◊ II.6

Geometry for N-Dimensional Graphics

Andrew J. Hanson
Computer Science Department
Indiana University
Bloomington, IN 47405
*hanson@cs.indiana.edu*

◊   Introduction   ◊

Textbook graphics treatments commonly use special notations for the geometry of 2 and 3 dimensions that are not obviously generalizable to higher dimensions. Here we collect a family of geometric formulas frequently used in graphics that are easily extendible to N dimensions as well as being helpful alternatives to standard 2D and 3D notations.

What use are such formulas? In mathematical visualization, which commonly must deal with higher dimensions — 4 real dimensions, 2 complex dimensions, etc. — the utility is self-evident (see, e.g., (Banchoff 1990, Francis 1987, Hanson and Heng 1992b, Phillips et al. 1993)). The visualization of statistical data also frequently utilizes techniques of N-dimensional display (see, e.g., (Noll 1967, Feiner and Beshers 1990a, Feiner and Beshers 1990b, Brun et al. 1989, Hanson and Heng 1992a)). We hope that publicizing some of the basic techniques will encourage further exploitation of N-dimensional graphics in scientific visualization problems.

We classify the formulas we present into the following categories: basic notation and the N-simplex; rotation formulas; imaging in N-dimensions; N-dimensional hyperplanes and volumes; N-dimensional cross-products and normals; clipping formulas; the point-hyperplane distance; barycentric coordinates and parametric hyperplanes; N-dimensional ray-tracing methods. An appendix collects a set of obscure Levi-Civita symbol techniques for computing with determinants. For additional details and insights, we refer the reader to classic sources such as (Sommerville 1958, Coxeter 1991, Hocking and Young 1961) and (Banchoff and Werner 1983, Efimov and Rozendorn 1975).

◊   Definitions — What is a Simplex, Anyway?   ◊

In a nutshell, an N-simplex is a set of $(N + 1)$ points that together specify the simplest non-vanishing N-dimensional volume element (e.g., two points delimit a line segment in 1D, 3 points a triangle in 2D, 4 points a tetrahedron in 3D, etc.). From a mathematical point of view,
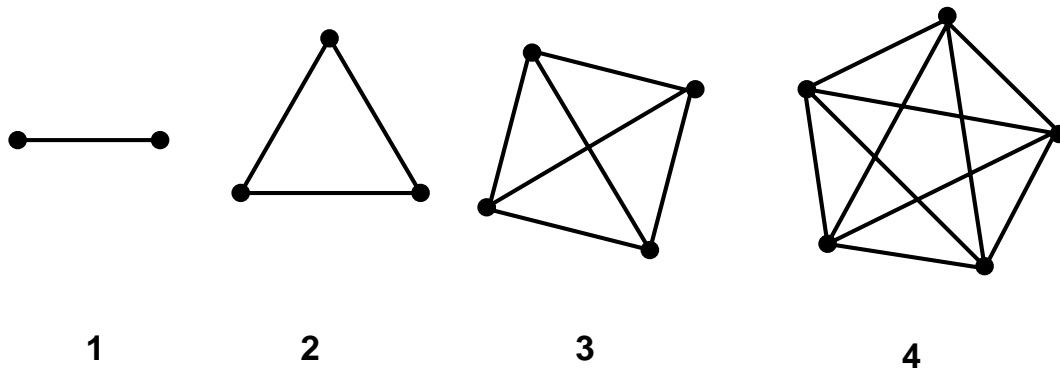
Figure 1.   2D projections of simplexes with dimension 1–4. An N-simplex is defined by $(N+1)$ linearly independent points and generalizes the concept of a line segment or a triangular surface patch.

there are lots of different N-dimensional spaces: here we will restrict ourselves to ordinary flat, real Euclidean spaces of N dimensions with global orthogonal coordinates that we can write as

$$\vec{x} = (x, y, z, \ldots, w)$$

or more pedantically as

$$\vec{x} = \left(x^{(1)}, x^{(2)}, x^{(3)}, \ldots, x^{(N)}\right) .$$

We will use the first, less cumbersome, notation whenever it seems clearer.

Our first type of object in N-dimensions, the 0-dimensional *point* $\vec{x}$, may be thought of as a vector from the origin to the designated set of coordinate values. The next type of object is the 1-dimensional *line*, which is determined by giving two points $(\vec{x}_0, \vec{x}_1)$; the line segment from $\vec{x}_0$ to $\vec{x}_1$ is called a 1-*simplex*. If we now take three noncollinear points $(\vec{x}_0, \vec{x}_1, \vec{x}_2)$, these uniquely specify a *plane*; the triangular area delineated by these points is a 2-*simplex*. A 3-simplex is a solid tetrahedron formed by a set of four noncoplanar points, and so on. In figure 1, we show schematic diagrams of the first few simplexes projected to 2D.

Starting with the $(N + 1)$ points $(\vec{x}_0, \vec{x}_1, \vec{x}_2, \ldots, \vec{x}_N)$ defining a simplex, one then connects all possible pairs of points to form edges, all possible triples to form faces, and so on, resulting in the structure of component "parts" given in table 1. The next higher object uses its predecessor as a building block: a triangular face is built from three edges, a tetrahedron is built from four triangular faces, a 4-simplex is built from 5 tetrahedra.

The general idea should now be clear: $(N + 1)$ linearly independent points define a *hyperplane* of dimension N and specify the boundaries of an N-dimensional coordinate patch comprising an N-*simplex* (Hocking and Young 1961). Just as the surfaces modeling a 3D object may be broken up (or *tessellated*) into triangular patches, N-dimensional objects may be tessellated into $(N - 1)$-dimensional simplexes that define their geometry.

Table 1. Numbers of component structures making up an N-simplex. For example, in 2D, the basic simplex is the triangle with 3 points, 3 edges, and one 2D face.

Type of Simplex	Dimension of Space					
	$N = 1$	$N = 2$	$N = 3$	$N = 4$	...	N
Points (0D)	2	3	4	5	...	$\begin{pmatrix} N+1 \\ 1 \end{pmatrix} = N+1$
Edges (1D simplex)	1	3	6	10	...	$\begin{pmatrix} N+1 \\ 2 \end{pmatrix}$
Faces (2D simplex)	0	1	4	10	...	$\begin{pmatrix} N+1 \\ 3 \end{pmatrix}$
Volumes (3D simplex)		0	1	5	...	$\begin{pmatrix} N+1 \\ 4 \end{pmatrix}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$(N-2)$D simplex					...	$\begin{pmatrix} N+1 \\ N-1 \end{pmatrix}$
$(N-1)$D simplex					...	$\begin{pmatrix} N+1 \\ N \end{pmatrix} = N+1$
ND simplex				1	...	$\begin{pmatrix} N+1 \\ N+1 \end{pmatrix} = 1$

◇ **Rotations** ◇

In N Euclidean dimensions, there are $\begin{pmatrix} N \\ 2 \end{pmatrix} = N(N-1)/2$ degrees of rotational freedom corresponding to the free parameters of the group $SO(N)$. In 2D, that means we only have one rotational degree of freedom given by the angle used to mix the x and y coordinates. In 3D, there are 3 parameters, which can be thought of as corresponding either to three Euler angles or to the three independent quaternion coordinates that remain when we represent rotations in terms of unit quaternions. In 4D, there are 6 degrees of freedom, and the familiar 3D picture of "rotating about an axis" is no longer valid; each rotation leaves an entire plane fixed, not just one axis.

General rotations in N dimensions may be viewed as a sequence of elementary rotations. Each elementary rotation acts in the plane of a particular pair, say (i, j), of coordinates, leaving an $(N-2)$-dimensional subspace unchanged; we may write any such rotation in the form

$$
\begin{aligned}
x'^{(i)} &= x^{(i)} \cos\theta \pm x^{(j)} \sin\theta \\
x'^{(j)} &= \mp x^{(i)} \sin\theta + x^{(j)} \cos\theta \\
x'^{(k)} &= x^{(k)} \ (k \neq i, j) \, .
\end{aligned}
$$

It is important to remember that *order matters* when doing a sequence of nested rotations; for example, two sequences of small 3D rotations, one consisting of a $(2, 3)$-plane rotation followed
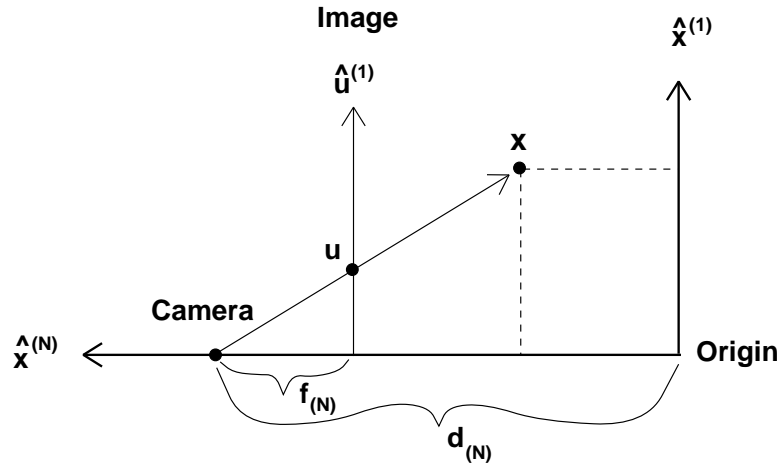
**Figure 2.** Schematic view of the projection process for an N-dimensional pinhole camera.

by a $(3, 1)$-plane rotation, and the other with the order reversed, will differ by a rotation in the $(1, 2)$-plane. (See any standard reference such as (Edmonds 1957).)

We then have a number of options for controlling rotations in N-dimensional Euclidean space. Among these are the following:

- (i, j)**-space pairs.** A brute-force choice would be just to pick a sequence of (i, j) planes in which to rotate using a series of matrix multiplications.

- (i, j, k)**-space triples.** A more interesting choice for an interactive system is to provide the user with a family of (i, j, k) triples having a 2D controller like a mouse coupled to two of the degrees of freedom, and having the 3rd degree of freedom accessible in some other way — with a different button, from context using the "virtual sphere" algorithm of (Chen et al. 1988), or implicitly using a context-free method like the "rolling-ball" algorithm (Hanson 1992). The simplest example is $(1, 2, 3)$ in 3D, with the mouse coupled to rotations about the $\hat{\mathbf{x}}$-axis $(2, 3)$ and the $\hat{\mathbf{y}}$-axis $(3, 1)$, giving $\hat{\mathbf{z}}$-axis $(1, 2)$ rotations as a side-effect. In 4D, one would have four copies of such a controller, $(1, 2, 3)$, $(2, 3, 4)$, $(3, 1, 4)$, and $(1, 2, 4)$, or two copies exploiting the decomposition of $SO(4)$ infinitesimal rotations into two independent copies of ordinary 3D rotations. In N dimensions, $\begin{pmatrix} N \\ 3 \end{pmatrix}$ sets of these controllers (far too many when N is large!) could in principle be used.

◇ N**-dimensional Imaging** ◇

The general concept of an "image" is a projection of a point $\vec{x} = (x^{(1)}, x^{(2)}, \ldots, x^{(N)})$ from dimension N to a point $\vec{u}$ of dimension $(N - 1)$ along a line. That is, the image of a 2D world
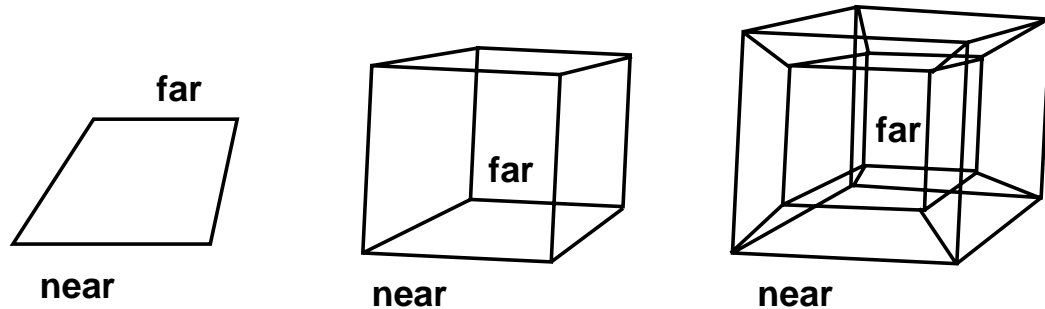
Figure 3.   Qualitative results of perspective projection of a wire-frame square, a cube, and a hypercube in 2D, 3D, and 4D, respectively.

is a projection to 1D film, 3D worlds project to 2D film, 4D worlds project to 3D film, and so on. Since we can rotate our coordinate system as we please, we lose no generality if we assume this projection is along the N-th coordinate axis. An orthographic or parallel projection results if we simply throw out the N-th coordinate $x^{(N)}$ of each point. A pinhole camera perspective projection (see figure 2) results when, in addition, we scale the first $(N-1)$ coordinates by dividing by $(d_N - x^{(N)})/f_N$, where d_N is the distance along the positive N-th axis to the camera focal point and f_N is the focal length. One may need to project this first image to successively lower dimensions to make it displayable on a 2D graphics screen; thus a hierarchy of up to $(N-2)$ parameter sets $\{(f_N, d_N), \ldots, (f_3, d_3)\}$ may be introduced if desired.

In the familiar 3D case, we replace a vertex (x, y, z) of an object by the 2D coordinates $(xf/(d-z), yf/(d-z))$, so that more distant objects (in the negative z direction) are shrunk in the 2D image. In 4D, entire solid objects are shrunk, thus giving rise to the familiar wire-frame hypercube shown in figure 3 that has the more distant cubic hyperfaces actually lying *inside* the projection of the nearest cube.

As we will see a bit later when we discuss normals and cross-products, the usual shading approaches allow only $(N-1)$-manifolds to interact uniquely with a light ray. That is, the generalization of a viewable "object" to N dimensions is a manifold of dimension $(N-1)$ that bounds an N-dimensional volume; only this boundary is visible in the projected image if the object is opaque. For example, curves in 2D reflect light toward the focal point to form images on a "film line," surface patches in 3D form area images on a 2D film plane, volume patches in 4D form volume images in the 3D film volume, etc. The image of this $(N-1)$-dimensional patch may be ray traced or scan converted. Objects are typically represented as tessellations which consist of a collection of $(N-1)$-dimensional simplexes; for example, triangular surface patches form models of the visible parts of 3D objects, while tetrahedral volumes form models of the visible parts of 4D objects. (An interesting side issue is how to display meaningful illuminated images of lower dimensional manifolds — lines in 3D, surfaces and lines in 4D, etc.; see (Hanson and Heng 1992b) for further discussion.)
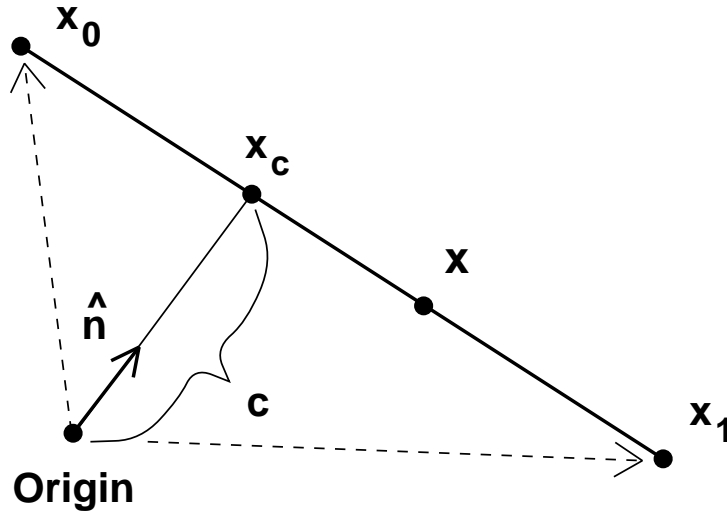
Figure 4. The line from $\vec{x}_0$ to $\vec{x}_1$ whose points obey the equation $\hat{\mathbf{n}} \cdot (\vec{x} - \vec{x}_0) = 0$. The constant c is just $\hat{\mathbf{n}} \cdot \vec{x}_0$.

◇ Hyperplanes and Volume Formulas ◇

Implicit Equation of a Hyperplane. In 2D, a special role is played by the single linear equation defining a line; in 3D, the analogous single linear equation defines a plane. In N-dimensions, the following implicit linear equation describes a set of points belonging to an $(N-1)$-dimensional hyperplane:

$$\hat{\mathbf{n}} \cdot (\vec{x} - \vec{x}_0) = 0 \,. \tag{1}$$

Here $\vec{x}_0$ is any point on the hyperplane and conventionally $\hat{\mathbf{n}} \cdot \hat{\mathbf{n}} = 1$. The geometric interpretation of this equation in 2D is the 1D line shown in figure 4. In general, $\hat{\mathbf{n}}$ is a normalized unit vector that is perpendicular to the hyperplane, and $\hat{\mathbf{n}} \cdot \vec{x}_0 = c$ is simply the (signed) distance from the origin to the hyperplane. The point $\vec{x}_c = c\hat{\mathbf{n}}$ is the point on the hyperplane closest to the origin; the point closest to some other point $\vec{P}$ is $\vec{x}_c = \vec{P} + \hat{\mathbf{n}}\{\hat{\mathbf{n}} \cdot (\vec{x}_0 - \vec{P})\}$.

Simplex Volumes and Subvolumes. The volume (by which we always mean the N-dimensional hypervolume) of an N-simplex is determined in a natural way by a determinant of its $(N+1)$ defining points (Sommerville 1958):

$$V_N = \frac{1}{N!} \det \begin{bmatrix} x_1 & x_2 & \cdots & x_N & x_0 \\ y_1 & y_2 & \cdots & y_N & y_0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ w_1 & w_2 & \cdots & w_N & w_0 \\ 1 & 1 & \cdots & 1 & 1 \end{bmatrix} \,. \tag{2}$$

The bottom row of 1's in eq. (2) corresponds to the familiar homogeneous coordinate used with 4×4 projection matrices in 3D graphics. We will attempt to convince the reader in a moment that disastrous sign inconsistencies result unless the global origin $\vec{x}_0$ of the N-simplex's coordinate system is in the last column as shown.

The expression for the volume in eq. (2) is *signed*, which means that it implicitly defines the N-dimensional generalization of the *Right-Hand Rule* typically adopted to determine triangle orientation in 3D geometry. For example, we observe that if $\vec{x}_0 = (0, 0, \ldots, 0)$ is the origin and we choose $\vec{x}_1 = (1, 0, \ldots, 0)$, $\vec{x}_2 = (0, 1, 0, \ldots, 0)$, and so on, the value of the determinant is $+1$. If we had put $\vec{x}_0$ in the first row in eq. (2), the sign would alternate from dimension to dimension! We will exploit this signed determinant shortly to define N-dimensional normal vectors, and again later to formulate N-dimensional clipping.

First, we use the standard column-subtraction identity for determinants to reduce the dimension of the determinant in eq. (2) by one, expressing it in a form that is manifestly *translation-invariant*:

$$
\begin{aligned}
V_N &= \frac{1}{N!} \det
\begin{bmatrix}
(x_1 - x_0) & (x_2 - x_0) & \cdots & (x_N - x_0) & x_0 \\
(y_1 - y_0) & (y_2 - y_0) & \cdots & (y_N - y_0) & y_0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
(w_1 - w_0) & (w_2 - w_0) & \cdots & (w_N - w_0) & w_0 \\
0 & 0 & \cdots & 0 & 1
\end{bmatrix} \\
&= \frac{1}{N!} \det
\begin{bmatrix}
(x_1 - x_0) & (x_2 - x_0) & \cdots & (x_N - x_0) \\
(y_1 - y_0) & (y_2 - y_0) & \cdots & (y_N - y_0) \\
\vdots & \vdots & \ddots & \vdots \\
(w_1 - w_0) & (w_2 - w_0) & \cdots & (w_N - w_0)
\end{bmatrix} .
\end{aligned}
\tag{3}
$$

These formulas for V_N can be intuitively understood as generalizations of the familiar 3D triple scalar product,

$$
[(\vec{x}_1 - \vec{x}_0) \times (\vec{x}_2 - \vec{x}_0)] \cdot (\vec{x}_3 - \vec{x}_0) ,
$$

which gives the volume of the parallelepiped with sides $((\vec{x}_1 - \vec{x}_0), (\vec{x}_2 - \vec{x}_0), (\vec{x}_3 - \vec{x}_0))$. The corresponding tetrahedron with vertices at the points $(\vec{x}_0, \vec{x}_1, \vec{x}_2, \vec{x}_3)$ has one-sixth the volume of the parallelepiped. The analogous observation in N dimensions is that the factor of $1/N!$ in eq. (3) is the proportionality factor between the volume of the N-simplex and the volume of the parallelepiped whose edges are given by the matrix columns.

Invariance. The volume determinant is invariant under rotations. To see this explicitly, let $|X|$ be the matrix in eq. (3) and let $|R|$ be any orthonormal rotation matrix (i.e., one whose columns are of unit length and are mutually perpendicular, with unit determinant); then, letting $|X'| = |R| \cdot |X|$, we find

$$
\det |X'| = \det(|R| \cdot |X|) = \det |R| \det |X| = \det |X| = N! \, V_N ,
$$

since the determinant of a product is the product of the determinants.

A manifestly translation *and* rotation invariant form for the square of the volume element is

$$
\begin{aligned}
(V_N)^2 &= \left(\frac{1}{N!}\right)^2 \det |X^t \cdot X| \\
&= \left(\frac{1}{N!}\right)^2 \det \begin{bmatrix} v(1,1) & v(1,2) & \cdots & v(1,N) \\ v(2,1) & v(2,2) & \cdots & v(2,N) \\ \vdots & \vdots & \ddots & \vdots \\ v(N,1) & v(N,2) & \cdots & v(N,N) \end{bmatrix},
\end{aligned}
\tag{4}
$$

where $v(i,j) = (\vec{x}_i - \vec{x}_0) \cdot (\vec{x}_j - \vec{x}_0)$.

This invariant form is not presented as an idle observation; we now exploit it to show how to construct volume forms for *subspaces* of N-dimensional spaces, for which the defining vertices of the desired simplex cannot form square matrices!

The trick here is to note that while V_K, for $K < N$, is not expressible in terms of a square matrix of coordinate differences the way V_N is, we may write V_K as the determinant of a square matrix in one particular coordinate frame, and multiply this matrix by its transpose to get a form like eq. 4, which does not depend on the frame. Since the form is invariant, we can transform back to an arbitrary frame to find the following expression for V_K in terms of its K basis vectors $(\vec{x}_k - \vec{x}_0)$ of dimension N:

$$
\begin{aligned}
(V_K)^2 &= \left(\frac{1}{K!}\right)^2 \det \begin{bmatrix} \vec{x}_1 - \vec{x}_0 \\ \vec{x}_2 - \vec{x}_0 \\ \vdots \\ \vec{x}_K - \vec{x}_0 \end{bmatrix} \cdot \begin{bmatrix} \vec{x}_1 - \vec{x}_0 & \vec{x}_2 - \vec{x}_0 & \cdots & \vec{x}_K - \vec{x}_0 \end{bmatrix} \\
&= \left(\frac{1}{K!}\right)^2 \det \begin{bmatrix} v(1,1) & v(1,2) & \cdots & v(1,K) \\ v(2,1) & v(2,2) & \cdots & v(2,K) \\ \vdots & \vdots & \ddots & \vdots \\ v(K,1) & v(K,2) & \cdots & v(K,K) \end{bmatrix}.
\end{aligned}
\tag{5}
$$

That is, to compute a volume of dimension K in N dimensions, find the K independent basis vectors spanning the subspace, and form a square $K \times K$ matrix of dot products related to V_K^2 by multiplying the $N \times K$ matrix of column vectors by its transpose on the left. When $K = 1$, we see that we have simply the squared Euclidean distance in N dimensions, $v(1,1) = (\vec{x}_1 - \vec{x}_0) \cdot (\vec{x}_1 - \vec{x}_0)$.

◇   **Normals and the Cross-Product**   ◇

A frequently asked question in N-dimensional geometry concerns how to define a normal vector as a cross-product of edges for use in geometry and shading calculations. To begin with,

you must have an $(N - 1)$-manifold (a line in 2D, surface in 3D, volume in 4D) in order to have a well-defined normal *vector*; otherwise, you may have a normal *space* (a plane, a volume, etc.). Suppose you have an ordered set of $(N - 1)$ edge vectors $(\vec{x}_k - \vec{x}_0)$ tangent to this $(N - 1)$-manifold at a point $\vec{x}_0$; typically these vectors are the edges of one of the $(N - 1)$-simplexes in the tessellation. Then the normal $\vec{N}$ at the point is a *generalized cross-product* whose components are cofactors of the last column in the following (notationally abusive!) determinant:

$$
\begin{aligned}
\vec{N} &= N_x\hat{\mathbf{x}} + N_y\hat{\mathbf{y}} + N_z\hat{\mathbf{z}} + \cdots + N_w\hat{\mathbf{w}} \\
&= \det \begin{bmatrix}
(x_1 - x_0) & (x_2 - x_0) & \cdots & (x_{N-1} - x_0) & \hat{\mathbf{x}} \\
(y_1 - y_0) & (y_2 - y_0) & \cdots & (y_{N-1} - y_0) & \hat{\mathbf{y}} \\
(z_1 - z_0) & (z_2 - z_0) & \cdots & (z_{N-1} - z_0) & \hat{\mathbf{z}} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
(w_1 - w_0) & (w_2 - w_0) & \cdots & (w_{N-1} - w_0) & \hat{\mathbf{w}}
\end{bmatrix} .
\end{aligned} \tag{6}
$$

As usual, we can normalize using $\|\vec{N}\|$, the square root of the sum of the squares of the cofactors, to form the normalized normal $\hat{\mathbf{n}} = \vec{N}/\|\vec{N}\|$. A quick check shows that if the vectors $(\vec{x}_k - \vec{x}_0)$ are assigned to the first $(N - 1)$ coordinate axes in order, this normal vector points in the direction of the positive N-th axis. For example, in 2D, we want the normal to the vector $(x_1 - x_0,\ y_1 - y_0)$ to be $\vec{N} = (-(y_1 - y_0),\ (x_1 - x_0))$ so that a vector purely in the x direction has a normal in the positive y direction; placing the column of unit vectors $(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}, \ldots, \hat{\mathbf{w}})$ in the *first* column fails this test. The 3D case can be done either way because an even number of columns are crossed! It is tempting to move the column of unit vectors to the first column instead of the last, but one must resist: the choice given here is the one to use for consistent behavior across different dimensions!

The qualitative interpretation of eq. (6) can now be summarized as follows:

- **2D:** Given two points $(\vec{x}_0, \vec{x}_1)$ determining a line in 2D, the cross-product of a *single vector* is the normal to the line.
- **3D:** Given three points defining a plane in 3D, the cross-product of the two 3D vectors outlining the resulting triangle is the familiar formula $(\vec{x}_1 - \vec{x}_0) \times (\vec{x}_2 - \vec{x}_0)$ for the normal $\vec{N}$ to the plane.
- **4D:** In four dimensions, we use four points to construct the three vectors $(\vec{x}_1 - \vec{x}_0), (\vec{x}_2 - \vec{x}_0), (\vec{x}_3 - \vec{x}_0)$; the cross product of these vectors is a *four-vector* that is perpendicular to each vector and thus is interpretable as the normal to the tetrahedron specified by the original four points.

From this point on, the relationship to standard graphics computations should be evident: If, in N-dimensional space, the $(N - 1)$-manifold to be rendered is tessellated into $(N - 1)$-simplexes, use eq. (6) to compute the normal of each simplex for flat shading. For interpolated shading, compute the normal at each vertex (e.g., by averaging the normals of all neighboring

simplexes and normalizing or by computing the gradient of an implicit function specifying the vertex). Compute the intensity at a point for which you know the normal by taking the dot product of the appropriate illumination vector with the normal (e.g, by plugging it into the last column of eq. (6)). If appropriate, set the dot product to zero if it is negative (pointing away from the light). Back face culling, to avoid rendering simplexes pointing away from the camera, is accomplished in exactly the same way: plug the camera view vector into the last column of eq. (6) and discard the simplex if the result is negative.

Dot Products of Cross Products. We conclude this section with the remark that sometimes computing the dot product between a normal and a simple vector is not enough; if we need to know the relative orientation of two face normals (e.g., to determine whether a finer tessellation is required), we must compute the dot products of normals. In principle, this can be done by brute force directly from eq. (6). Here we note an alternative formulation that is the N-dimensional generalization of the 3D formula for the decomposition of the dot product of two cross products; in the 3D case, if one normal is given by the cross product $\vec{X} = \vec{A} \times \vec{B}$ and the other by $\vec{Y} = \vec{C} \times \vec{D}$, we can write

$$\vec{X} \cdot \vec{Y} = (\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) = (\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{D}) - (\vec{A} \cdot \vec{D})(\vec{B} \cdot \vec{C}) \ . \tag{7}$$

We note that the degenerate case for the square of a cross product is

$$(\vec{A} \times \vec{B}) \cdot (\vec{A} \times \vec{B}) = (\vec{A} \cdot \vec{A})(\vec{B} \cdot \vec{B}) - (\vec{A} \cdot \vec{B})^2 \ ,$$

which, if θ is the angle between $\vec{A}$ and $\vec{B}$, reduces to the identity $\|\vec{A}\|^2 \|\vec{B}\|^2 \sin^2 \theta = \|\vec{A}\|^2 \|\vec{B}\|^2 - \|\vec{A}\|^2 \|\vec{B}\|^2 \cos^2 \theta$.

The generalization of this expression to N dimensions can be derived from the product of two Levi-Civita symbols (see the Appendix). If $\vec{X}$ and $\vec{Y}$ are two cross products formed from the sets of vectors $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_{N-1}$ and $\vec{y}_1, \vec{y}_2, \ldots, \vec{y}_{N-1}$, then

$$\vec{X} \cdot \vec{Y} = \sum_{\text{all indices}} x_1^{(i_1)} x_2^{(i_2)} \ldots x_{N-1}^{(i_{N-1})} y_1^{(j_1)} y_2^{(j_2)} \ldots y_{N-1}^{(j_{N-1})}$$

$$\det \begin{bmatrix} \delta_{i_1 j_1} & \delta_{i_1 j_2} & \cdots & \delta_{i_1 j_{N-1}} \\ \delta_{i_2 j_1} & \delta_{i_2 j_2} & \cdots & \delta_{i_2 j_{N-1}} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{i_{N-1} j_1} & \delta_{i_{N-1} j_2} & \cdots & \delta_{i_{N-1} j_{N-1}} \end{bmatrix} , \tag{8}$$

where the Kronecker delta, δ_{ij}, is defined as

$$\begin{aligned} \delta_{ij} &= 1 & i = j \\ &= 0 & i \neq j \ . \end{aligned}$$

It is easy to verify that for $N = 3$ this reduces to eq. (7).

More remarkable, however, is the fact that this formula shows that the square magnitude of the normal $\vec{N}$ of a hyperplane given in eq. (6) is the *subvolume* of the corresponding parallelepiped specified by eq. (5). That is, not only does the *direction* of eq. (6) have an important geometric meaning with respect to the $(N-1)$-simplex specifying the hyperplane, but so does its *magnitude*! We find

$$\vec{N} \cdot \vec{N} = \det \begin{bmatrix} v(1,1) & v(1,2) & \cdots & v(1,N-1) \\ v(2,1) & v(2,2) & \cdots & v(2,N-1) \\ \vdots & \vdots & \ddots & \vdots \\ v(N-1,1) & v(N-1,2) & \cdots & v(N-1,N-1) \end{bmatrix} = ((N-1)!\, V_{N-1})^2 \ .$$

◇ **Clipping Tests in N Dimensions** ◇

Now we can exploit the properties of the volume formula to define clipping ("which side") tests in any dimension. If we replace $(\vec{x}_N - \vec{x}_0)$ by $(\vec{x} - \vec{x}_0)$, eq. (3) becomes a *function* $V_N(\vec{x})$. Furthermore, this function has the remarkable property that it is an alternative form for the hyperplane equation, eq. (1), when $V_N(\vec{x}) = 0$.

We can furthermore determine *on which side* of the $(N-1)$-dimensional hyperplane determined by $(\vec{x}_0, \vec{x}_1, \ldots, \vec{x}_{N-1})$ an arbitrary point $\vec{x}$ lies simply by checking the sign of $V_N(\vec{x})$. That is,

- $V_N(\vec{x}) = 0 \Rightarrow$ the point $\vec{x}$ lies on a hyperplane and solves an equation of the form eq. (1).
- $V_N(\vec{x}) > 0 \Rightarrow$ the point $\vec{x}$ lies above the hyperplane.
- $V_N(\vec{x}) < 0 \Rightarrow$ the point $\vec{x}$ lies below the hyperplane.

Note: The special case $V_N = 0$ is of course just the general criterion for discovering *linear dependence* among a set of $(N+1)$ vector variables. This has the following elegant geometric interpretation: In 2D, we use the formula to compute the area of the triangle formed by 3 points $(\vec{x}_0, \vec{x}_1, \vec{x})$; if the area vanishes, the 3 points lie on a single line. In 3D, if the volume of the tetrahedron formed by 4 points $(\vec{x}_0, \vec{x}_1, \vec{x}_2, \vec{x})$ vanishes, all 4 points are coplanar, and so on. Vanishing N-volume means the points lie in a hyperplane of dimension no greater than $(N-1)$.

These relationships between the sign of $V_N(\vec{x})$ and the relative position of $\vec{x}$ are precisely those we are accustomed to examining when we *clip* vectors (e.g., edges of a triangle) to lie on one side of a plane in a viewing frustum or within a projected viewing rectangle. For example, a 2D clipping line defined by the vector $\vec{x}_1 - \vec{x}_0 = (x_1 - x_0,\, y_1 - y_0)$ has a right-handed (unnormalized) normal $\vec{N} = (-(y_1 - y_0),\, (x_1 - x_0))$. Writing the 2D volume as the area A, eq. (3) becomes

$$A(\vec{x}) = \frac{1}{2} \det \begin{bmatrix} (x_1 - x_0) & (x - x_0) \\ (y_1 - y_0) & (y - y_0) \end{bmatrix} = \frac{1}{2} \left[\vec{N} \cdot (\vec{x} - \vec{x}_0) \right]$$
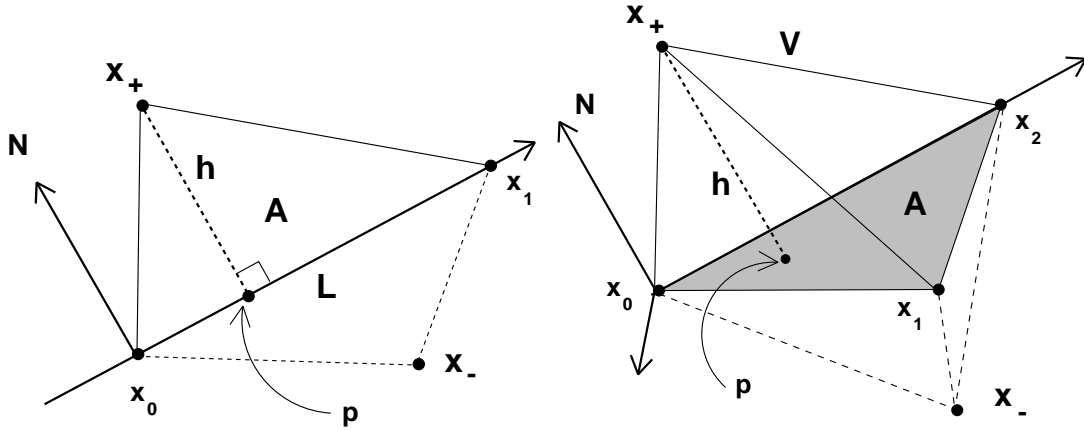
Figure 5. In 2D, the line through $\vec{x}_0$ to $\vec{x}_1$ defined by $\hat{\mathbf{n}} \cdot (\vec{x} - \vec{x}_0) = 0$ partitions the plane into two regions, one where this expression is positive (e.g., for $\vec{x}_+$) and another where it is negative (e.g., for $\vec{x}_-$). In 3D, the analogous procedure uses the plane defined by $(\vec{x}_0, \vec{x}_1, \vec{x}_2)$ to divide 3-space into two half spaces. The same pictures serve to show how the distance h from a point to a hyperplane is computable from the ratio of the simplex volume to the lower-dimensional volume of its base, i.e., $2A/L$ or $3V/A$.

for some arbitrary point $\vec{x}$, and so we recover the form of eq. (1) as

$$\hat{\mathbf{n}} \cdot (\vec{x} - \vec{x}_0) = \frac{2A}{\|\vec{x}_1 - \vec{x}_0\|} \,,$$

where $\hat{\mathbf{n}} = \vec{N}/\|\vec{N}\|$; the relationship of $\vec{x}$ to the clipping line is determined by the sign.

  In 3D, when clipping a line against a plane, everything reduces to the traditional form, namely the dot product between a 3D cross-product and a vector from a point $\vec{x}_0$ in the clipping plane to the point $\vec{x}$ being clipped. The normal to the plane through $(\vec{x}_0, \vec{x}_1, \vec{x}_2)$ is

$$
\begin{aligned}
\vec{N} &= (\vec{x}_1 - \vec{x}_0) \times (\vec{x}_2 - \vec{x}_0) \tag{9}\\
&= \left(+\det \begin{bmatrix} (y_1 - y_0) & (y_2 - y_0) \\ (z_1 - z_0) & (z_2 - z_0) \end{bmatrix} \right., \\
&\qquad \left. -\det \begin{bmatrix} (x_1 - x_0) & (x_2 - x_0) \\ (z_1 - z_0) & (z_2 - z_0) \end{bmatrix}, +\det \begin{bmatrix} (x_1 - x_0) & (x_2 - x_0) \\ (y_1 - y_0) & (y_2 - y_0) \end{bmatrix} \right) ,
\end{aligned}
$$

and we again find the same general form,

$$\hat{\mathbf{n}} \cdot (\vec{x} - \vec{x}_0) = \frac{6V}{\|\vec{N}\|} \,,$$

whose sign determines where $\vec{x}$ falls. Figure 5 summarizes the relationship of the signed volume to the clipping task in 2D and 3D.

Hyperplanes for clipping applications in any dimension are therefore easily defined and checked by choosing $\vec{x}_N$ to be the test point $\vec{x}$ and checking the sign of eq. (3). If $\vec{N}$ and a point $\vec{x}_0$ are easy to determine directly, then the procedure reduces to checking the sign of the left hand side of eq. (1).

The final step is to find the desired point on the truncated, clipped line. Since the clipped form of a triangle, tetrahedron, etc., can be determined from the clipped forms of the component lines, we need only consider the point at which a line straddling the clipping hyperplane intersects this hyperplane. If the line to be clipped is given parametrically as $\vec{x}(t) = \vec{x}_a + t(\vec{x}_b - \vec{x}_a)$, where $\vec{x}_a$ and $\vec{x}_b$ are on opposite sides of the clipping hyperplane so $0 \le t \le 1$, then we simply plug $\vec{x}(t)$ into $V(\vec{x}) = 0$ and solve for t:

$$t = \frac{\det\left[\begin{array}{ccccc} \vec{x}_1 - \vec{x}_0 & \vec{x}_2 - \vec{x}_0 & \cdots & \vec{x}_a - \vec{x}_0 \end{array}\right]}{\det\left[\begin{array}{ccccc} \vec{x}_1 - \vec{x}_0 & \vec{x}_2 - \vec{x}_0 & \cdots & \vec{x}_a - \vec{x}_b \end{array}\right]} = \frac{\hat{\mathbf{n}} \cdot (\vec{x}_a - \vec{x}_0)}{\hat{\mathbf{n}} \cdot (\vec{x}_a - \vec{x}_b)} \, . \tag{10}$$

Here $\hat{\mathbf{n}}$ is of course just the normal to the clipping hyperplane, discussed in detail above.

◇　**Point-Hyperplane Distance**　◇

The general formula for the volume of a parallelepiped is the product of the base and the height, $W = Bh$. In N dimensions, if we take $W_N = N!\, V_N$ to be the volume of the parallelepiped with edges $(\vec{x}_1 - \vec{x}_0), (\vec{x}_2 - \vec{x}_0), \ldots, (\vec{x}_{N-1} - \vec{x}_0), (\vec{x} - \vec{x}_0)$, this generalizes to

$$W_N = hW_{N-1} \, ,$$

where h is the perpendicular distance from the point $\vec{x}$ to the $(N-1)$-dimensional parallelepiped with volume $W_{N-1} = (N-1)!\, V_{N-1}$ and edges $(\vec{x}_1 - \vec{x}_0), (\vec{x}_2 - \vec{x}_0), \ldots, (\vec{x}_{N-1} - \vec{x}_0)$. We may thus immediately compute the distance h from a point to a hyperplane as

$$h = \frac{W_N}{W_{N-1}} = \frac{N!\, V_N}{(N-1)!\, V_{N-1}} = \frac{N\, V_N}{V_{N-1}} \, . \tag{11}$$

Note! Here one must use the trick of eq. 4 to express W_{N-1} in terms of the square root of a square determinant given by the product of two non-square matrices.

Thus in 2D, the area of a triangle $(\vec{x}_0, \vec{x}_1, \vec{x})$ is

$$A = V_2 = \frac{1}{2} W_2 = \frac{1}{2} \det \left[\begin{array}{cc} (x_1 - x_0) & (x - x_0) \\ (y_1 - y_0) & (y - y_0) \end{array} \right]$$

and the length-squared of the base is $L^2 = (\vec{x}_1 - \vec{x}_0) \cdot (\vec{x}_1 - \vec{x}_0)$ so, with $A = (1/2)hL$, the height becomes $h = 2A/L = W_2/L = W_2/W_1$. In 3D, the volume of the tetrahedron $(\vec{x}_0, \vec{x}_1, \vec{x}_2, \vec{x})$ is $V = V_3 = (1/6)W_3$ and the area $A = (1/2)W_2$ of the triangular base may be written

$$(2A)^2 = (W_2)^2 = \det \left[\begin{array}{cc} (\vec{x}_1 - \vec{x}_0) \cdot (\vec{x}_1 - \vec{x}_0) & (\vec{x}_1 - \vec{x}_0) \cdot (\vec{x}_2 - \vec{x}_0) \\ (\vec{x}_2 - \vec{x}_0) \cdot (\vec{x}_1 - \vec{x}_0) & (\vec{x}_2 - \vec{x}_0) \cdot (\vec{x}_2 - \vec{x}_0) \end{array} \right] \, .$$
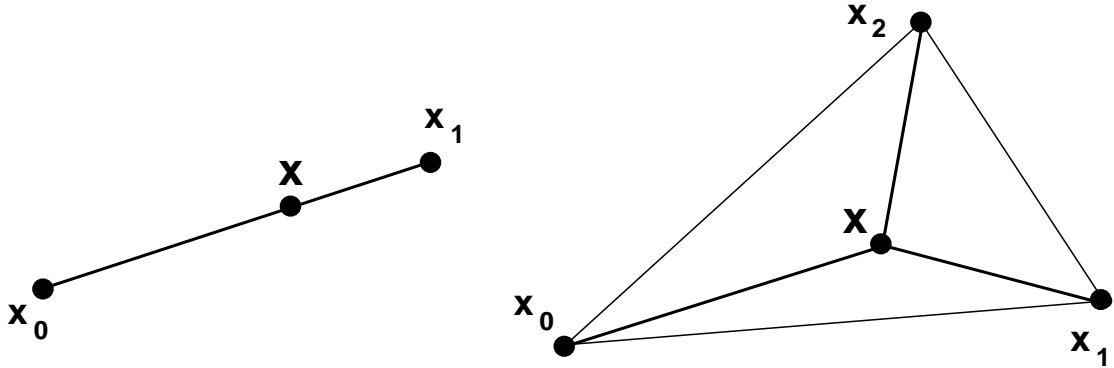
Figure 6.   Barycentric coordinates in N dimensions.

We know $V = (1/3)hA$, and so $h = 3V/A = 6V/2A = W_3/W_2$. (See figure 5.) We note for reference that, as we showed earlier, the base $(N-1)$-volume is related to its normal by $\vec{N} \cdot \vec{N} = W_{N-1}^2$.

   Here we also typically need to answer one last question, namely *where* is the point $\vec{p}$ on the base hyperplane closest to the point $\vec{x}$ whose distance h we just computed? This can be found by parameterizing the line from $\vec{x}$ to the base hyperplane along the normal $\hat{\mathbf{n}}$ to the hyperplane as $\vec{x}(t) = \vec{x} + t\hat{\mathbf{n}}$, writing the implicit equation for the hyperplane as $\hat{\mathbf{n}} \cdot (\vec{x}(t) - \vec{x}_0) = 0$, and solving for the mutual solution $t_p = \hat{\mathbf{n}} \cdot (\vec{x}_0 - \vec{x}) = -h$. Thus

$$\begin{aligned} \vec{p} &= \vec{x} + \hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot (\vec{x}_0 - \vec{x})) \\ &= \vec{x} - h\hat{\mathbf{n}} \ . \end{aligned}$$

◇ **Barycentric Coordinates** ◇

Barycentric coordinates (see, e.g., (Hocking and Young 1961), chapter 5) are a practical way to parameterize lines, surfaces, etc., for applications that must compute where various geometric objects intersect. In practice, the barycentric coordinate method reduces to specifying two points $(\vec{x}_0, \vec{x}_1)$ on a line, three points $(\vec{x}_0, \vec{x}_1, \vec{x}_2)$ on a plane, four points $(\vec{x}_0, \vec{x}_1, \vec{x}_2, \vec{x}_3)$ in a volume, etc., and parameterizing the line segment, enclosed triangular area, and enclosed tetrahedral volume, etc., respectively, by

$$\begin{aligned} \vec{x}(t) &= \vec{x}_0 + t(\vec{x}_1 - \vec{x}_0) & (12) \\ \vec{x}(t_1, t_2) &= \vec{x}_0 + t_1(\vec{x}_1 - \vec{x}_0) + t_2(\vec{x}_2 - \vec{x}_0) & (13) \\ \vec{x}(t_1, t_2, t_3) &= \vec{x}_0 + t_1(\vec{x}_1 - \vec{x}_0) + t_2(\vec{x}_2 - \vec{x}_0) + t_3(\vec{x}_3 - \vec{x}_0) & (14) \\ & \qquad \cdots \ . \end{aligned}$$

The line and plane geometries are shown in figure 6. The interpolated point then lies within the N-simplex defined by the specified points provided

$$0 \le t \le 1$$
$$0 \le t_1 \le 1, \ 0 \le t_2 \le 1, \ 0 \le (1 - t_1 - t_2) \le 1$$
$$0 \le t_1 \le 1, \ 0 \le t_2 \le 1, \ 0 \le t_3 \le 1, \ 0 \le (1 - t_1 - t_2 - t_3) \le 1$$

$$\dots$$

Center of What?   However, this is really only half the story of barycentric coordinates. For the other half, we seek a geometric interpretation of the parameters t_i when we are *given* the value of $\vec{x}$.

First let us look at the simple case when $\vec{x}$ lies on the line segment between $\vec{x}_0$ and $\vec{x}_1$. Solving eq. (12) for t directly gives

$$t = \frac{(\vec{x} - \vec{x}_0) \cdot (\vec{x}_1 - \vec{x}_0)}{(\vec{x}_1 - \vec{x}_0) \cdot (\vec{x}_1 - \vec{x}_0)} \ .$$

That is, t is the fraction of the distance that $\vec{x}$ has traveled along the line, the *ratio* between the length from $\vec{x}_0$ to $\vec{x}$ and the total length. But, since $\vec{x}_1 - \vec{x}_0 = \vec{x}_1 - \vec{x} + \vec{x} - \vec{x}_0$, we easily see that an alternative parameterization would be to take $t_1 = t$ and

$$t_0 = \frac{(\vec{x}_1 - \vec{x}) \cdot (\vec{x}_1 - \vec{x}_0)}{(\vec{x}_1 - \vec{x}_0) \cdot (\vec{x}_1 - \vec{x}_0)}$$

so that $1 = t_0 + t_1$ and eq. (12) for $\vec{x}$ becomes

$$\vec{x}(t_0, t_1) = t_0 \vec{x}_0 + t_1 \vec{x}_1 \ .$$

If $t_0 = 1$, then the entire fraction of the distance from $\vec{x}_1$ to $\vec{x}$ is assigned to t_0 and $\vec{x} = \vec{x}_0$. If $t_1 = 1$, then the entire fraction of the distance from $\vec{x}_0$ to $\vec{x}$ is assigned to t_1 and $\vec{x} = \vec{x}_1$.

Next, suppose we know $\vec{x}$ in a plane and wish to compute its barycentric coordinates by solving eq. (13) for (t_1, t_2). Once we realize that $(\vec{x}_1 - \vec{x}_0)$ and $(\vec{x}_2 - \vec{x}_0)$ form the basis for an affine coordinate system for the plane specified by $(\vec{x}_0, \vec{x}_1, \vec{x}_2)$ in *any* dimension, we see that we may measure the relative barycentric coordinates by taking the dot product with each basis vector:

$$(\vec{x} - \vec{x}_0) \cdot (\vec{x}_1 - \vec{x}_0) = t_1 \|\vec{x}_1 - \vec{x}_0\|^2 + t_2 (\vec{x}_2 - \vec{x}_0) \cdot (\vec{x}_1 - \vec{x}_0)$$
$$(\vec{x} - \vec{x}_0) \cdot (\vec{x}_2 - \vec{x}_0) = t_1 (\vec{x}_1 - \vec{x}_0) \cdot (\vec{x}_2 - \vec{x}_0) + t_2 \|\vec{x}_2 - \vec{x}_0\|^2 \ .$$

Extending the previously introduced abbreviation to the form $v(x, j) = (\vec{x} - \vec{x}_0) \cdot (\vec{x}_j - \vec{x}_0)$ and solving this pair of equations by Cramer's rule, we get

$$t_1 = \frac{\det \begin{bmatrix} v(x, 1) & v(1, 2) \\ v(x, 2) & v(2, 2) \end{bmatrix}}{\det \begin{bmatrix} v(1, 1) & v(1, 2) \\ v(2, 1) & v(2, 2) \end{bmatrix}}$$

$$t_2 \;\; = \;\; \frac{\det \begin{bmatrix} v(1,1) & v(x,1) \\ v(1,2) & v(x,2) \end{bmatrix}}{\det \begin{bmatrix} v(1,1) & v(1,2) \\ v(2,1) & v(2,2) \end{bmatrix}} \; .$$

The denominator is clearly proportional to the *square* of the area of the triangle $(\vec{x}_0, \vec{x}_1, \vec{x}_2)$, and the numerators have the form of squared areas as well. In N dimensions, the numerators reduce to determinants of products of non-square matrices, and so may *not* be expressed as two separate determinants! However, if we transform to a coordinate system that contains the triangle within the plane of two coordinate axes, or if $N = 2$, an effectively square matrix is recovered; one factor of area in the denominator then cancels out, giving the intuitively expected result that the barycentric coordinates are ratios of two areas: $t_1 = A(\vec{x}, \vec{x}_0, \vec{x}_1)/A(\vec{x}_0, \vec{x}_1, \vec{x}_2)$, $t_2 = A(\vec{x}, \vec{x}_2, \vec{x}_0)/A(\vec{x}_0, \vec{x}_1, \vec{x}_2)$. This leads us to introduce the generalized version of t_0 for the line, namely,

$$\begin{aligned} t_0 \;\; &= \;\; 1 - t_1 - t_2 = \frac{A(\vec{x}_1, \vec{x}_2, \vec{x})}{A(\vec{x}_0, \vec{x}_1, \vec{x}_2)} \\ &= \;\; \frac{\det \begin{bmatrix} (\vec{x}_1 - \vec{x}_0) \cdot (\vec{x}_1 - \vec{x}) & (\vec{x}_1 - \vec{x}_0) \cdot (\vec{x}_2 - \vec{x}) \\ (\vec{x}_2 - \vec{x}_0) \cdot (\vec{x}_1 - \vec{x}) & (\vec{x}_2 - \vec{x}_0) \cdot (\vec{x}_2 - \vec{x}) \end{bmatrix}}{\det \begin{bmatrix} v(1,1) & v(1,2) \\ v(2,2) & v(2,2) \end{bmatrix}} \; . \end{aligned}$$

Here we used the squaring argument given above to extend t_0 from its special-coordinate-system interpretation as the fraction of the area contributed by the triangle $(\vec{x}, \vec{x}_1, \vec{x}_2)$ to the invariant form. This form obviously has the desired property that $t_0 = 1$ when $\vec{x} = \vec{x}_0$, and we finally have the sought equation (with $1 = t_0 + t_1 + t_2$)

$$\vec{x}(t_0, t_1, t_2) = t_0 \vec{x}_0 + t_1 \vec{x}_1 + t_2 \vec{x}_2 \; .$$

It is amusing to note that the determinant identity $1 = t_0 + t_1 + t_2$ and its higher analogs, which are nontrivial to derive, generalize the simple identity $\vec{x}_1 - \vec{x}_0 = \vec{x}_1 - \vec{x} + \vec{x} - \vec{x}_0$ that we used in the 1D case.

   Thus we can construct barycentric coordinates in any dimension which intuitively correspond to *fractions of hypervolumes*; each barycentric coordinate is the hypervolume of an N-simplex defined by the point $\vec{x}$ and all but one of the other simplex-defining points divided by the volume of the whole simplex. The actual computation, however, is best done using the squared-volume form because only that form is independent of the chosen coordinate system.

   Note: The volumes are *signed*; even if $\vec{x}$ lies outside the N-simplex volume, the ratios remain correct due to the cancellation between the larger volumes and the negative volumes. We also remark that the generalized formulas for t_i in any dimension, with $1 = \sum_{i=0}^{N} t_i$, give an elegant geometric interpretation of Cramer's rule as ratios of simplex volumes.
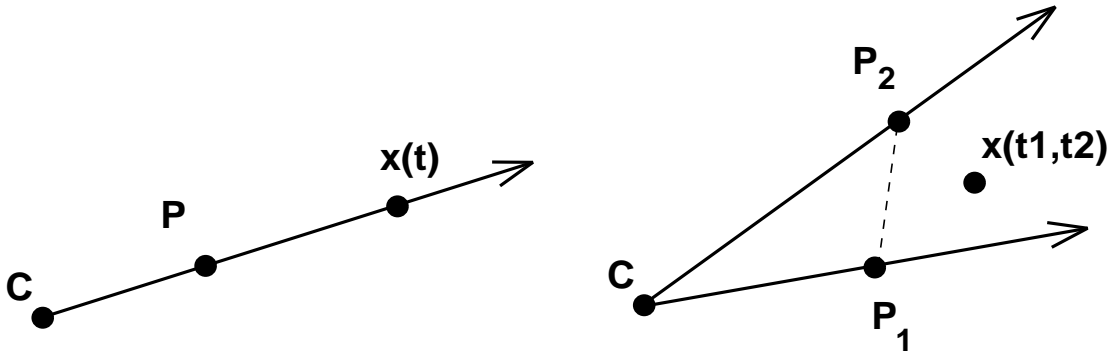
Figure 7.   Schematic diagram comparing an ordinary camera ray and a planar "thick ray" used in N-dimensional ray-tracing methods.

◇   **Ray Tracing**   ◇

It is often useful to compute the intersection of a ray passing through two points (typically the camera focal point $\vec{C}$ and an image point $\vec{P}$) with a geometrical object. In N dimensions, this object will typically be an $(N-1)$-simplex defining an oriented visible "face" with a normal vector computable as described above. We need to do several things: compute the intersection of the ray with the hyperplane containing the "face," check to see whether the point lies within the simplex's boundaries (observe that this is a clipping problem), and see whether the normal vector points in the direction of the ray (making it visible).

We formulate this procedure by first writing

$$\vec{X}(t) = \vec{C} + t(\vec{P} - \vec{C})$$

for the position of a point on the camera ray, as illustrated in figure 7. Then we consider a single $(N-1)$-simplex of the tessellation to be described either by a known normal or by using the set of N points giving its vertices to define its normal via eq. (6); in either case, we can write the equation of any *other* point $\vec{x}$ lying within the simplex as

$$\hat{\mathbf{n}} \cdot (\vec{x} - \vec{x}_0) = 0 \ .$$

Plugging in the parametric ray equation, we solve for the point $\vec{X}(t)$ in the simplex that lies on the ray:

$$t = \frac{\hat{\mathbf{n}} \cdot \left(\vec{x}_0 - \vec{C}\right)}{\hat{\mathbf{n}} \cdot \left(\vec{P} - \vec{C}\right)} \ .$$

A useful generalization of ray-tracing to N-dimensions follows from the observation that a "thick ray" is cast into space by an open-ended simplex that is essentially a barycentric coordinate form with the restriction $0 \le (1 - t_1 - t_2 - \ldots) \le 1$ relaxed (see, e.g., (Hanson and Cross 1993)).

A planar ray such as that shown in figure 7 then has two parameters,

$$\vec{X}(t_1, t_2) = \vec{C} + t_1(\vec{P}_1 - \vec{C}) + t_2(\vec{P}_2 - \vec{C}) \,,$$

with obvious generalizations to volume rays, etc. Intersecting such a planar ray with an $(N-2)$-dimensional manifold (describable using $(N-2)$ barycentric parameters) results in N equations with N unknown parameters, and thus a unique *point* is determined as the mutual solution. In 3D, a plane intersects a line in one point, in 4D two planes intersect in a single point, while in 5D a plane intersects a volume in a point. Other generalizations, including rays that intersect particular geometries in lines and surfaces, can easily be constructed. For example, the intersection of a planar ray with the single hyperplane equation for a 3-manifold in 4D leaves one undetermined parameter, and is therefore a line.

◇   **Conclusion**   ◇

Geometry is an essential tool employed in the creation of computer graphics images of everyday objects. Statistical data analysis, mathematics, and science, on the other hand, provide many problems where N-dimensional generalizations of the familiar 2D and 3D formulas are required. The N-dimensional formulas and insights into the nature of geometry that we have presented here provide a practical guide for extending computer graphics into these higher-dimensional domains.

◇   **Appendix: Determinants and the Levi-Civita Symbol**   ◇

One of the unifying features that has permitted us throughout this treatment to extend formulas to arbitrary dimensions has been the use of *determinants*. But what if you encounter an expression involving determinants that has not been given here and you wish to work out its algebraic properties for yourself? In this appendix, we outline for the reader a useful mathematical tool for treating determinants, the Levi-Civita symbol. References for this are hard to locate; the author learned these techniques by apprenticeship while studying general relativity, but even classic texts like Møller (Møller 1972) contain only passing mention of the methods; somewhat more detail is given in hard-to-find sources such as (Efimov and Rozendorn 1975).

First we define two basic objects, the Kronecker delta, δ_{ij},

$$\begin{aligned} \delta_{ij} &= 1 & i = j \\ &= 0 & i \neq j \end{aligned}$$

and the Levi-Civita symbol, $\epsilon_{ijk\ldots}$, which is the totally antisymmetric pseudotensor with the properties

$$\begin{aligned} \epsilon_{ijk\ldots} &= 1 & i, j, k, \ldots \text{ in an even permutation of cyclic order} \\ &= -1 & i, j, k, \ldots \text{ in an odd permutation of cyclic order} \\ &= 0 & \text{when any two indices are equal.} \end{aligned}$$

All indices are assumed to range from 1 to N, e.g., $i = \{1, 2, \ldots, (N-1), N\}$, so that, for example, (1234,1342,4132,4321), are even permutations and (1324,2134,1243,4312) are odd permutations.

We can use the Kronecker delta to write the dot product between two N-dimensional vectors as a matrix product with the Kronecker delta representing the unit matrix,

$$\vec{A} \cdot \vec{B} = \sum_{i=1}^{N} \sum_{j=1}^{N} A_i \delta_{ij} B_j = \sum_{i=1}^{N} A_i \left(\sum_{j=1}^{N} \delta_{ij} B_j \right) = \sum_{i=1}^{N} A_i B_i \ , \tag{15}$$

and the Levi-Civita symbol to write the determinant of a matrix $|M|$ as

$$\det [M] \quad = \sum_{\text{all } i_k \text{ indices}} \epsilon_{i_1 i_2 \ldots i_N} M_{1,i_1} M_{2,i_2} \cdots M_{N,i_N} \ .$$

The fundamental formula for the product of two Levi-Civita symbols is:

$$\epsilon_{i_1 i_2 \ldots i_N} \epsilon_{j_1 j_2 \ldots j_N} \quad = \quad \det \begin{bmatrix} \delta_{i_1 j_1} & \delta_{i_1 j_2} & \cdots & \delta_{i_1 j_N} \\ \delta_{i_2 j_1} & \delta_{i_2 j_2} & \cdots & \delta_{i_2 j_N} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{i_N j_1} & \delta_{i_N j_2} & \cdots & \delta_{i_N j_N} \end{bmatrix} \ .$$

(Note that if we set $\{j_1 j_2 \ldots j_N\} = \{1, 2, \ldots, N\}$, the second Levi-Civita symbol reduces to $+1$, and the resulting determinant is an explicit realization of the antisymmetry of the Levi-Civita symbol itself as a determinant of Kronecker deltas!)

With this notation, the generalized cross product $\vec{N}$ of eq. (6), simplified by setting $\vec{x}_0 = 0$, can be written

$$\vec{N} \quad = \sum_{\text{all indices}} \epsilon_{i_1 i_2 \ldots i_{N-1} i_N} x_1^{(i_1)} x_2^{(i_2)} \cdots x_{N-1}^{(i_{N-1})} \hat{\mathbf{x}}^{(i_N)} \ ,$$

where $\hat{\mathbf{x}}^{(i_N)}$ are the unit vectors $(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \ldots, \hat{\mathbf{w}})$ of the coordinate system. The dot product between the normal and another vector simply becomes

$$\vec{N} \cdot \vec{L} \quad = \sum_{\text{all indices}} \epsilon_{i_1 i_2 i_3 \ldots i_{N-1} i_N} x_1^{(i_1)} x_2^{(i_2)} x_3^{(i_3)} \cdots x_{N-1}^{(i_{N-1})} L^{(i_N)} \ .$$

The reader can verify that, in 2D, $N_k = \sum_{i=1}^{2} x^{(i)} \epsilon_{ik} = (-y, +x)$, and so on. We conclude with two examples of applications:

Rotations of Normals. Is the normal $\vec{N}$ a vector? *Almost*. To check this, we must rotate each column vector in the cross product formula using $x'^{(i)} = \sum_{j=1}^{N} R_{ij} x^{(j)}$ and compute the behavior of $\vec{N}$. Using the identity ((Efimov and Rozendorn 1975), p. 203),

$$\epsilon_{i_1 i_2 \ldots i_{N-1} i_N} \det[R] \quad = \sum_{\text{all } j_k \text{ indices}} \epsilon_{j_1 j_2 \ldots j_{N-1} j_N} R_{j_1 i_1} R_{j_2 i_2} \cdots R_{j_{N-1} i_{N-1}} R_{j_N i_N} \ ,$$

we find

$$N'^{(i)} \quad = \sum_{\substack{\text{all indices} \\ \text{except } i}} \epsilon_{i_1 i_2 \ldots i_{N-1} i} R_{i_1 j_1} x_1^{(j_1)} R_{i_2 j_2} x_2^{(j_2)} \cdots R_{i_{N-1} j_{N-1}} x_{N-1}^{(j_{N-1})}$$

$$= \sum_{j=1}^{N} R_{ij} N^{(j)} \det[R] \ .$$

Therefore $\vec{N}$ is a *pseudotensor*, and behaves as a vector for ordinary rotations (which have $\det[R] = 1$), but changes sign if $[R]$ contains an odd number of reflections.

Contraction Formula. The contraction of two partial determinants of $(N-K)$ N-dimensional vectors can expanded in terms of products of Kronecker deltas as follows:

$$\sum_{i_{N-K+1} \ldots i_N} \epsilon_{i_1 i_2 \ldots i_{N-K} i_{N-K+1} \ldots i_N} \epsilon_{j_1 j_2 \ldots j_{N-K} i_{N-K+1} \ldots i_N} =$$

$$K! \det \begin{bmatrix} \delta_{i_1 j_1} & \delta_{i_1 j_2} & \cdots & \delta_{i_1 j_{N-K}} \\ \delta_{i_2 j_1} & \delta_{i_2 j_2} & \cdots & \delta_{i_2 j_{N-K}} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{i_{N-K} j_1} & \delta_{i_{N-K} j_2} & \cdots & \delta_{i_{N-K} j_{N-K}} \end{bmatrix} \ .$$

The expression eq. (8) for the dot product of two normals is a special case of this formula.

Bibliography

(Banchoff and Werner 1983) T. Banchoff and J. Werner. *Linear Algebra through Geometry*. Springer-Verlag, 1983.

(Banchoff 1990) Thomas F. Banchoff. *Beyond the Third Dimension: Geometry, Computer Graphics, and Higher Dimensions*. Scientific American Library, New York, NY, 1990.

(Brun et al. 1989) R. Brun, O. Couet, C. Vandoni, and P. Zanarini. *PAW – Physics Analysis Workstation, The Complete Reference*. CERN, Geneva, Switzerland, October 1989. Version 1.07.

(Chen et al. 1988) Michael Chen, S. Joy Mountford, and Abigail Sellen. A study in interactive 3-d rotation using 2-d control devices. In *Proceedings of Siggraph 88*, volume 22, pages 121–130, 1988.

(Coxeter 1991) H.S.M. Coxeter. *Regular Complex Polytopes*. Cambridge University Press, second edition, 1991.

(Edmonds 1957) A.R. Edmonds. *Angular Momentum in Quantum Mechanics*. Princeton University Press, Princeton, New Jersey, 1957.

(Efimov and Rozendorn 1975) N.V. Efimov and E.R. Rozendorn. *Linear Algebra and Multi-Dimensional Geometry*. Mir Publishers, Moscow, 1975.

(Feiner and Beshers 1990a) S. Feiner and C. Beshers. Visualizing n-dimensional virtual worlds with n-vision. *Computer Graphics*, 24(2):37–38, March 1990.

(Feiner and Beshers 1990b) S. Feiner and C. Beshers. Worlds within worlds: Metaphors for exploring n-dimensional virtual worlds. In *Proceedings of UIST '90, Snowbird, Utah*, pages 76–83, October 1990.

(Francis 1987) G.K. Francis. *A Topological Picturebook*. Springer-Verlag, New York, 1987.

(Hanson and Cross 1993) A.J. Hanson and R.A. Cross. Interactive visualization methods for four dimensions. In *IEEE Visualization '93*, 1993. To be published.

(Hanson and Heng 1992a) Andrew J. Hanson and Pheng A. Heng. Four-dimensional views of 3-D scalar fields. In Arie E. Kaufman and Gregory M. Nielson, editors, *Proceedings of Visualization 92*, pages 84–91, Los Alamitos, CA, October 1992. IEEE Computer Society Press.

(Hanson and Heng 1992b) Andrew J. Hanson and Pheng A. Heng. Illuminating the fourth dimension. *IEEE Computer Graphics and Applications*, 12(4):54–62, July 1992.

(Hanson 1992) Andrew J. Hanson. The rolling ball. In David Kirk, editor, *Graphics Gems III*, pages 51–60. Academic Press, San Diego, CA, 1992.

(Hocking and Young 1961) John G. Hocking and Gail S. Young. *Topology*. Addison-Wesley, 1961.

(Møller 1972) C. Møller. *The Theory of Relativity*. Oxford, Clarendon Press, 1972.

(Noll 1967) Michael A. Noll. A computer technique for displaying n-dimensional hyperobjects. *Communications of the ACM*, 10(8):469–473, August 1967.

(Phillips et al. 1993) Mark Phillips, Silvio Levy, and Tamara Munzner. Geomview: An interactive geometry viewer. *Notices of the Amer. Math. Society*, 40(8):985–988, October 1993. Available by anonymous ftp from geom.umn.edu, The Geometry Center, Minneapolis MN.

(Sommerville 1958) D.M.Y. Sommerville. *An Introduction to the Geometry of N Dimensions*. Reprinted by Dover Press, 1958.