of  $W_{j,k,-j-k}$  is tessellated by twelve 3-cells; each of them is the union of one of the two lower half boundary edges of a  $w$ -region over one of the two triangles in the corresponding base region. It is easy to see that these 3-cells are in fact exactly the following twelve pyramids in (16):

$$(21) \quad \begin{array}{cccc} A12^1_{k-1,-j-k} , & A12^2_{k-1,-j-k} , & A12^1_{k,-j-k-1} , & A12^2_{k,-j-k-1} , \\ A20^1_{-j-k-1,j} , & A20^2_{-j-k-1,j} , & A20^1_{-j-k,j-1} , & A20^2_{-j-k,j-1} , \\ A01^1_{j-1,k} , & A01^2_{j-1,k} , & A01^1_{j,k-1} , & A01^2_{j,k-1} . \end{array}$$

It is also easy to see there are six 3-cells contained inside  $W_{j,k,-j-k}$ ; each of them is the union of one of the two triangles in a  $w$ -region over the middle edge of the corresponding base region. These 3-cells are the following six pyramids in (17):

$$(22) \quad \begin{array}{cc} B12^{10}_{k-1,-j-k} , & B12^{01}_{k,-j-k-1} \\ B20^{10}_{-j-k-1,j} , & B20^{01}_{-j-k,j-1} , \\ B01^{10}_{j-1,k} , & B01^{01}_{j,k-1} . \end{array}$$

These 3-cells divide  $W_{j,k,-j-k}$  into six 4-cells; they are the following six in (19):

$$(23) \quad \begin{aligned} & \{ A01_{j,k-1}^1, \quad A12_{k-1,-j-k}^1, \quad B01_{j,k-1}^{01}, \quad B12_{k-1,-j-k}^{10} \}, \\ & \{ A12_{k-1,-j-k}^2, \quad A20_{-j-k,j-1}^2, \quad B12_{k-1,-j-k}^{10}, \quad B20_{-j-k,j-1}^{01} \}, \\ & \{ A20_{-j-k,j-1}^1, \quad A01_{j-1,k}^1, \quad B20_{-j-k,j-1}^{01}, \quad B01_{j-1,k}^{10} \}, \\ & \{ A01_{j-1,k}^2, \quad A12_{k,-j-k-1}^2, \quad B01_{j-1,k}^{10}, \quad B12_{k,-j-k-1}^{01} \}, \\ & \{ A12_{k,-j-k-1}^1, \quad A20_{-j-k-1,j}^1, \quad B12_{k,-j-k-1}^{01}, \quad B20_{-j-k-1,j}^{10} \}, \\ & \{ A20_{-j-k-1,j}^2, \quad A01_{j,k-1}^2, \quad B20_{-j-k-1,j}^{10}, \quad B01_{j,k-1}^{01} \}. \end{aligned}$$

The structure of (21), (22) and (23) together can be illustrated by the diagram in Figure 9.

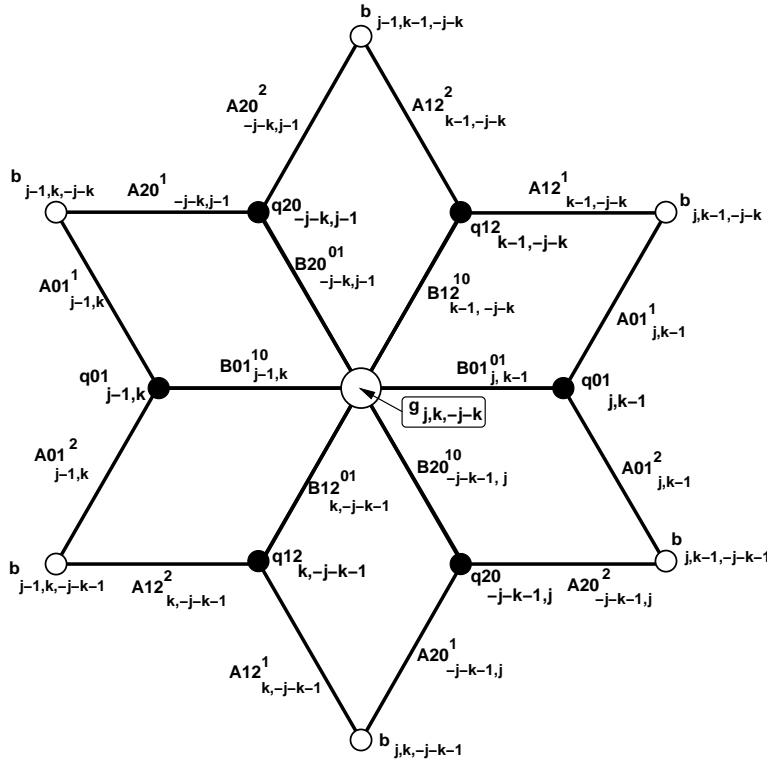


FIGURE 9

Now it is clear that the  $\mathbb{C}P^2$  is well tessellated.

**Remark.** If all we need is a tessellation of the  $\mathbb{C}P^2$  (and hence  $F_n$ ), then the triangles in (14) and the 3-cells in (17) are not needed. The pyramids in (16) are paired into  $3n^2$  octahedrons, and the 4-cells of the tessellation are precisely the  $n^2$   $W_{j,k,-j-k}$ 's. However, when this tessellation is lifted to  $F_n$ , it is not  $\Gamma_n$ -invariant.

3. TESSELLATION OF  $F_n$ 

Through the  $n$ -fold regular branched covering (2), the tessellation of  $\mathbb{CP}^2$  in §2 now lifts to a well defined tessellation for the  $F_n$ . The numbers of vertices, edges, 2-cells, 3-cells and 4-cells are as indicated in the introduction. In this section, we examine this tessellation more closely and show that it is  $\Gamma_n$ -invariant. Recall that  $\Gamma_n$  is the group of the isomorphisms of  $F_n$  induced from permuting and/or phase multiplying the homogeneous coordinates of  $\mathbb{CP}^3$ .

The intersection of  $F_n$  with each of the projective planes  $z_k = 0$ ,  $k = 0, 1, 2, 3$ , is the  $S_n$  in that plane, triangulated as described in §1. The tessellation of the  $F_n$  is an extension of the triangulations on these four  $S_n$ 's. In fact, the four  $S_n$ 's contain all the vertices, edges, and triangles lifted from those in (8) and (12). The other  $4n^3$  triangles are lifted from (13) and (14) and are characterized by the fact that for each of them, the three edges lie on three distinct  $S_n$ 's. Notice then that for any three  $S_n$ 's of the four, any three different pairwise intersections are vertices of a unique triangle lifted from (13) or (14).

The formation of the quadrilaterals can be described as follows. Start with any edge on one of the four  $S_n$ 's, say, the one in the projective plane  $z_0 = 0$ ; its two end points, denoted by  $q_1$  and  $q_2$ , must then also lie in two other distinct projective planes, say,  $z_1 = 0$  and  $z_2 = 0$ , respectively. Then there are  $n$  distinct edges on  $z_2 = 0$  joining  $q_2$  and the  $n$  distinct intersections of the projective planes  $z_2 = 0$  and  $z_3 = 0$ . Any one of these edges plus  $\overline{q_1 q_2}$ , the edge we started with, form two adjacent sides of a unique quadrilateral. Hence one sees that there are in all  $3n^3$  quadrilaterals.

For two opposite sides, say, lying in the projective planes  $z_0 = 0$  and  $z_3 = 0$ , respectively, of a given quadrilateral, as in the example above, there are exactly two vertices,  $v_1, v_2$ , in the intersection of  $z_0 = 0$  and  $z_3 = 0$  that are the opposite vertices of the given edges in triangles lying in  $z_0 = 0$  and  $z_3 = 0$ , respectively (see Figure 10). Each of these two vertices forms a pyramid with the quadrilateral. Notice that if one starts with the other pair of opposite sides of the quadrilateral, the two vertices will be different. One sees that there are in all  $12n^3$  pyramids.

Finally, every 4-cell is bounded by four pyramids, and each pyramid is shared by two 4-cells, thus there are  $6n^3$  4-cells.

From the description above, one can see that if, instead of (2) which is induced from the projection  $[z_0, z_1, z_2, z_3] \mapsto [z_0, z_1, z_2]$ , we use the branched covering induced from, say,  $[z_0, z_1, z_2, z_3] \mapsto [z_1, z_2, z_3]$ , the lifted tessellation will be the same. This, combined with the invariance for  $S_n$  demonstrated in

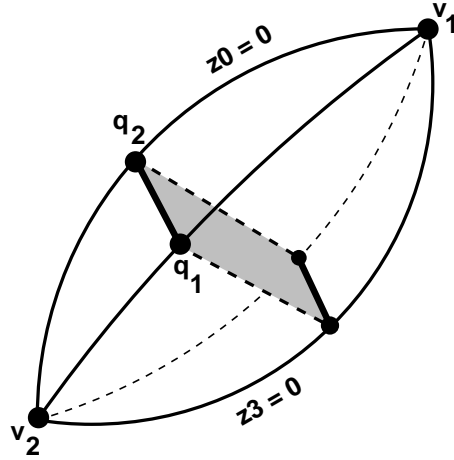


FIGURE 10

§1, shows that the tessellation of the  $F_n$  is  $\Gamma_n$ -invariant and the action of  $\Gamma_n$  is transitive on the set of 4-cells.

The list of all the vertices in the tessellation is:

$$\begin{aligned}
 p01_k &:= [0, 0, 1, e^{i(\pi+2k\pi)/n}], & p02_k &:= [0, 1, 0, e^{i(\pi+2k\pi)/n}], \\
 p03_k &:= [0, 1, e^{i(\pi+2k\pi)/n}, 0], & p12_k &:= [1, 0, 0, e^{i(\pi+2k\pi)/n}], \\
 p13_k &:= [1, 0, e^{i(\pi+2k\pi)/n}, 0], & p23_k &:= [1, e^{i(\pi+2k\pi)/n}, 0, 0],
 \end{aligned}$$

for  $k = 0, \dots, n - 1$ .

One can then list the edges, 2-cells, 3-cells, and 4-cells in terms of the vertices. We now write down a few lists of edges and 2-cells for illustration (see Figure 11).

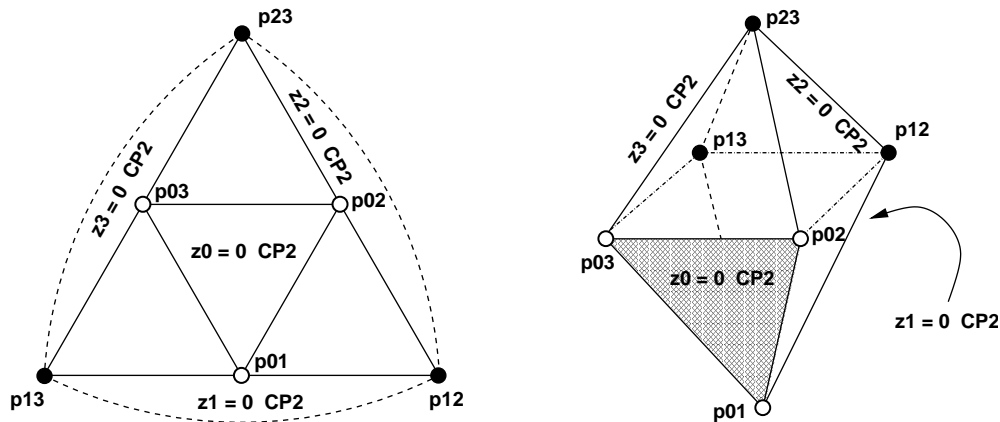


FIGURE 11

Every edge lies on one of the four  $\mathbb{C}\mathbf{P}^2$ 's defined by  $z_k = 0$ . For example, the  $3n^2$  edges lying on  $z_0 = 0$  are:

$$\{p01_i, p02_j\}, \quad \{p01_i, p03_j\}, \quad \{p02_i, p03_j\},$$

$i, j = 0, \dots, n-1$ .

The 2-cells are divided into three groups: triangles each of which lies on one of the four  $\mathbb{C}\mathbf{P}^2$ 's defined by  $z_k = 0$ ; triangles each of which has three sides on three different  $\mathbb{C}\mathbf{P}^2$ 's; rectangles.

For example, the  $2n^2$  triangles lying on  $z_0 = 0$  are:

$$\{p01_i, p02_j, p03_k\}, \quad i - j + k = 0 \text{ or } -1 \pmod{n}.$$

The  $n^3$  triangles whose edges lie on three different  $\mathbb{C}\mathbf{P}^2$ 's,  $z_1 = 0$ ,  $z_2 = 0$ ,  $z_3 = 0$ , respectively, are:

$$\{p12_i, p13_j, p23_k\}, \quad i, j, k = 0, \dots, n-1.$$

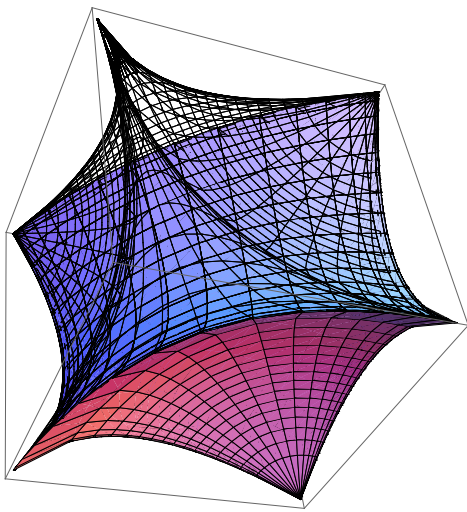
The rectangles can be divided into three groups: each of them has one edge lying on the  $\mathbb{C}\mathbf{P}^2$  labeled by  $z_0 = 0$  and an opposite edge on  $z_m = 0$ ,  $m = 1, 2, 3$ . The group with  $m = 1$ , for example, contains the following  $n^3$  rectangles:

$$\{p02_i, p03_j, p13_k, p12_l\}, \quad i - j + k - l = 0 \pmod{n}.$$

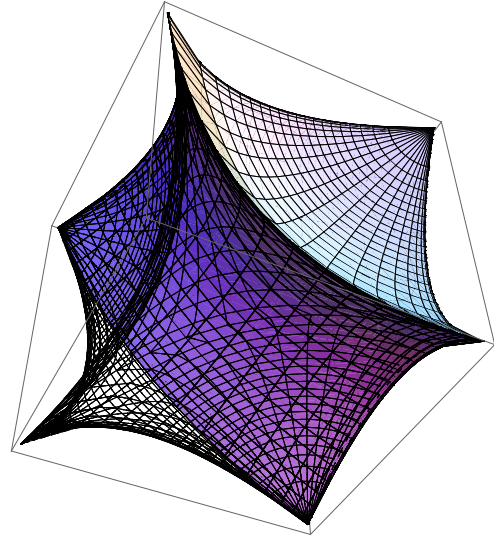
Figure 12 is an actual image of a generic 4-cell using an explicit embedding of  $\mathbb{C}\mathbf{P}^3$  into  $\mathbb{R}^{16}$  (see, e.g., [Hanson & Sha, 2006]). (a) and (b) depict the 3-balls that are the upper and lower hemispheres of the  $S^3$  bounding the 4-cell. Note the distinct rectangles, which cut across the middle of the two 3-balls, dividing each into two pyramids; one pyramid in each 3-ball has been made transparent using wire-frame rendering to make the rectangle visible. (c) shows a complete partially transparent shaded rendering of the entire embedded 4-cell projected to 3D, with the outer octahedron being essentially the equator  $S^2$  that is shared by the two hemispheres (a) and (b) of the  $S^3$ .

Finally, we can use computer algebra tools to produce a representative from each equivalence class of the group  $\Gamma_n$  modulo the isotropy group to generate the explicit forms of all distinct  $12n^3$  3-cells and all distinct  $6n^3$  4-cells. Figure 13 represents the K3 surface ( $F_4$ ) as a 3D projection from its embedding in  $\mathbb{R}^{16}$  in terms of all the edges bounding the  $6n^3 = 384$  4-cell equators corresponding to Figure 12(c). The function of this figure is mainly to illustrate qualitatively how to use the action of  $\Gamma_n$  to produce the full manifold; more sophisticated interactive visualization tools are required to expose and explain the structure, e.g., by interactively selecting and reprojecting subsets of the tessellation.

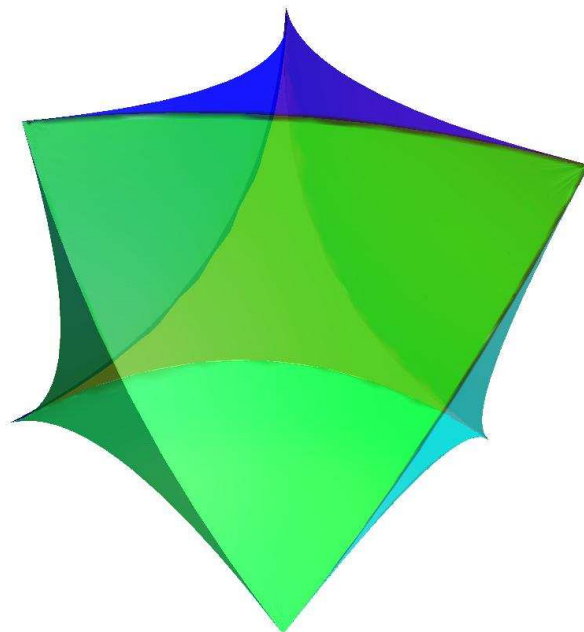




(a)



(b)



(c)

FIGURE 12

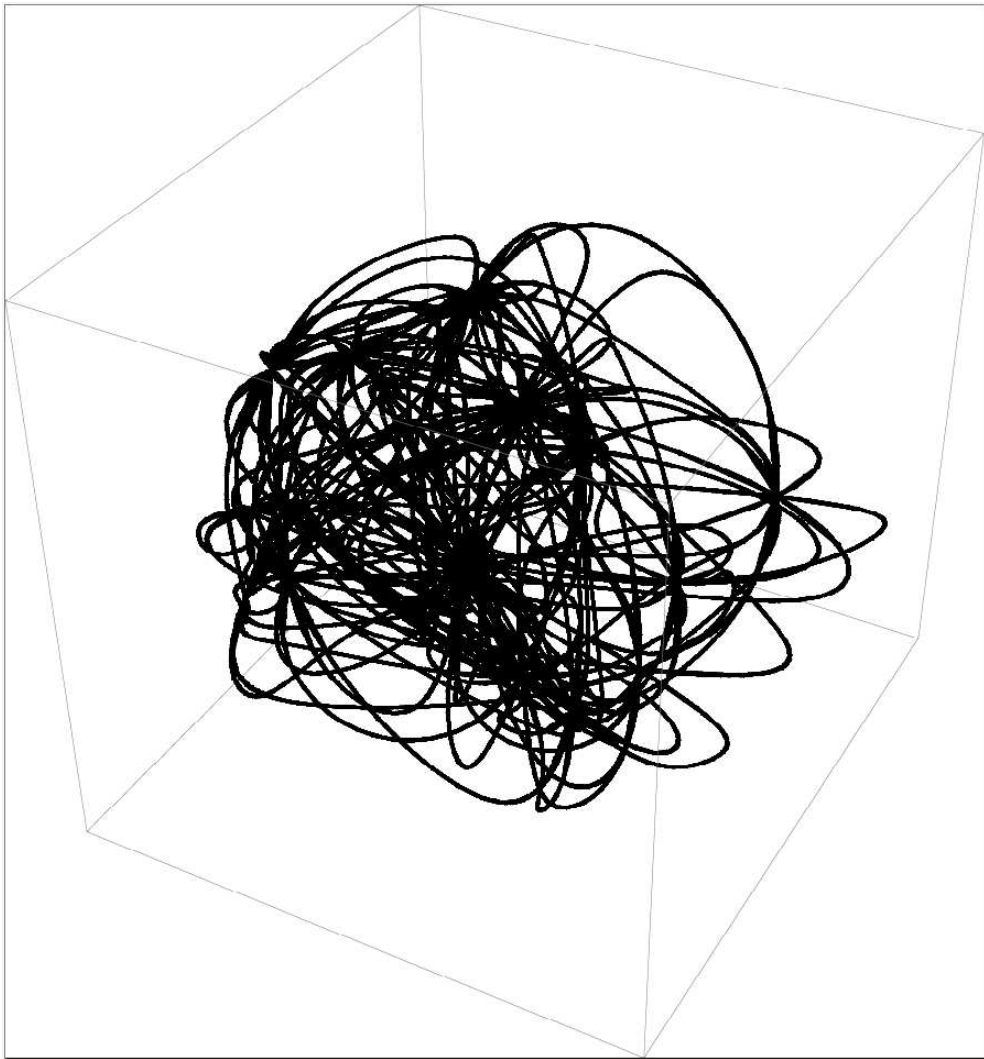


FIGURE 13

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