Visualizing Quaternions

Course Notes for SIGGRAPH 2005

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Abstract

This intermediate-level tutorial provides a comprehensive approach to the visualization of quaternions and their relationships to computer graphics and scientific visualization. The introduction focuses on a selection of everyday phenomena involving rotating objects whose explanation for an audience that is technically trained but not pure mathematicians is essentially impossible without a quaternion visualization. The course will then pursue selected examples of quaternion-based visualization methods to help explain the behavior of *quaternion manifolds*: quaternion representations of orientation frames attached to curves, surfaces, and volumes.

Presenter's Biography

Andrew J. Hanson is a professor of computer science at Indiana University, and has regularly taught courses in computer graphics, computer vision, and scientific visualization. He received a BA in chemistry and physics from Harvard College in 1966 and a PhD in theoretical physics from MIT in 1971. Before coming to Indiana University, he did research in theoretical physics at the Institute for Advanced Study, Stanford, and Berkeley, and then in computer vision at the SRI Artificial Intelligence Center in Menlo Park, CA. He has published in IEEE Computer, CG&A, TVCG, ACM Computing Surveys, and has over a dozen papers in the IEEE Visualization Proceedings. He has also contributed three articles to the Graphics Gems series dealing with user interfaces for rotations and with techniques of N-dimensional geometry. Previous experience with conference tutorials includes a Siggraph '98 tutorial on N-dimensional graphics, a Visualization '98 course on Clifford Algebras and Quaternions, and tutorials on Visualizing Quaternions presented at Siggraph '99, Siggraph 2000, and again at Siggraph 2001 in tandem with a course on Visualizing Relativity for a graphics audience. Major research interests include scientific visualization, machine vision, computer graphics, perception, and the design of interactive user interfaces for virtual reality and visualization applications. Particular visualization applications currently being studied include an astrophysical treatment of the local galactic neighborhood of the sun, the exploitation of constrained navigation for visualization environments. Mathematical visualization interests include the depiction of Calabi-Yau spaces and the general problems of graphics and visualization in dimensions greater than three and their applications to mathematics and theoretical physics.

Contents

General Information on the Tutorial

Course Syllabus

Summary: This tutorial will deal with visualizable representations of quaternions, their features, technology, folklore, and applications. The introduction will focus on visually understanding quaternions themselves by exploiting parallels to complex variables and 2D rotations. Starting from this basis, the tutorial will proceed to give visualizations of advanced quaternion applications.

Prerequisites: Participants should be comfortable with and have an appreciation for conventional mathematical methods of 3D computer graphics and geometry used in geometric transformations and polygon rendering. The material will be of most interest to those wishing to deepen their intuitive understanding of moving coordinate frames and quaternion-based animation techniques.

Objectives: Participants will learn the basic facts relating quaternions to ordinary 3D rotations, as well as methods for examining the properties of quaternion constructions using interactive visualization methods. A variety of applications, including the use of quaternions to smaples coordinate frames for curves, surfaces, and volumes, will be explored.

Outline: This tutorial will last approximately one and a half hours plus time for questions and discussion. The material will be arranged as follows:

I. *(45 min)* **Twisting Belts, Rolling Balls, and Locking Gimbals:** *Explaining Rotation Sequences with Quaternions*

   Sequences of orientations are manifestly evident in our everyday lives. While we can immediately observe strange things that happen when we twist a leather belt, roll a baseball, or push on a gyroscope, if you ask "why" and expect a real explanation, most of us hit a dead end. Quaternion visualization provides satisfying answers to such questions.

II. *(45 min)* **Quaternion Fields:** *Curves, Surfaces, and Volumes*

   Once we have mastered the visualization of quaternion paths, we have the tools to take a fresh look at many problems in graphics and visualization. The quaternion *field* is a continuous map from a set of orientation frames such as framed curves, surfaces, and volumes into the corresponding quaternions. We examine a family of examples showing how quaternion curves, surfaces, and volumes can solve new problems and reveal new properties.

1 Overview

Practitioners of computer graphics and animation frequently represent 3D rotations using the quaternion formalism, a mathematical tool that originated with William Rowan Hamilton in the 19th century, and is now an essential part of modern analysis, group theory, differential geometry, and even quantum physics. Quaternions are in many ways very simple, and yet there are enormous subtleties to address in the process of fully understanding and exploiting their properties. The purpose of this Tutorial is to construct an intuitive bridge between our intuitions about 2D and 3D rotations and the quaternion representation.

The Tutorial will begin with an introduction to various natural phenomena that can be understood using quaternions. Rotations in 2D, which will be found to have surprising richness, will lead the way to the construction of the relation between 3D rotations and quaternions. Quaternion visualization methods of various sorts will be introduced, followed by applications of the quaternion frame representation to problems of interest by graphicists and visualization scientists. An extensive bibliography of related literature is included, as well as several relevant reprints and technical reports, a Mathematica implementation of the Quaternion Frenet Equations, and a basic GLUT quaternion visualization application.

2 Twisting Belts, Rolling Balls, and Locking Gimbals

We will begin with a basic introduction to the ways in which sequences of rotations enter our lives in surprising ways. We will then proceed to look at a variety of methods for understanding quaternions and making meaningful pictures of constructs involving them. These methods will range from some of the concepts pointed out by Hart, Francis, and Kauffman [54] to theoretical methods given in [47, 48, 40, 51].

Traditional treatments of quaternions range from the original works of Hamilton and Tait [35, 85] to a variety of recent studies such as those of Altmann, Pletincks, Juttler, and Kuipers [2, 73, 63, 67].

In our pedagogical treatment, we will focus on the use of 2D rotations as a rich but algebraically simple proving ground in which we can see many of the key features of quaternion geometry in a very manageable context. The relationship between 3D rotations and quaternions is then introduced as a natural extension of the 2D systems. Quaternion visualization itself utilizes a basic trick: since a four-vector quaternion $q = (q_0, \mathbf{q})$ obeying $q \cdot q = 1$, then the four-vector lies on the three-sphere S^3 and has only three independent components: if we display just $\mathbf{q}$, we can in principle *infer* the value of $q_0 = \sqrt{1 - \mathbf{q} \cdot \mathbf{q}}$.

3 Quaternion Fields

After the conceptual introduction, we proceed to study the nature of quaternions as representations of frames in 3D. The now-traditional application of quaternion animation splines was introduced to the graphics community originally by Shoemake [77]. Our visualizations of these and other applications exploit the fact that quaternions are points on the three-sphere embedded in 4D; the three-sphere (S^3) is analogous to an ordinary ball or two-sphere (S^2) embedded in 3D, except that the three-sphere is a solid object instead of a surface. To manipulate, display, and visualize rotations in 3D, we may convert 3D rotations to 4D quaternion points and treat the entire problem in the framework of 4D geometry.

We pursue three main applications, which involve the identification of quaternion frames with sampled curves, surfaces, and volumes. The curve methods follow closely techniques introduced in Hanson and Ma [47, 48] for representing families of coordinate frames on curves in 3D as curves in the 4D quaternion space. The extension to surfaces and the corresponding induced surfaces in quaternion space follow the treatment by Hanson [40, 51], and volumetric quaternions are studied using the methods of Herda, et al. [56, 57, 55].

4 Demonstration Software

We provide an elementary OpenGL-based interactive quaternion visualization application, *QuatRot*, that should be essentially system-independent and run on any platform. In addition, we supply our own version, `quatutils.nb`, of some basic Mathematica routines for quaternions (which serve as the basis for a number of the illustrations in the notes), as well as a Mathematica notebook `qfrmint.nb` that explicitly implements a numerical integration of the Frenet frame equations in quaternion form, vastly improving the exactly equivalent calculation for the standard Frenet equations implemented by Gray [33].

Acknowledgments

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Republished in the Course Notes are two key papers from IEEE Transactions on Visualization and Computer Graphics [48], and from the Proceedings of IEEE Visualization [40]; we thank the IEEE Computer Society Press for permitting us to include this material.

Finally we would like to thank the National Science Foundation for their support: the research incorporated in portions of this work was supported in part by NSF grant CCR-0204112.

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Siggraph 2005 Tutorial

OUTLINE

I: (45min) **Twisting Belts, Rolling Balls, and Locking Gimbals:**

Explaining Rotation Sequences with Quaternions

II: (45 min) **Quaternion Fields:**

Curves, Surfaces, and Volumes

Part I

Twisting Belts, Rolling Balls, and Locking Gimbals

Explaining Rotation Sequences with Quaternions

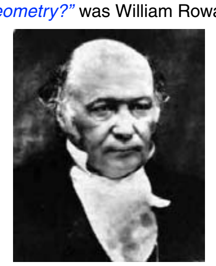
Where Did Quaternions Come From?

. . . from the discovery of *Complex Numbers:*

- $\boxed{z = x + iy}$ Complex numbers = realization that $z^2 + 1 = 0$ cannot be solved unless you have an “imaginary” number with $i^2 = -1$.
- **Euler’s formula**: $e^{i\theta} = \cos\theta + i \sin\theta$ allows you to do most of 2D geometry.

Hamilton

The first to ask *“If you can do 2D geometry with complex numbers, how might you do 3D geometry?”* was William Rowan Hamilton, circa 1840.



Sir William Rowan Hamilton
4 August 1805 — 2 September 1865

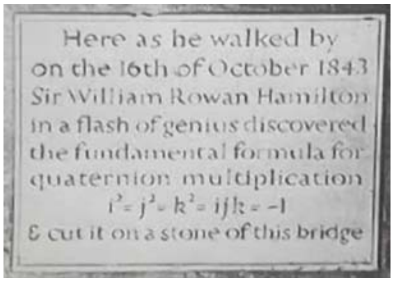
Hamilton’s epiphany: 16 October 1843

“An electric circuit seemed to close; and a spark flashed forth . . . Nor could I resist the impulse – unphilosophical as it may have been – to cut with a knife on a stone of Brougham Bridge, as we passed it, the fundamental formula with the symbols, i, j, k; namely,

$$i^2 = j^2 = k^2 = ijk = -1$$

which contains the Solution of the Problem...”

...at the site of Hamilton's carving


Here as he walked by
on the 16th of October 1843
Sir William Rowan Hamilton
in a flash of genius discovered
the fundamental formula for
quaternion multiplication
$i^2 = j^2 = k^2 = ijk = -1$
& cut it on a stone of this bridge

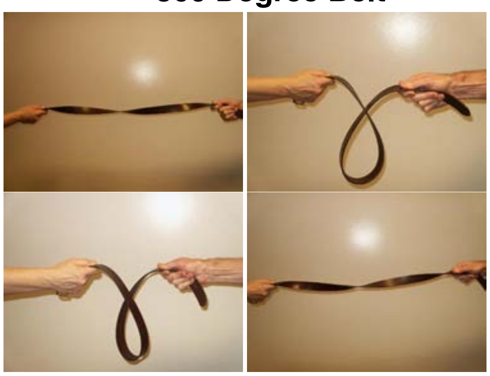
The plaque on Broome Bridge in Dublin, Ireland, commemorating the legendary location where Hamilton conceived of the idea of quaternions. (Hamilton apparently misspelled it as "Brougham Bridge" in his letter.)

The Belt Trick

Quaternion Geometry in our daily lives

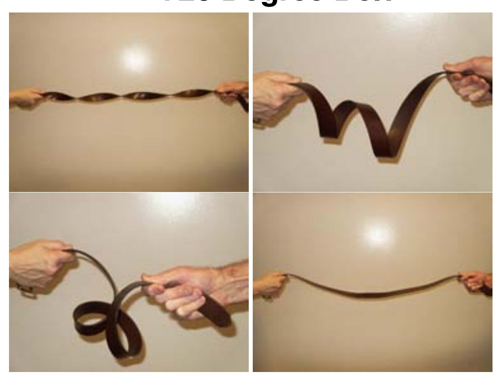
- Two people hold ends of a belt.
- Twist the belt either 360 degrees or 720 degrees.
- **Rule:** *Move belt ends any way you like but do not change orientation of either end.*
- Try to straighten out the belt.

360 Degree Belt



360 twist: stays twisted, can change DIRECTION!

720 Degree Belt

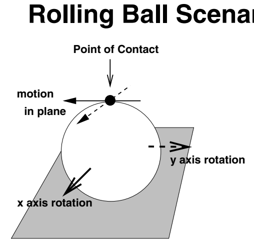


720 twist: CAN FLATTEN OUT WHOLE BELT!

Rolling Ball Puzzle

1. Put a ball on a flat table.
2. Place hand flat on top of the ball
3. Make circular rubbing motion, as though polishing the tabletop.
4. Watch a point on the equator of the ball.
5. *small clockwise circles* → **equator goes counterclockwise**
6. *small counterclockwise circles* → **equator goes clockwise**

Rolling Ball Scenario



Gimbal Lock

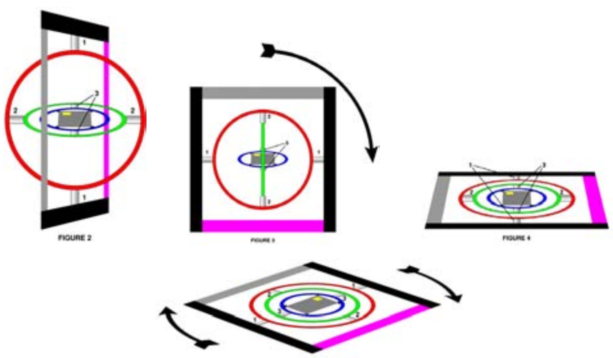
Gimbal Lock occurs when a mechanical or computer system experiences an anomaly due to an (x, y, z)-based orientation control sequence.

- *Mechanical systems cannot avoid all possible gimbal lock situations .*

- *Computer orientation interpolation systems can avoid gimbal-lock-related glitches* **by using quaternion interpolation.**

Gimbal Lock — Apollo Systems

Red-painted area = Danger of real Gimbal Lock



Mechanical Gimbal Lock: Using x, y, z axes to encode orientation gives singular situations.

2D Rotations

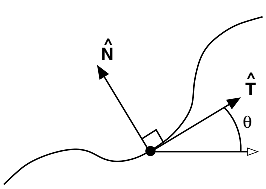
- 2D rotations $\leftrightarrow$ *complex numbers.*
- Why? $e^{i\theta}\,(x + iy) = (x' + iy')$

$$\begin{aligned} x' &= x \cos\theta - y \sin\theta \\ y' &= x \sin\theta + y \cos\theta \end{aligned}$$

- **Complex numbers** are a subspace of quaternions — so exploit 2D rotations to **introduce us to quaternions** and their geometric meaning.

Frames in 2D

The tangent and normal to 2D curve move continuously along the curve:



Frames in 2D

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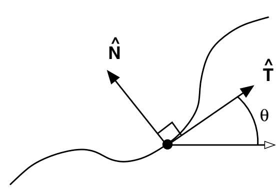

Frames in 2D

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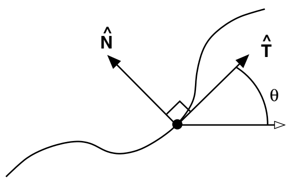


Frame Matrix in 2D

This motion is described at each point (or time) by the matrix:

$$
\begin{aligned}
R_2(\theta) &= \left[\, \hat{\mathbf{T}} \;\; \hat{\mathbf{N}} \,\right] \\
&= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} .
\end{aligned}
$$

The Belt Trick Says:

There is a Problem...at least in 3D

How do you get $\cos\theta$ to know about 720 degrees?

The Belt Trick Says:

There is a Problem...at least in 3D

How do you get $\cos\theta$ to know about 720 degrees?

Hmmmmm. $\cos(\theta/2)$ knows about 720 degrees, right?

Half-Angle Transform:

A Fix for the Problem?

Let $a = \cos(\theta/2)$, $b = \sin(\theta/2)$,
(i.e., $\cos\theta = a^2 - b^2$, $\sin\theta = 2ab$),
and parameterize 2D rotations as:

$$
R_2(a,b) \;=\; \begin{bmatrix} a^2 - b^2 & -2ab \\ 2ab & a^2 - b^2 \end{bmatrix} .
$$

where orthonormality implies

$$
(a^2 + b^2)^2 = 1
$$

which reduces back to $a^2 + b^2 = 1$.

Frame Evolution in 2D

Examine time-evolution of 2D frame (on our way to 3D): First in $\theta(t)$ coordinates:

$$
\left[\, \hat{\mathbf{T}} \;\; \hat{\mathbf{N}} \,\right] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} .
$$

Differentiate to find frame equations:

$$
\begin{aligned}
\dot{\hat{\mathbf{T}}}(t) &= +\kappa \hat{\mathbf{N}} \\
\dot{\hat{\mathbf{N}}}(t) &= -\kappa \hat{\mathbf{T}} ,
\end{aligned}
$$

where $\kappa(t) = d\theta/dt$ is the **curvature**.

Frame Evolution in (a, b):

The basis $(\hat{\mathbf{T}}, \hat{\mathbf{N}})$ is nasty — **Four equations** with **Three constraints** from orthonormality, but just **One** true degree of freedom.

Major Simplification occurs in (a, b) coordinates!!

$$\dot{\hat{\mathbf{T}}} = 2 \begin{bmatrix} a\dot{a} - b\dot{b} \\ a\dot{b} + b\dot{a} \end{bmatrix} = 2 \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} \dot{a} \\ \dot{b} \end{bmatrix}$$

Frame Evolution in (a, b):

But this formula for $\dot{\hat{\mathbf{T}}}$ is just $\kappa\hat{\mathbf{N}}$, where

$$\kappa\hat{\mathbf{N}} = \kappa \begin{bmatrix} -2ab \\ a^2 - b^2 \end{bmatrix} = \kappa \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} -b \\ a \end{bmatrix}$$

or

$$\kappa\hat{\mathbf{N}} = \kappa \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

2D Quaternion Frames!

Rearranging terms, *both* $\dot{\hat{\mathbf{T}}}$ and $\dot{\hat{\mathbf{N}}}$ eqns reduce to

$$\begin{bmatrix} \dot{a} \\ \dot{b} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & -\kappa \\ +\kappa & 0 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix}$$

$\boxed{\text{This is the square root of frame equations.}}$

2D Quaternions . . .

So *one equation* in the two *"quaternion"* variables (a, b) with the constraint $a^2 + b^2 = 1$ contains *both* the frame equations

$$\dot{\hat{\mathbf{T}}} = +\kappa\hat{\mathbf{N}}$$

$$\dot{\hat{\mathbf{N}}} = -\kappa\hat{\mathbf{T}}$$

$\Rightarrow$ this is much better for computer implementation, etc.

Rotation as Complex Multiplication

If we let $(a + ib) = \exp(i\,\theta/2)$ we see that rotation is complex multiplication!

"Quaternion Frames" in 2D are just complex numbers, with

$\boxed{\text{Evolution Eqns = derivative of } \exp(i\,\theta/2)!}$

Rotation with no matrices!

Due to an extremely deep reason in Clifford Algebras,

$$a + ib = e^{i\theta/2}$$

represents rotations "more nicely" than the matrices $R(\theta)$.

$$(a' + ib')(a + ib) = e^{i(\theta' + \theta)/2} = A + iB$$

where if we *want* the matrix, we write:

$$R(\theta')R(\theta) = R(\theta' + \theta) = \begin{bmatrix} A^2 - B^2 & -2AB \\ 2AB & A^2 - B^2 \end{bmatrix}$$

The Algebra of 2D Rotations

The algebra corresponding to 2D rotations is easy: just complex multiplication!!

$$\begin{aligned}(a', b') * (a, b) &\cong (a' + ib')(a + ib)\\ &= a'a - b'b + i(a'b + ab')\\ &\cong (a'a - b'b,\, a'b + ab')\\ &= (A,\, B)\end{aligned}$$

2D Rotations are just **complex multiplication**, and take you around the unit circle!

Quaternion Frames

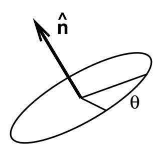
In 3D, repeat our trick: take square root of the frame, but now use *quaternions*:

- **Write down the 3D frame.**
- **Write as double-valued quadratic form.**
- **Rewrite frame evolution equations linearly in the new variables.**

The Geometry of 3D Rotations

We begin with a basic fact:

Euler theorem: *every* 3D frame can be written as a spinning by θ about a fixed axis $\hat{\mathbf{n}}$, the eigenvector of the rotation matrix:



Quaternion Frames ...

The Matrix $R_3(\theta, \hat{\mathbf{n}})$ giving 3D rotation by θ about axis $\hat{\mathbf{n}}$ is :

$$\begin{bmatrix} c + (n_1)^2(1-c) & n_1 n_2(1-c) - s n_3 & n_3 n_1(1-c) + s n_2 \\ n_1 n_2(1-c) + s n_3 & c + (n_2)^2(1-c) & n_3 n_2(1-c) - s n_1 \\ n_1 n_3(1-c) - s n_2 & n_2 n_3(1-c) + s n_1 & c + (n_3)^2(1-c) \end{bmatrix}$$

where $c = \cos\theta$, $s = \sin\theta$, and $\hat{\mathbf{n}} \cdot \hat{\mathbf{n}} = 1$.

Can we find a 720-degree form?

Remember 2D: $a^2 + b^2 = 1$
then substitute $1 - c = (a^2 + b^2) - (a^2 - b^2) = 2b^2$
to find the remarkable expression for $\mathbf{R}(\theta, \hat{\mathbf{n}})$:

$$\begin{bmatrix} a^2 - b^2 + (n_1)^2(2b^2) & 2b^2 n_1 n_2 - 2ab n_3 & 2b^2 n_3 n_1 + 2ab n_2 \\ 2b^2 n_1 n_2 + 2ab n_3 & a^2 - b^2 + (n_2)^2(2b^2) & 2b^2 n_2 n_3 - 2ab n_1 \\ 2b^2 n_3 n_1 - 2ab n_2 & 2b^2 n_2 n_3 + 2ab n_1 & a^2 - b^2 + (n_3)^2(2b^2) \end{bmatrix}$$

Rotations and Quadratic Polynomials

Remember $(n_1)^2 + (n_2)^2 + (n_3)^2 = 1$ and $a^2 + b^2 = 1$; letting

$$q_0 = a = \cos(\theta/2) \qquad \mathbf{q} = b\hat{\mathbf{n}} = \hat{\mathbf{n}}\sin(\theta/2)$$

We find a matrix $R_3(q)$

$$\begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2q_1 q_2 - 2q_0 q_3 & 2q_1 q_3 + 2q_0 q_2 \\ 2q_1 q_2 + 2q_0 q_3 & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2q_2 q_3 - 2q_0 q_1 \\ 2q_1 q_3 - 2q_0 q_2 & 2q_2 q_3 + 2q_0 q_1 & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix}$$

Quaternions and Rotations . . .

HOW does $q = (q_0, \mathbf{q})$ represent rotations?

LOOK at

$$R_3(\theta, \hat{\mathbf{n}}) \stackrel{?}{=} R_3(q_0, q_1, q_2, q_3)$$

THEN we can verify that choosing

$$q(\theta, \hat{\mathbf{n}}) = (\cos\frac{\theta}{2},\ \hat{\mathbf{n}}\sin\frac{\theta}{2})$$

makes the R_3 equation an *IDENTITY*.

Quaternions and Rotations . . .

WHAT happens if you do **TWO** rotations?

EXAMINE the action of two rotations

$$R(q')R(q) = R(Q)$$

EXPRESS in quadratic forms in q and LOOK FOR an analog of complex multiplication:

Quaternions and Rotations . . .

RESULT: the following multiplication rule $\boxed{q' * q = Q}$ yields **exactly** the correct 3×3 rotation matrix $R(Q)$:

$$\begin{bmatrix} Q_0 = [q' * q]_0 \\ Q_1 = [q' * q]_1 \\ Q_2 = [q' * q]_2 \\ Q_3 = [q' * q]_3 \end{bmatrix} = \begin{bmatrix} q'_0 q_0 - q'_1 q_1 - q'_2 q_2 - q'_3 q_3 \\ q'_0 q_1 + q'_1 q_0 + q'_2 q_3 - q'_3 q_2 \\ q'_0 q_2 + q'_2 q_0 + q'_3 q_1 - q'_1 q_3 \\ q'_0 q_3 + q'_3 q_0 + q'_1 q_2 - q'_2 q_1 \end{bmatrix}$$

This is Quaternion Multiplication.

Algebra of Quaternions
= 3D Rotations!

2D rotation matrices are represented by **complex multiplication**

3D rotation matrices are represented by **quaternion multiplication**

Algebraic 2D/3D Rotations

Therefore in 3D, the 2D complex multiplication

$$(a', b') * (a, b) = (a'a - b'b,\ a'b + ab')$$

is replaced by 4D quaternion multiplication:

$$\begin{aligned} q' * q \ = \ (&q'_0 q_0 - q'_1 q_1 - q'_2 q_2 - q'_3 q_3, \\ &q'_0 q_1 + q'_1 q_0 + q'_2 q_3 - q'_3 q_2, \\ &q'_0 q_2 + q'_2 q_0 + q'_3 q_1 - q'_1 q_3, \\ &q'_0 q_3 + q'_3 q_0 + q'_1 q_2 - q'_2 q_1) \end{aligned}$$

Algebra of Quaternions . . .

The is easier to remember by dividing it into the *scalar* piece q_0 and the *vector* piece $\vec{\mathbf{q}}$:

$$\begin{aligned} q' * q \ = \ (&q'_0 q_0 - \vec{\mathbf{q}}' \cdot \vec{\mathbf{q}}, \\ &q'_0 \vec{\mathbf{q}} + q_0 \vec{\mathbf{q}}' + \vec{\mathbf{q}}' \times \vec{\mathbf{q}}) \end{aligned}$$

Now we can SEE quaternions!

Since $(q_0)^2 + \mathrm{q} \cdot \mathrm{q} = 1$ then

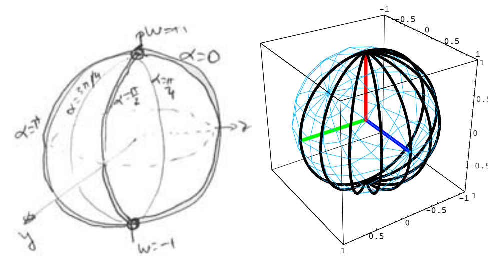
$$q_0 = \sqrt{1 - \mathrm{q} \cdot \mathrm{q}}$$

$\boxed{\textbf{Plot just the 3D vector: } \mathrm{q} = (q_x,\ q_y,\ q_z)}$

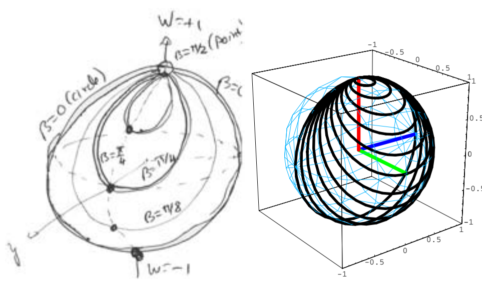
q_0 is KNOWN! Can also use any other triple: the fourth component is dependent.

DEMO

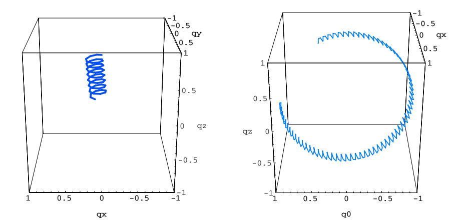
360° Belt Trick in Quaternion Form



720° Belt Trick in Quaternion Form

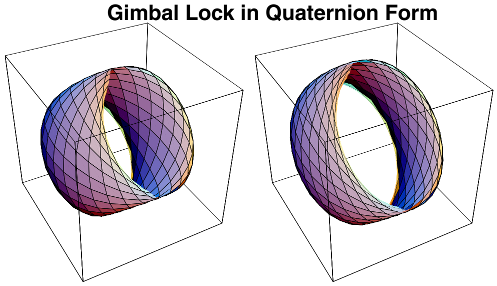


Rolling Ball in Quaternion Form



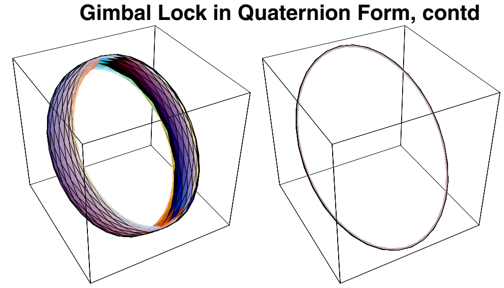
q vector-only plot. (q_0, q_x, q_z) plot

Gimbal Lock in Quaternion Form



Quaternion Plot of the *remaining* orientation degrees of freedom of $\mathbf{R}(\theta, \hat{\mathbf{x}}) \cdot \mathbf{R}(\phi, \hat{\mathbf{y}}) \cdot \mathbf{R}(\psi, \hat{\mathbf{z}})$ at $\phi = 0$ and $\phi = \pi/6$

Gimbal Lock in Quaternion Form, contd



Choosing ϕ and plotting the *remaining* orientation degrees in the rotation sequence $\mathbf{R}(\theta, \hat{\mathbf{x}}) \cdot \mathbf{R}(\phi, \hat{\mathbf{y}}) \cdot \mathbf{R}(\psi, \hat{\mathbf{z}})$, we see degrees of freedom **decrease from TWO to ONE as** $\phi \to \pi/2$

Quaternion Interpolations

- Shoemake (Siggraph '85) proposed using quaternions instead of Euler angles to get smooth frame interpolations without **Gimbal Lock**:

  BEST CHOICE: Animate objects and cameras using rotations represented on S^3 by quaternions

Interpolating on Spheres

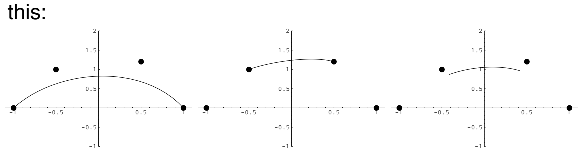
General quaternion spherical interpolation employs the "SLERP," a constant angular velocity transition between two directions, $\hat{q}_1$ and $\hat{q}_2$:

$$\begin{aligned} \hat{q}_{12}(t) &= \text{Slerp}(\hat{q}_1, \hat{q}_2, t) \\ &= \hat{q}_1 \frac{\sin((1-t)\theta)}{\sin(\theta)} + \hat{q}_2 \frac{\sin(t\theta)}{\sin(\theta)} \end{aligned}$$

where $\cos\theta = \hat{q}_1 \cdot \hat{q}_2$.

Plane Interpolations

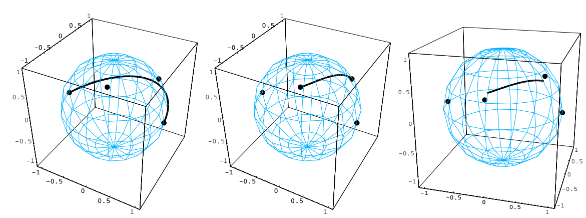
In Euclidean space, these three basic cubic splines look like this:



Bezier Catmull-Rom Uniform B

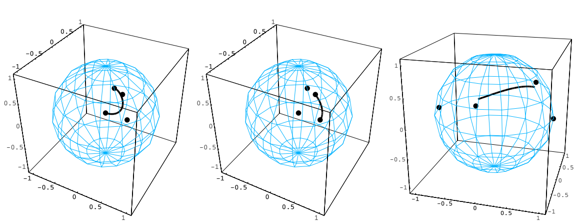
The differences are in the derivatives: Bezier has to start matching all over at every fourth point; Catmull-Rom matches the first derivative; and B-spline is the cadillac, matching all derivatives but no *control points*.

Spherical Interpolations



Bezier Catmull-Rom Uniform B

Quaternion Interpolations



Bezier Catmull-Rom Uniform B

Exp Form of Quaternion Rotations

In Hamilton's notation, we can generalize the 2D equation

$$a + ib = e^{i\theta/2}$$

Just set

$$\begin{aligned} q &= (q_0,\ q_1,\ q_2,\ q_3) \\ &= q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3 \\ &= e^{(\mathbf{I}\cdot\hat{\mathbf{n}}\theta/2)} \end{aligned}$$

with $q_0 = \cos(\theta/2)$ and $\vec{q} = \hat{\mathbf{n}}\sin(\theta/2)$ and $\mathbf{I} = (\mathbf{i}, \mathbf{j}, \mathbf{k})$, with $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$, and $\mathbf{i} * \mathbf{j} = \mathbf{k}$ (cyclic),

Key to Quaternion Intuition

Fundamental Intuition: We know

$$q_0 = \cos(\theta/2), \quad \vec{q} = \hat{n}\sin(\theta/2)$$

We also know that *any coordinate frame* M can be written as $M = R(\theta, \hat{n})$.

Therefore

$\vec{q}$ points exactly along the axis we have to rotate around to go from identity I to M, and the length of $\vec{q}$ tells us how much to rotate.

Summarize Quaternion Properties

- **Unit four-vector.** Take $q = (q_0, q_1, q_2, q_3) = (q_0, \vec{q})$ to obey constraint $q \cdot q = 1$.
- **Multiplication rule.** The quaternion product q and p is $q * p = (q_0 p_0 - \vec{q} \cdot \vec{p},\ q_0 \vec{p} + p_0 \vec{q} + \vec{q} \times \vec{p})$,
  or, alternatively,

$$\begin{bmatrix} [q*p]_0 \\ [q*p]_1 \\ [q*p]_2 \\ [q*p]_3 \end{bmatrix} = \begin{bmatrix} q_0 p_0 - q_1 p_1 - q_2 p_2 - q_3 p_3 \\ q_0 p_1 + q_1 p_0 + q_2 p_3 - q_3 p_2 \\ q_0 p_2 + q_2 p_0 + q_3 p_1 - q_1 p_3 \\ q_0 p_3 + q_3 p_0 + q_1 p_2 - q_2 p_1 \end{bmatrix}$$

Quaternion Summary . . .

- **Rotation Correspondence.** The unit quaternions q and $-q$ correspond to a single 3D rotation R_3:

$$\begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2q_1q_2 - 2q_0q_3 & 2q_1q_3 + 2q_0q_2 \\ 2q_1q_2 + 2q_0q_3 & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2q_2q_3 - 2q_0q_1 \\ 2q_1q_3 - 2q_0q_2 & 2q_2q_3 + 2q_0q_1 & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix}$$

If

$$q = (\cos\frac{\theta}{2}, \hat{n}\sin\frac{\theta}{2}),$$

with $\hat{n}$ a unit 3-vector, $\hat{n} \cdot \hat{n} = 1$. Then $R(\theta, \hat{n})$ is usual 3D rotation by θ in the plane $\perp$ to $\hat{n}$.

SUMMARY

- **Quaternions represent $\boxed{\text{3D frames}}$**
- **Quaternion multiplication represents 3D rotation**
- **Quaternions are]fboxpoints on a hypersphere**
- **Quaternions paths can be visualized with 3D display**
- **Belt Trick, Rolling Ball, and Gimbal Lock can be understood as Quaternion Paths.**

Visualizing Quaternions

Part II

Quaternion Fields

Curves, Surfaces, and Volumes

1

OUTLINE

- **Quaternion Curves:** generalize the Frenet Frame, optimize in quaternion space
- **Quaternion Surfaces:** generalize Gauss map, optimize in quaternion space
- **Quaternion Volumes:** visualize degrees of freedom of a joint

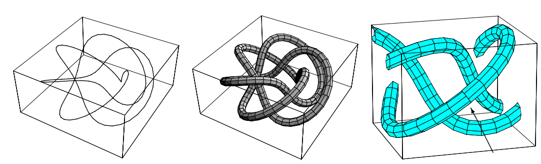
2

What are Frames used For?

- Move objects and object parts in an animated scene.
- Move the camera generating the rendered viewpoint of the scene.
- Attach tubes and textures to thickened lines, oriented textures to surfaces.
- Compare shapes of similar curves.
- Collect orientation data of moving object (e.g., a joint)

3

Motivating Problem: Framing Curves

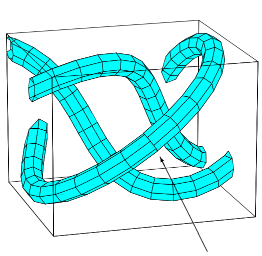


The (3,5) torus knot.

- Line drawing $\approx$ useless.
- Tubing based on parallel transport, **not periodic.**
- Closeup of the non-periodic mismatch.

4

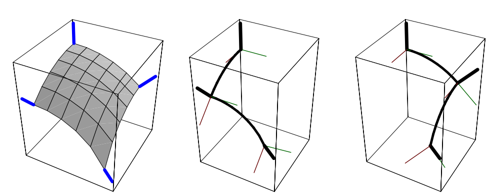
Motivating Problems: Curves



Closeup of the non-periodic mismatch.
Can't apply texture.

5

Motivating Problems: Surfaces



A smooth 3D surface patch: two ways to get bottom frame.

No unique orthonormal frame is derivable from the parameterization.

6

3D Curves: Frenet and PT Frames

Now give more details of 3D frames: Classic Moving Frame:

$$\begin{bmatrix}\mathbf{T}'(t)\\ \mathbf{N}'(t)\\ \mathbf{B}'(t)\end{bmatrix} = \begin{bmatrix}0 & k_1(t) & k_2(t)\\ -k_1(t) & 0 & \sigma(t)\\ -k_2(t) & -\sigma(t) & 0\end{bmatrix}\begin{bmatrix}\mathbf{T}(t)\\ \mathbf{N}(t)\\ \mathbf{B}(t)\end{bmatrix}.$$

Serret-Frenet frame: $k_2 = 0$, $k_1 = \kappa(t)$ is the curvature, and $\sigma(t) = \tau(t)$ is the classical torsion. LOCAL.

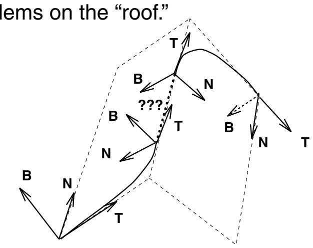
Parallel Transport frame (Bishop): $\sigma = 0$ to get minimal turning. NON-LOCAL = an INTEGRAL.

3D curve frames, contd

Frenet frame is *locally* defined, e.g., by

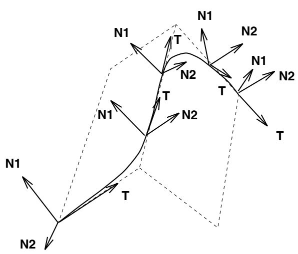
$$\mathbf{B}(t) = \frac{\mathbf{x}'(t) \times \mathbf{x}''(t)}{\|\mathbf{x}'(t) \times \mathbf{x}''(t)\|}$$

but has problems on the "roof."

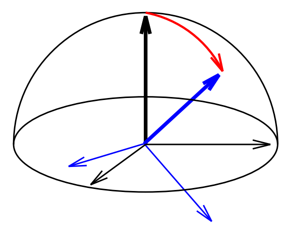


3D curve frames, contd

Bishop's Parallel Transport frame is *integrated over whole curve*, non-local, but no problems on "roof:"

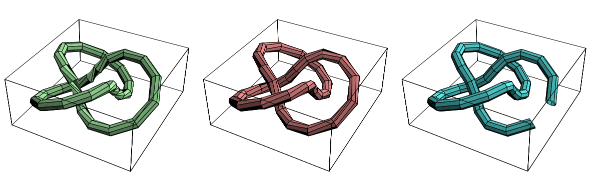


3D curve frames, contd



Geodesic Reference Frame is the frame found by tilting North Pole of "canonical frame" along a great circle until it points in desired direction (tangent for curves, normal for surfaces).

Sample Curve Tubings and their Frames



Frenet Geodesic Reference Parallel Transport

Easily see PT has least "Twist," but lacks periodicity.

3D Frames to Quaternion Frames

- **Quaternion Correspondence.** The unit quaternions q and $-q$ correspond to a single 3D rotation $R_3(q)$:

$$\begin{bmatrix} q_0^2+q_1^2-q_2^2-q_3^2 & 2q_1q_2-2q_0q_3 & 2q_1q_3+2q_0q_2\\ 2q_1q_2+2q_0q_3 & q_0^2-q_1^2+q_2^2-q_3^2 & 2q_2q_3-2q_0q_1\\ 2q_1q_3-2q_0q_2 & 2q_2q_3+2q_0q_1 & q_0^2-q_1^2-q_2^2+q_3^2\end{bmatrix}$$

- **Rotation Correspondence.** $q = (\cos\frac{\theta}{2}, \hat{\mathbf{n}}\sin\frac{\theta}{2})$, with $\hat{\mathbf{n}}$ a unit 3-vector, $\hat{\mathbf{n}}\cdot\hat{\mathbf{n}} = 1$. $R(\theta, \hat{\mathbf{n}})$ is usual 3D rotation by θ in the plane perpendicular to $\hat{\mathbf{n}}$.

- **Extract quaternion:** Either directly from sequence of quaternion multiplications, or indirectly from $R_3(q)$.

Quaternion Frame Evolution

Just as in 2D, let columns of $R_3(q)$ be a **9-part frame**: $(\mathbf{T}, \mathbf{N}, \mathbf{B})$.

Derivatives of the i-th column R_i in quaternion coordinates have the form:

$$\boxed{\dot{R}_i = 2W_i \cdot [\dot{q}(t)]}$$

$$\text{e.g. } W_1 = \begin{bmatrix} q_0 & q_1 & -q_2 & -q_3 \\ q_3 & q_2 & q_1 & q_0 \\ -q_2 & q_3 & -q_0 & q_1 \end{bmatrix}$$

where $i = 1, 2, 3$ and rows form mutually orthonormal basis.

When we simplify by eliminating W_i . . .

we find the *square root* of the 3D frame eqns!

Tait (1890) derived the quaternion equation that makes **all 9 3D frame equations reduce to:** $\boxed{\dot{q} = (1/2)q * (0, \mathrm{k})}$ or:

$$\begin{bmatrix} \dot{q}_0 \\ \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & k_2 & -k_1 & -\sigma \\ -k_2 & 0 & \sigma & -k_1 \\ k_1 & -\sigma & 0 & -k_2 \\ \sigma & k_1 & k_2 & 0 \end{bmatrix} \cdot \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

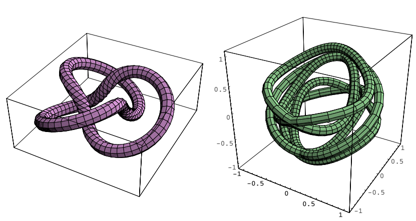
Properties of Tait's quaternion frame equations:

- Antisymmetry $\Rightarrow$ $q(t) \cdot \dot{q}(t) = 0$ as required to keep constant unit radius on 3-sphere.
- *Nine equations and six constraints* become *four equations and one constraint*, keeping quaternion on the 3-sphere. $\Rightarrow$ **Good for computer implementation.**
- MATHEMATICA code implementing this differential equation is provided.

- Analogous treatment (given in Hanson Tech Note in Course Notes) applies also to the Weingarten equations, allowing a direct quaternion treatment of the classical differential geometry of *surfaces* as well.

Example of a Quaternion Frame Curve

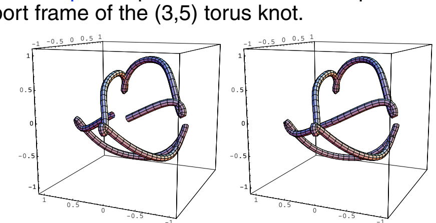
Left Curve = torus knot tubed with Frenet frame; Right Curve is projection from 4D of (twice around) quaternion Frenet frames:



see Notes: Hanson and Ma, "Quaternion Frame Approach to Streamline Visualization," *IEEE Trans. on Visualiz. and Comp. Graphics*, **1**, No. 2, pp. 164–174 (June, 1995).

Minimizing Quaternion Length Solves Periodic Tube

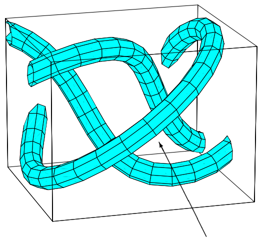
Quaternion space optimization of the non-periodic parallel transport frame of the (3,5) torus knot.



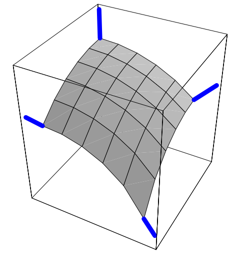
see Notes: "Constrained Optimal Framings of Curves and Surfaces using Quaternion Gauss Maps," *Proceedings of Visualization '98*, pp. 375–382 (1998).

Minimizing Quaternion Length Works

Result of Quaternion space optimization of the (3,5) torus knot frame.

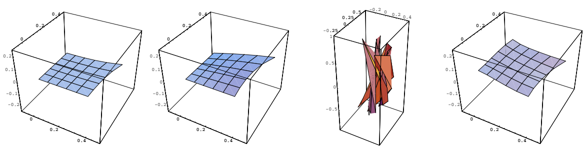


Return to Frames on Surface Patch



Remember: no unique way to disambiguate bottom frame.

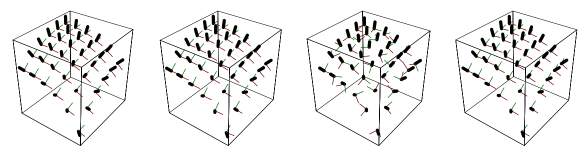
Can also Optimize Quaternion Frames on Patch:



(a) (b) (c) (d)

Quaternion frames for (a) Geodesic Ref. (b) One edge Parallel Transport. (c) Random. (d) Minimal area result.

3D Frames for Patch



(a) (b) (c) (d)

Quaternion frames for (a) Geodesic Ref. (b) One edge Parallel Transport. (c) Random. (d) Minimal area result.

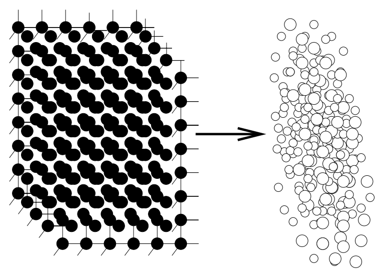
Quaternion Volumes

Last possible orientation field = Volumes:

- Collections of oriented objects in a volume.
- 3 degree-of-freedom control monitoring
- 3 degree-of-freedom biological and robotic joints

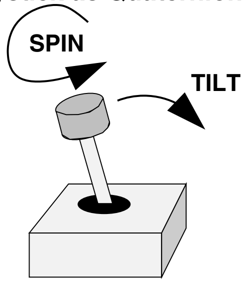
$\Rightarrow$ all map to $\boxed{\text{Quaternion Volumes}}$

Quaternion Volumes



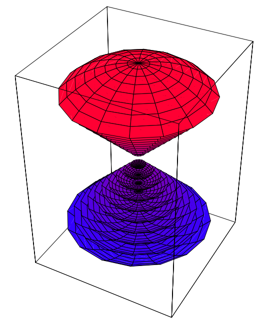
Lattice with Frames **Quaternion Points**

Joystick as Quaternion volume



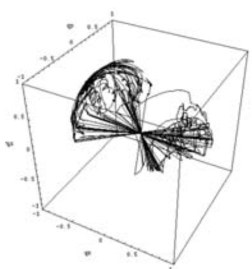
Motion of joystick maps to quaternion volume.

Joystick as quaternion volume



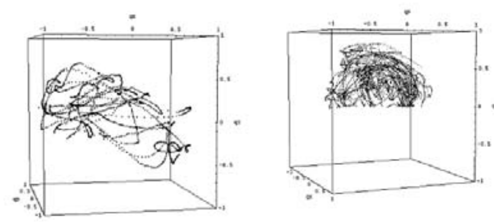
"Solid cone" describes the joystick access space as a quaternion volume

Quaternion volumes: Shoulder data



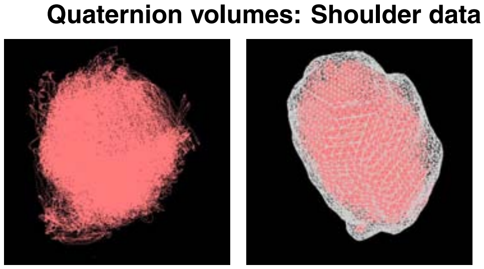
Quaternion shoulder joint data before correction for doubling.

Quaternion volumes: Shoulder data



Shoulder data with neighbors forced to be in same hemisphere of quaternion space as their predecessors.

Quaternion volumes: Shoulder data



(a) (b)

(a) A dense sample of shoulder orientation data in quaternion space.

(b) Implicit surface model fitted to the data. (Herda et al.)

SUMMARY

- **Quaternions nicely represent frame sequences.**
- **Curve frames ⇒ quaternion curves.**
- **Surface patch frames ⇒ quaternion surface patches.**
- **Minimizing quaternion length or area finds parallel transport "minimal turning" set of frames.**
- **Volume sampled frames ⇒ quaternion volumes.**

Use Quaternions for Global Picture of any orientation sequence or collection!